Efficient Propagation of Shocks and the Optimal Return on Money*

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Abstract

We study optimal allocations in an environment in which money is essential due to lack of commitment and anonymity of individuals. Because the economy features aggregate preference shocks, we apply a notion of implementability that allows for allocations with non-trivial business-cycle dynamics for the propagation of shocks. We show that history-dependence is predicted by the theory of second best and becomes necessary for optimality when the degree of patience is neither too low nor too high. Our analysis concludes with a discussion of whether there is a role for the propagation of shocks in alternative economic environments.

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1 Introduction

The view that monetary transactions have particular implications for business cycles, with shocks propagated in a persistent fashion, or even according to sticky aggregate prices, is an organizing principle in many macroeconomic models. That principle, for instance, can be found in the Phillips-curve correlations derived by Lucas [[10]]. Price adjustments in his model follow lags in the flow of information about the nature or magnitude of aggregate shocks. Regardless of whether shocks are nominal or real, theoretical derivations of aggregate persistence have, in common with [[10]], an appeal to assumptions that do not seem necessary for money to have a role in the underlying world. In other words, economists have not linked the dependence of macro aggregates on histories to the fundamental properties of money.

In this paper, we provide such a link in a setup best described as a standard model of money as a medium of exchange, one without monitoring of individuals. By formulating a planner's problem, subject to participation constraints, we postulate that if price and output persistence is a fundamental property of monetary economies, then that property must appear in the optimal return on money. We then ask what the theory of second best predicts when the economy is hit with aggregate shocks, and a candidate allocation has participation constraints binding in some states but not in others. The prediction is that welfare can be improved by having the return on money distorted in all states and histories.

Our findings point towards an *envelope* argument as follows: Suppose that output is only allowed to vary with the last realization of the shock, in what we call a *state-dependent* allocation. Suppose further that such an allocation has output in one state at the first-best level, with participation constraints slacking in that state, but has output constrained at a different state. We would then argue that there exists a *history-dependent* allocation, one that is allowed to vary with the whole history of realizations, and that improves welfare. That is so because small changes around the first-best output level have negligible welfare costs, but can alleviate constraints in the binding state. We emphasize that in our economic environment history-dependence is not an artifact of incomplete information about aggregate shocks.[1]

We find that the heterogeneity created by monetary transactions introduce discrepancies between social welfare and the incentives necessary to convince individuals to accept money, which is the case when buyers and sellers value money holdings differently. This property puts monetary models in the same category as models with distortions for which second-best allocations are relevant. However, history-dependence is not a direct implication of the theory of the second best, as we show by briefly describing a gift-giving economy, based on [[3]]. Our particular example is constructed so that individual heterogeneity is assumed away and has monitoring substituting for money. As we show, the optimum does not feature history-dependence. For some parameters, the optimum is a second best, with participation constraints slacking in one state, not in the other, but still without friction between incentives and welfare.

There is a voluminous literature that derives history-dependent allocations in principal-agent problems with imperfect risk sharing, but the results in this literature are not directly relevant to our monetary context.[2] Our monetary economy, with anonymous individuals, has no role for

[1]By contrast, the message in [[15]] is that persistence/stickiness à la Lucas [[10]] blends well with the Kiyotaki and Wright [[7]] model, insofar information about shocks can be acquired slowly through pairwise meetings. See [[5]] for extensions of this argument. See also [[2]] for a search model, with alternating changes in preferences, in which the optimum features a cyclical supply of money.

[2]Our basic environment has no room for insurance, in contrast to models of optimal risk sharing, such as the principal-agent economy of [[12]], and the no-aggregate-uncertainty economy of [[4]]. As a recursive treatment of aggregate distortions, our main debt is to [[6]] and [[1]]. An alternative view of macroeconomic history dependence, apart from the foundations of money, can be found in [[16]].

mechanisms that rely on monitoring or reports sent to a planner or principal. Moreover, the focus of our paper is on macroeconomic persistence and not on history-dependence at the individual level. Since our results are novel in the macroeconomic literature, special consideration must be made for our simplifying assumptions, in particular the restriction to 0-1 holdings of money. We now briefly describe why such assumptions are necessary, and leave for the conclusion a discussion of their role in the results.

The task of using a model of heterogeneous agents to assess properties of monetary business cycles has challenges of its own. In the case of random-matching models, a general recursive approach requires three state variables: the distribution of money, the value function for money, and the last realization of the shock (see [[1]]). By restricting individual holdings to 0 or 1 unit, however, we can work here with a problem of lower dimensionality since the value function of money holdings can be replaced by a scalar variable that we label *return* (of money), which captures the relevant utility promises. Even in this limited setting, there are two kinds of participation constraints: one for the producer who acquires money, and one for the consumer who acquires goods. We are able to show that only the producer constraint matters when shocks are sufficiently small.

More specifically, trade takes place at random, pairwise meetings, and the probability that money is spent in a given period is proportional to $\frac{1}{N}$, where N, the number of individual types, describes the degree of absence of the double coincidence of wants. We allow allocations to respond to the history of realizations of a common shock, assumed *iid*. The equilibrium concept is that of mechanism design: allocations are implementable if they respect individual rationality, which, due to the absence of commitment, and to a lack of differential information, takes the form of simple participation constraints. By ranking allocations according to ex-ante average utility, we identify parameters for which the optimum is history dependent. We find, loosely speaking, that the transition from old to new returns has a memory parameter $1 - \frac{1}{N}$, which captures the odds of realizing the return on money in a given period. That parameter reflects the fact that monetary returns can be written as a sum of future payoffs, discounted at a higher rate than that of time preferences.[3]

The rest of the paper is organized as follows. In Section 2, we describe the random-matching environment and its implementable set of non-stationary allocations. In Section 3, we derive a stochastic difference equation for the multiplier of the producer constraint. We also describe examples of allocations that are candidates for optima. In one such allocation, the economy may start in a "recession", that is, in a situation such that the nominal price is sufficiently low for accommodating the trade of goods in exchange for a fiat promise (goods in the future with some probability). That is not the case when the aggregate shock sends the economy into a "boom", when the expected value of money constrains output. The alternative is then to improve returns by inducing expectations that, in the future, bad shocks will be dealt with in ways that increase output and thus the value of money. The relevance of the examples is proved in Section 4. When shocks are removed, the optimal output is constant and the consumer constraint slacks. When shocks are sufficiently small, the dynamics are described by simple extensions of the deterministic case. We then proceed, in Section 5, with the discussion of the economy with monitoring, and an assessment of the role of our simplifying assumptions.

[3] We conjecture that the envelope argument applies to the random-matching model with general and observable money holdings if the support of aggregate shocks is sufficiently large.

2 The random-matching environment

Time is discrete and the horizon is infinite. A measure-one continuum of individuals lives forever and is specialized in consumption and production. Goods are divisible and perishable. Production and consumption take place in bilateral trade according to random meetings. People are equally divided according to specialization types, $i = 1, 2, ..., N$, so that type i consumes good i but produces good $i + 1$ (modulo N). We assume that $N > 2$, so that the probability of a single coincidence of wants is $\frac{1}{N} \in (0, 1)$ and barter is not possible. As usual, the setup is chosen so that outcomes are symmetric across types, and the notation i can be dispensed with.

At the beginning of every period, the economy is hit by an aggregate shock s, with support in $\{l, h\}$. The notations l (*low*) and h (*high*) are motivated by our interest in outcomes with the property that aggregate output is constrained with the realization of the high shock, but not with that of the low shock. The shock s is *iid* over time. The probability that the realization in the next period is s is strictly positive and denoted π_s.

People meet in pairs according to random meetings. The utility from consumption of $y \geq 0$ units is $u_s(y)$ when the shock is s, where u_s is continuous, strictly concave, differentiable, and such that $u_s(0) = 0$, $u_s'(0) = \infty$, and $u_s(y) < y$ for y sufficiently high. The utility for the producer in a meeting, producing $y \geq 0$, is $-y$, independently of the aggregate shock. We assume that production is bounded above by an arbitrarily large maximum output. The discount factor is $\beta \in (0, 1)$.

People cannot commit to future actions, and individual histories are private. Money holdings are restricted to $\{0, 1\}$, and are observable by the pair in a meeting. Let s_0 denote the date-0 realization of the aggregate shock. The welfare criteria, used for selecting optimal allocations, is the ex-ante expected utility conditional on s_0, where the expectation is taken with regards to the invariant distribution of money holdings. The *first-best* level of output in realization s is y_s^* such that $u'_s(y^*_s) = 1$. Thus, first-best levels of output maximize the sum of utilities of the consumer and the producer in a single-coincidence meeting conditional on the realization of the aggregate state.

An event for this economy at date t is denoted s^t. We often use the expression (s^{t-1}, s_t) for s^t when we want to emphasize that the last realization of s^t is s_t. Hence $s_t \in \{l, h\}$, and s^t is the list, of size $t+1$, $\{s_0, s_1, ..., s_t\}$. We assume that the history of shocks s^t is public information. Let us denote by S^t the set of possible histories, up to t, and let us denote by $p_t(\cdot)$ the probability measure on S^t induced by the stochastic process for s. We assume that in period-0 a fraction $m \in (0, 1)$ of individuals holds one unit of money. We consider the fraction m as being part of the description of the random-matching environment. Since the value of money will be non-negative in an optimal allocation, individuals will not have incentives to dispose of their money holdings. With the restriction to either 0 or 1 holdings, as assumed, trade preserves the distribution of money, which can also be ignored in the analysis.[4] Since one unit of money is exchanged for y units of a good, the nominal price is simply $\frac{1}{y}$, and is thus implicit in the analysis of output.

An *allocation* for this economy is a sequence of output functions $y_t : S^t \to \mathbb{R}_+$, or simply $y(s^t)$, that is exchanged for money in meetings in which the consumer in single-coincidence meetings has money, and the producer has not, for each s^t in S^t, and each $t = 0, 1, ...$[5] By assumption, $p(s_0) = 1$. The welfare criterion is

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t p(s^t) z_{s_t}(y(s^t)) \tag{1}$$

[4]In the random-matching environment, our goal is to describe the necessary conditions for optimality for arbitrary $m \in (0, 1)$. Such conditions hold for the particular case in which m is also chosen optimally.

[5]Throughout the paper we follow the convention of writing $y(s^t)$ to denote a function of histories at time t, which abstracts from the fact that the domain S^t of the function depends on t. The same convention applies to other variables.

where

$$z_s(y) \equiv m(1-m)\frac{1}{N}(u_s(y)-y). \tag{2}$$

The function z represents the average social gain from monetary exchange, insofar as $m(1-m)\frac{1}{N}$ is the aggregate measure of potential trade (meetings in which one individual has money, the other has not, and where there is a single coincidence of wants) and $u_s(y)-y$ is the average payoff for an exchange for y when the shock is s.

2.1 Implementability and optimality

Let $v_j(s^t)$ denote the expected, discounted utility of an individual with money holdings $j \in \{0,1\}$, at date t and history s^t, before meetings occur. The expected utilities for this economy, given an allocation, are described by

$$\begin{aligned} v_1(s^t) &= (1-m)\frac{1}{N}[u_{s_t}(y(s^t)) + \beta(\pi_l v_0(s^t,l) + \pi_h v_0(s^t,h))] + \\ &\quad (1-(1-m)\frac{1}{N})\beta(\pi_l v_1(s^t,l) + \pi_h v_1(s^t,h)) \end{aligned} \tag{3}$$

and

$$\begin{aligned} v_0(s^t) &= m\frac{1}{N}[-y(s^t) + \beta(\pi_l v_1(s^t,l) + \pi_h v_1(s^t,h))] + \\ &\quad (1-m\frac{1}{N})\beta(\pi_l v_0(s^t,l) + \pi_h v_0(s^t,h)), \end{aligned} \tag{4}$$

provided that the sequence $v(s^t) = (v_0(s^t), v_1(s^t))$ is bounded.

There are two kinds of participation constraints, one for the producer and one for the consumer. The producer constraint is

$$y(s^t) \le \beta(\pi_l \partial v(s^t,l) + \pi_h \partial v(s^t,h)) \tag{5}$$

where

$$\partial v(s^t) \equiv v_1(s^t) - v_0(s^t). \tag{6}$$

The consumer constraint is

$$u_s(y(s^t)) \geq \beta(\pi_l \partial v(s^t, l) + \pi_h \partial v(s^t, h)). \tag{7}$$

Definition 1 *An output allocation $y(s^t)$ is implementable if there exists a bounded $v(s^t)$ satisfying (3-7) for all $s^t \in S^t$ and all $t = 0, 1, 2, \ldots.$ An allocation is optimal if it maximizes (1) among the set of implementable allocations.*

We emphasize that our notion of implementation only requires that agents have rational expectations and behave rationally. Since individuals understand the world they live in, they can rationally forecast the expected utility gain of holding money at different histories s^t. In particular individuals can solve the planner's problem that gives the optimal terms of trade in bilateral meetings (which we state in the next section). As a result, our implementation does not require the planner to send any message stating the terms of trade and expected return on money. In fact, we assume that the economic environment is such that there is no possible communication between the planner and individuals.

2.2 The sequential planner's problem

We now write a simple planner's problem that we shall use to characterize optimal allocations. To this end, we assume that the support of s, together with (u_s, π_s), is such that the satisfaction of the producer constraint implies the satisfaction of the consumer constraint (an assumption which must be verified later).

We start by writing an expression linking $\partial v(s^t)$ with the conditional expectation of $\partial v(s^t, s_{t+1})$. We shall argue that $\partial v(s^t)$ and s_t are the key state variables for the planner's problem. Using (3), (4) and the definition of ∂v, (6), it follows that, for $s^t = (s^{t-1}, s_t)$,

$$\partial v(s^t) = f_{s_t}(y(s^t)) + (1 - \frac{1}{N})\beta(\pi_l \partial v(s^t, l) + \pi_h \partial v(s^t, h)), \tag{8}$$

where

$$f_s(y) \equiv \frac{1}{N}((1-m)u_s(y) + my).$$

Notice that $f_s(y)$ depends on history only by way of s, the current realization, and y, the current level of output, and that $f_s(\cdot)$ is continuous, increasing, concave, differentiable, and such that $f_s(0) = 0$.

As noticed, we anticipate that the consumer's constraint is irrelevant. We also anticipate that as long as upper bounds on the return on money are verified, the condition (8) can be replaced by an inequality constraint on the minimum return in a given history.[6] Using the notation $\alpha \equiv (1 - \frac{1}{N})\beta$, the planner's problem is then given by

$$\max_{\{y(s^t), \partial v(s^t)\}} \sum_{t=0}^{\infty} \sum_{s_t \in S^t} p(s^t)\beta^t z_{s_t}(y(s^t)) \tag{9}$$

s.t.

$$\begin{aligned} y(s^t) &\leq \beta(\pi_l \partial v(s^t, l) + \pi_h \partial v(s^t, h)) \\ \partial v(s^t) &\leq f_{s_t}(y(s^t)) + \alpha(\pi_l \partial v(s^t, l) + \pi_h \partial v(s^t, h)) \\ 0 &\leq \partial v(s^t) \leq B \text{ for all } s^t \text{ and all } t. \end{aligned}$$

The constraint $\partial v(s^t) \leq B$ rules out "Ponzi schemes" so that the return on money is bounded above by the discounted expected-utility gain of having one unit of money

$$\partial v(s^t) \leq f_{s_t}(y_{\max}) + \sum_{\tau > t} \sum_{s_\tau \in S^\tau} \alpha^{\tau - t} p(s^\tau)\ f_{s_\tau}(y_{\max}).$$

where $y_{\max}$ is an exogenous bound on output. We thus assume that B is large enough not to bind in the solution to the maximization problem. The next proposition shows that the non-Ponzi

[6] Working with an inequality constraint for the return to money facilitates the description of the planner's value function in the recursive version. It is trivial to check that optimal plans for relevant histories will satisfy the constraint with equality.

condition implies a state-dependent maximum sustainable return on money. We shall refer to the (promised) return on money, $\partial v(s^t)$, as *return*, and use the variable r for its counterpart in the recursive approach. We end this subsection with a derivation of upper bounds for return.

Proposition 2 ***(Maximum Sustainable Return)*** *Any sequence $\{y(s^t), \partial v(s^t)\}$, satisfying the constraints of the planner's problem, is such that $\partial v(s^t) \leq \bar{r}_s$ for all s^t, where $\bar{r}_s$ solves $\bar{r}_s = f_s(\beta\bar{r}) + \alpha\bar{r}$ and $\bar{r} = \pi_l\bar{r}_l + \pi_h\bar{r}_h$ for $s \in \{l, h\}$.*

Proof. Deriving the sustainable-return interval in the deterministic case is quite a simple task, which we build on for the stochastic case. We start by noticing that we can simplify the analysis by assuming that the producer's constraints hold with equality, since the set of sustainable return is non-decreasing in output levels. In this case, allocations are fully described by a contingent sequence $\partial v(s^t)$, with the understanding that output is pinned down by the producer's constraint. We then associate with each allocation in this class a sequence of expected return, and investigate under which conditions such a sequence is bounded.

We use the shorter notation $r(s^t)$ for return, instead of $\partial v(s^t)$. We start by studying restrictions on expected return by considering the function $g(x) = \pi_l f_l(\beta x) + \pi_h f_h(\beta x) + \alpha x$, for $x \in \mathbb{R}_+$. Since the derivative of u_s is infinite at 0, $\pi_l u_l'(0) + \pi_h u_h'(0) > \frac{1-\beta+\beta\frac{1}{N}(1-m)}{\beta\frac{1}{N}(1-m)}$ so that $g'(0) > 1$. Then, let $\bar{r}$ be the unique positive fixed point of g, that is, the value of $x > 0$ for which $g(x) = x$. Note that $g(x)$ is strictly positive for $x > 0$, increasing, and concave. That $\bar{r}$ is well defined and unique follows from properties of f (concavity, strictly increasing function, and the conditions assumed on u_s around 0). The values of maximum sustainable return in the statement of the proposition satisfy $\bar{r}_s = f_s(\beta\bar{r}) + \alpha\bar{r}$ for $s \in \{l, h\}$.

Let now $r(\cdot)$ be given and suppose, by way of a contradiction, that $r(s^t) > \bar{r}_{s_t}$, for some s^t. Define the state-dependent function $g_{s_t}(x) = f_{s_t}(\beta x) + \alpha x$. Feasibility requires that the next period expected return r^e at history s^t satisfies $r(s^t) \leq f_{s_t}(\beta r^e) + \alpha r^e = g_{s_t}(r^e)$. Because g_{s_t} is an increasing function with $g_{s_t}(\bar{r}) = \bar{r}_{s_t}$ and $r(s^t) > \bar{r}_{s_t}$, it follows that $r^e > \bar{r}$.

Notice that, by construction, the function g is a recursion expressing expected return at history s^t as a function of expected return at history s^{t+1}. As a result, the inverse of the function g defines a lower bound for the expected return at node s^{t+1} in terms of the expected return at node s^t. Since g is an increasing concave function satisfying $g(\bar{r}) = \bar{r}$, it follows that g^{-1} is an increasing, convex function satisfying $g^{-1}(\bar{r}) = \bar{r}$. As a result, any feasible sequence $r(s^t)$ for which $r^e(s^t) \equiv \pi_l r(s^t, l) + \pi_h r(s^t, h) > \bar{r}$ should satisfy $r^e(s^{t+1}) \geq g^{-1}(r^e(s^t)) > r^e(s^t) > \bar{r}$. Convexity of g^{-1} implies that the sequence $\left\{r^e(s^t)\right\}$ grows without bounds. Now, if the expected return grows unboundedly, it follows that $r(s^t) > B$ for some history s^t. We thus contradict feasibility of the sequence $\{y(s^t), r(s^t)\}$. ■

2.3 The recursive planner's problem

As a simple application of Bellman's optimality principle, the sequential planner's problem has a recursive representation in which the triple (s, r_l, r_h) is the state.[7] Since only return r_s is the relevant promise in realization s, the planner's value function is constant in $r_{s'}$ for $s' \neq s$. Thus the Bellman equation of interest has only the pair (s, r) as a state (it helps, as we do in some contexts,

[7] Cavalcanti and Monteiro ([1]) derive a recursive formulation for general money holdings. The state of the general formulation is the triple (μ, v_s, s), where μ is the distribution of money holdings, and v_s is the value function in state s, describing expected utilities associated with each level of money holdings. In our application, μ is constant, as a result of 0-1 holdings, and only the difference in 0-1 values appears in the participation constraints.

to think of r as a short for r_s). We thus consider the following dynamic programming problem:

$$\begin{aligned} w(s,r) &= \max z_s(y) + \beta[\pi_l w(l,r_l') + \pi_h w(h,r_h')] \quad (10) \\ & \text{s.t.} \\ y &\leq \beta(\pi_l r_l' + \pi_h r_h') \quad (11) \\ r &\leq f_s(y) + \alpha(\pi_l r_l' + \pi_h r_h') \quad (12) \end{aligned}$$

where the maximization is taken over $(y, r_l', r_h') \in \mathbb{R}_+ \times R_l \times R_h$, and where $R_s \equiv [0, \bar{r}_s]$ is defined by Proposition-2 bounds. The Bellman equation (10) can be interpreted as follows. At each date, after the realization of the shock takes place as s, the decision unit chooses output y, taking into account the return function z_s and the limits imposed by the participation constraint (11), as well as the impact of y on the return on money at the current period, which is given by the right-hand side of (12). Since the contingent return, r, guided producer's choices in the previous period at the time of exchange, then rational expectations require that the current production, as well as future expectations about the value of money, (r_h', r_h'), are sufficiently high to justify the expected value r. In this problem, the decision unit inherits a promise r, which must be honored by an action y, together with future promises (r_h', r_h') which are to be debited from the future according to the value function w.

Alternatively, the Bellman equation can be written as a fixed point of an operator T. For such a formal description, we define W, the set of functions w mapping pairs (s, r) into real numbers, such that $w(s, \cdot) : R_s \to \mathbb{R}$ is bounded, weakly decreasing, concave, and continuous. We anticipate that the following operator maps W into itself:

$$Tw(s,r) = \max_{y, r_l', r_h' \in \Gamma(s,r)} z_s(y) + \beta[\pi_l w(l, r_l') + \pi_h w(h, r_h')],$$

where $\Gamma(s,r) = \{(y, r_l', r_h') \in \mathbb{R}_+ \times R_l \times R_h$ such that $y \leq \beta(\pi_l r_l' + \pi_h r_h')$ and $r \leq f_s(y) + \alpha(\pi_l r_l' +$

$\pi_h r'_h)\}$. We now prove that T is well behaved.

Proposition 3 *Let $w \in W$. Then, Tw is bounded, continuous, weakly decreasing and concave in r.*

Proof. Boundedness follows trivially. Moreover, since $\Gamma(s,r)$ is decreasing in r_s, it follows that Tw is also decreasing in r. Note that $\Gamma(s,.)$ is a compact-valued, non-empty and continuous correspondence. Then, for $w \in W$ it follows from the Theorem of the Maximum that Tw is continuous. To establish the concavity of Tw, note that using the concavity of f, it is easy to show that $\Gamma(s,.)$ is a convex correspondence. Then, arguments similar to the ones in Proposition 4.8 in [[13]] (just follow the arguments for establishing weak concavity) imply that Tw is a concave function. ■

Proposition 4 *There exists an unique solution w to the Bellman equation (10). Moreover, if $\{y(s^t)\}$ is the allocation generated by the optimal policy functions for w, from the initial state $(s,r) = (s_0, 0)$, then $\{y(s^t)\}$ solves the sequential planner's problem.*

Proof. Blackwell's sufficient conditions for a contraction implies that T is a contraction. It follows from the Contraction Mapping Theorem that there exists a unique $w \in W$ such that $w = Tw$. Proposition 4.2 in [[13]] implies that $w(s,r) = w_s^*(r_s)$, where $w_s^* : R_s \to R$ gives the utility attained in Problem (9) when the initial state is s_0 and the planner is constrained to deliver a return on money $r_{s_0} \in R_{s_0}$. Given that $w(s,.)$, and hence that w_s^*, is a non-increasing function of r, the solution to the planner's problem gives welfare equal to $w(s,0) = w_{s_0}^*(0)$, and the optimal allocation is implied by the policy functions attaining w from the initial state $(s,r) = (s_0, 0)$. ■

It is instructive to rewrite Bellman's equation as follows

$$
\begin{aligned}
w(s,r) &= \max_{y,r'}\{z_s(y)+\beta v(r')\} \qquad (13)\\
&\text{s.t.}\\
y &\leq \beta r',\\
r &\leq f_s(y)+\alpha r',
\end{aligned}
$$

where v is a concave non-increasing function defined as

$$
\begin{aligned}
v(r) &= \max_{r_l,r_h}\{\pi_l w(l,r_l)+\pi_h w(h,r_h)\}\\
&\text{s.t.}\\
r &= \pi_l r_l+\pi_h r_h.
\end{aligned}
$$

The maximization problem in (13) is, for a given v, a one-period problem and has a simple graphical representation. It involves a trade-off between current output y and the level of expected return r' for tomorrow. The FOC for output is given by $z_s'(y)-\mu+\lambda f_s'(y)=0$, where μ and λ represent the Lagrange multipliers on the producer and return constraints. The solution to the one-period problem can take one of three possible forms: (a) return constraint binds and producer constraint slacks, (b) return and producer constraints bind, and (c) return constraint does not bind. Case (c) is irrelevant because we can assume, without loss of generality that r is high enough so that the return constraint binds.[8]

[8] In case (a), optimal output satisfies $z_s'(y)=-\lambda f_s'(y)<0$. The concavity of z together with the fact that $z_s'(y_s^*)=0$ implies that the optimum satisfies $y>y_s^*$. As a result, output is above the first-best level when the producer constraint binds. Case (b) corresponds to a corner solution in which the indifference curve going through the optimal choice (y,d') is not tangent to the boundary of the constraint set. In this case, output y could be either above or below the first-best level, depending on the sign of $\mu-\lambda f_s'(y)$.

It should be clear that in the one-period problem, the state of the economy as given by the current realization of the shock s and the return level r, determines which particular form the solution takes (among the three possibilities just described). It is natural to expect that as the state of the economy changes, the solution to the one-period problem could change in type. In the next section, we investigate how the state of the economy changes over time and how these changes affect the solution to the one-period problem.

3 Basic allocations

We say that an allocation is state dependent when the current-period output and return plans for next period depend on history only through the current realization of the preference shock. Otherwise, we say that the allocation is history-dependent. In the case of solutions that are history-dependent, the past promise to producers is a relevant state variable, and the realization of the current preference shock is not a sufficient statistic to determine the optimal choices in a given period. Note that in a history-dependent solution, there is always a positive probability that the economy reaches a state in which output and state-contingent return (to money) are constrained by (past) promises to producers. In other words, in a history-dependent allocation, the economy reaches with positive probability states in which the return constraint is active. As we shall see, history-dependence also requires that producer constraints do not bind forever (with probability 1). With the first-best allocation, output y_s^* is state-dependent because it only varies with the current realization of the preference shock. The first-best allocation however may not be implementable. In this case, we want to know whether the second-best allocation is history dependent and, if the answer is yes, under what conditions.

We can gain some insights about conditions necessary for history-dependence, as well as the memory of the system, by considering the first-order conditions (FOC) of the sequential planner's problem (9) (note that it is a standard convex optimization problem). If we associate the multipliers $\beta^t p(s^t)\,\mu(s^t)$ and $\beta^t p(s^t)\,\lambda(s^t)$ to the producer and return constraints, the FOC for optimality imply

$$\begin{aligned}
y(s^t) &: \quad z'_{s_t}(y(s^t)) - \mu(s^t) + \lambda(s^t) f'_{s_t}(y) = 0 \\
\partial v(s^t, s_{t+1}) &: \quad \beta\mu(s^t) + \alpha\lambda(s^t) - \beta\lambda(s^t, s_{t+1}) = 0 \\
\mu(s^t) &: \quad \beta(\pi_l \partial v(s^t, l) + \pi_h \partial v(s^t, h)) - y(s^t) \geq 0 \\
\lambda(s^t) &: \quad f_{s_t}(y(s^t)) + \alpha(\pi_l \partial v(s^t, l) + \pi_h \partial v(s^t, h)) - \partial v(s^t) \geq 0
\end{aligned}$$

The condition for $\mu(s^t)$ (as well as that for $\lambda(s^t)$) becomes an equality if $\mu(s^t) > 0$ (if $\lambda(s^t) > 0$). We have already used the fact that in any optimal solution, y and ∂v are strictly positive. The FOC with respect to $\partial v(s^t, s_{t+1})$ yields

$$\lambda(s^t, s_{t+1}) = \mu(s^t) + (1 - \frac{1}{N})\lambda(s^t), \tag{14}$$

which implies that the value of the multiplier on the return constraint at $t+1$ does not depend on the realization of s_{t+1}. Intuitively, the planner chooses a state-contingent return on money so that the marginal burden of the return constraint is equalized across states of nature.

In period-0 the initial return level is unrestricted. It is optimal, without loss of generality, to set $r(s_0) = 0$. It follows that the return constrain does not bind in period-0 and $\lambda(s_0) = 0$. Now, when s^t is such that the producer's constraint does not bind ($\mu(s^t) = 0$), it follows from (14) that next-period's return constraint will not bind. Moreover, once the producer constraint binds for the first time, the following period the return constraint will bind for the first time. History-dependence, hence, requires that the producer constraint binds. Intuitively, in order to relax a binding producer constraint, the planner makes promises to induce producers to increase output. This comes at the cost of restricting next period's (and all future) choices.

Consider now a history s^t with a binding return constraint $\lambda(s^t) > 0$. If the producer constraint does not bind ($\mu(s^t) = 0$), it follows from (14) that $\lambda(s^t, s_{t+1}) = (1 - \frac{1}{N})\lambda(s^t) < \lambda(s^t)$. Thus, the return constraint is less binding following periods in which the producer constraint did not bind. The rate of decay of $\lambda(s^t)$ depends on $\frac{1}{N}$, the matching friction that gives a role to money. Since $\frac{1}{N} < 1$, the return on money will be linked across periods. When past promises to producers bind, the return on money becomes *sticky*.

In the next section, we demonstrate the existence of history-dependent allocations. In particular, we show that in this monetary economy, history-dependence requires aggregate uncertainty and a discount factor that is neither too high nor too low. The following basic result is useful for predicting state dependence.

Lemma 5 ***(Envelope Property)*** *Suppose that the optimum is state dependent, with slacking consumer constraints, but it is not the first best. Then, the producer constraint binds in both states.*

Proof. The proof is a direct consequence of the FOC of the sequential problem. If the optimum is state dependent, then there are functions $(y, \lambda, \mu, \partial v)$ of $s \in \{l, h\}$ satisfying the FOC. But since λ must be a constant function of the last realization of the shock, then μ must also be constant in s. Since the optimum is not the first best, μ must be positive. ■

3.1 Examples

We end this section with examples of some basic allocations that are candidates for optima. When multiplier dynamics occur, they follow the law of motion (14), which is sufficiently general to cover all cases, including the extreme in which aggregate shocks are ignored. Such a case is considered because it helps to establish that the consumer constraints can be ignored when shocks are small, and also because an artificial transition can be described in the deterministic case as well. The transition is artificial insofar as it requires a high level of initial return which is never chosen for optimal allocations in the deterministic case.

In the examples that follow, $\bar{r}$ is the maximum expected return $\pi_l \bar{r}_l + \pi_h \bar{r}_h$, where $\bar{r}_s$ is the Proposition-2 bound in state s. We also use $\bar{y}$ for the maximum output $\beta\bar{r}$ (which is also well defined when shocks are removed in our deterministic case). In addition to the first-best levels of output, y_l^* and y_h^*, we also find it useful to define r_s^*, as well as $r^* = \pi_l r_l^* + \pi_h r_h^*$, which are the first-best return levels, given by the same equations in Proposition 1 for $\bar{r}_s$ and $\bar{r}$, but using as the argument of f_s the first-best output levels, instead of the maximum sustainable ones. We start with the unconstrained solution,

Example 1 (first-best allocation). The average return is r^* and output in state s is y_s^*, which satisfies $y_l^* < y_h^* < \beta r^*$.

Hence, this class of allocations is state-dependent. Another class for which policy functions are constant is

Example 2 (flat allocation). The average return is $\bar{r}$ and output in both states is $\bar{y} = \beta\bar{r}$.

We shall show that the example-2 allocation is optimal when $\beta\bar{r} < y_l^* < y_h^*$, and that the planner's value function is flat on the set of sustainable levels of return.

We now focus on a situation such that $y_l^* < \beta r^*$, and hence first-best output in l is feasible, but $y_h^* > \beta\bar{r}$. That is, first-best output is not feasible in h.

Example 3 (price/output persistence). The allocation is described with the help of threshold return levels $(\widehat{r}_l, \widehat{r}_h)$. When $s = l$ and $r \leq \widehat{r}_l$, output is y_l^* and the contingent promise is $(\widehat{r}_l, \widehat{r}_h)$, that is, a constant. If $s = l$ but r is in a right-neighborhood of $\widehat{r}_l$, output is greater than y_l^*, and each promise exceeds $\widehat{r}_s$, but an infinite sequence of repetitions of $s = l$ brings (r_l', r_h') down to $(\widehat{r}_l, \widehat{r}_h)$.

Now, if $s = h$, no matter the level of initial return, output belongs to the interval (y_l^*, y_h^*) while each promise exceeds $\widehat{r}_s$, without a tendency to approach $(\widehat{r}_l, \widehat{r}_h)$ with repetitions of state h.

Example 3 is such that the return multiplier evolves over time as output-persistence is used in order to reduce the growing return of high-shock periods. Examples 1 and 2 feature no dynamics in λ, either because the economy is in the first best (Example 1), or because output is constant (Example 2). In both cases, the average future return is a constant.

4 Main results

We now present the economies for which the examples of the previous section describe the solutions. For ease of exposition, we start the analysis by removing the aggregate shocks.

4.1 No aggregate-uncertainty

Here we show that the economy without aggregate uncertainty does not feature history-dependence. Intuitively, history-dependence requires that the promises made by the planner to producers vary over time. In the absence of aggregate uncertainty, the planner does not gain by varying promises to producers and, as a result, allocations do not depend on history.

Consider a sequential planner's problem in which the initial return on money is fixed at r_0, without aggregate uncertainty. This problem is given by

$$
\begin{aligned}
w^*(r_0) &\equiv \max_{\{y_t, r_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t z(y_t) \\
& \quad s.t. \\
y_t &\leq \beta r_{t+1}, \\
r_t &\leq f(y_t) + \alpha r_{t+1}, \\
r_t &\in R, \ r_0 \text{ given}
\end{aligned}
$$

where $R = [0, \bar{r}]$ is the sustainable-return interval (the deterministic case of the general definition). Again, the dependence of w^* on r_0 is just included in order to facilitate the connection with the recursive formulation, although it is understood that r_0 is not a true given and is instead freely chosen as the initial return on money is not affected by past history.

In the next proposition we characterize the recursive solution to the Bellman equation that w^* must satisfy. The proof of the proposition yields a partition of $(0, 1)$ into two intervals for β. If the discount factor belongs to the high interval, the first-best allocation is implementable. This allocation does not depend on history and features a non-biding producer and return constraints (the associated multipliers are 0). Otherwise, if the discount factor belongs to the complement, the first-best allocation is not implementable. Here, both the producer and return constraints bind but the optimal allocation is constant over time and does not depend on history. This formulation illustrates that a binding return constraint is not a sufficient condition for a history-dependent allocation.

Proposition 6 *In the economy without shocks, the optimum is a constant allocation, and the consumer constraint slacks.*

Proof. Note that $\bar{r}$ is an increasing and continuous function of β, with $\bar{r} = 0$ when $\beta = 0$ and $\bar{r}$

unbounded as β approaches 1. As a result, the intermediate value theorem implies that there exists β^* such that $y^* \leq \overline{y}(\beta)$ for all $\beta \geq \beta^*$(with equality only if $\beta = \beta^*$). Consider β sufficiently high so that $y^* < \overline{y}$. Then, $r^* = \frac{f(y^*)}{1-\alpha} < \frac{f(\overline{y})}{1-\alpha} = \frac{f(\beta\ \bar{r})}{1-\alpha} = \ \bar{r}$, which establishes that $R^* \in (0,\ \bar{r})$. Consider the sequence $\{y_t, r_t\}_{t=0}^{\infty} = \{y^*, r^*\}_{t=0}^{\infty}$. The definition of r^* ensures that the return constraint is satisfied with equality by the above sequence (that is, $r^* = f(y^*) + \alpha r^*$). To show that the sequence satisfies the producer's constraint for all t, consider the function $h(y) \equiv \frac{f(y)}{1-\alpha} - \frac{y}{\beta}$, which is a strictly concave function satisfying $h(0) = h(\overline{y}) = 0$. Since $0 < y^* < \overline{y}$, it follows from the properties of h that $h(y^*) = \frac{f(y^*)}{1-\alpha} - \frac{y^*}{\beta} > 0$. It follows that $y^* < \beta\frac{f(y^*)}{1-\alpha} = \beta r^*$, which establishes that the sequence $\{y^*, r^*\}_{t=0}^{\infty}$ also satisfies the producer constraint (in addition to the return constraint). Since $z(y) \leq z(y^*)$ for all y,it follows that $w^*(r) \leq z(y^*)/(1-\beta)$. Since the sequence $\{y^*, r^*\}_{t=0}^{\infty}$ achieves $z(y^*)/(1-\beta)$ and is feasible, we conclude that $w^*(r) = z(y^*)/(1-\beta)$ for all $r \leq r^*$. Now, consider $r > r^*$. Then, its easy to see that we cannot find a sequence of return levels $\{r_t\}$ such that the sequence $\{y^*, r_t\}_{t=0}^{\infty}$ is feasible with a starting return of r. Feasibility thus requires $y_t > y^*$ for some t, which implies that $w^*(r) < z(y^*)/(1-\beta)$.

Consider β sufficiently low so that $y^* > \overline{y}$. Any feasible sequence should have $y_t \leq \overline{y}$ for all t. Otherwise, the date$-t$ producer's constraint implies $r_{t+1} > \overline{y}/\beta = \bar{r}$ and our previous proposition about maximum sustainable return implies that this is not feasible. Since $z(y)$ is an increasing function of y when $y < y^*$, we conclude that $w^*(r) \leq \bar{w} = z(\bar{y})/(1-\beta)$. Since $\bar{w}$ is attained by the feasible sequence $\{\bar{y}, \bar{r}\}_{t=0}^{\infty}$ we conclude that $w^*(r) = \bar{w}$.

It remains to show that the consumer constraint slacks. By imposing stationarity to the Bellman equations and defining individual values v_j, which now do not vary with any aggregate state, it follows that $v_0 \geq 0$ if and only if $y \leq \beta(v_1 - v_0)$, and $v_1 \geq 0$ if and only if $u(y) \geq \beta(v_1 - v_0)$. It is straightforward to check that, in any optimum for the deterministic economy, the constant return is strictly positive and that $v_0 \geq 0$, so that $v_1 > 0$ and the consumer constraint slacks. ■

Based on the previous proposition, we assume from now on that shocks are sufficiently small so that the consumer constraint can be ignored. This is not to say that, even when the consumer constraint binds, the methods here cannot be applied. As a matter of fact, we conjecture that price stickiness is still robust in such a case, but the power that history-dependence has to weaken the *producer* constraint should be smaller. We learn the following about intertemporal trade-offs for a given initial return in R.

Proposition 7 ***(Dynamics)*** *Let β be large enough so that $y^* < \beta\overline{r}$. Then, for all $r \in R$ such that $r > r^*$ we have:*

1. *$w^*$is strictly decreasing.*

2. *There are policy functions for output and return attaining the optimum which are strictly increasing in r. Moreover, $y > y^*$.*

3. *The policy function satisfies $r' < r$. Thus, for any $r_0 \in (r^*, \bar{r})$ the sequences $\{r_{n+1}, y_n\}$ generated by the policy function converge to (r^*, y^*).*

Proof. We prove the proposition in various steps.

Step 1. We first show that on a right neighborhood of r^*, the return constraint binds, the producer constraint slacks, and w^* is differentiable and strictly decreasing.

Proof of Step 1. Note that in the maximization defining w^* the return constraint and the producer constraint do not bind for all $r \leq r^*$. By Proposition 5, we know that, for all $r \leq r^*$, $w^*(r)$ is constant and (y^*, r^*) is an optimal choice of output and return in the maximization problem defining w^*. Moreover, by Proposition 5, we also know that the producer constraint slacks when

$r = r^*$. Now, consider the problem of choosing y optimally to maximize

$$\begin{aligned} v(r) &= \max_y \{z(y) + \beta w^*(r^*)\} \\ y &\leq \beta r^*, r \leq f(y) + \alpha r^*. \end{aligned}$$

The concavity of f implies that the maximization problem defining v involves the choice of y belonging to a convex-valued set Γ_r. The concavity of z then implies that v is a concave function of r. Notice that $v(r) = w^*(r)$ for all $r \leq r^*$. When $r > r^*$ it follows that $v(r) \leq w^*(r)$ since the maximization problem defining v corresponds to the maximization problem defining $w^*$when the choice of tomorrow's return is fixed at r^* (and r^* is a feasible choice in the maximization problem of w^* for r sufficiently close to r^*). For r close to r^*, the return constraint binds in the maximization problem defining v, and the producer constraint slacks. Application of the envelope and implicit function theorems gives $v'(r) = z'(y(r)) \; f^{-1}\,'(y(r)) = \frac{z'(y(r))}{f\;'(y(r))} < 0$, where $y(r)$ gives the optimal choice of output y given return r and the last equality uses the fact that f is a monotone differentiable function. Note that $y(r) > y^*$since the return constraint binds for $r > r^*$ ($y \leq y^*$ is not a feasible choice in the problem defining v), which implies that $v'(r) < 0$ according to the formula above. We can then apply the Benveniste-Sheikman Theorem to conclude that $w^*(r)$ is differentiable and that $w^{*\prime}(r) = v'(r) = \frac{z'(y(r))}{f\;'(y(r))} < 0$ for r close to r^*.

Step 2. w^* is strictly decreasing and $\lambda(r) > 0$ for all $r > r^*$, where $\lambda(r)$ represents the Lagrange multiplier on the return constraint in the maximization problem defining $w^*(r)$.

Proof of Step 2. The weak concavity of w^* together with the fact that $w^*$is strictly decreasing on the right of r^* implies that w^* is strictly decreasing for all $r > r^*$. It also follows that, with regards to the multiplier of the return constraint, $\lambda(r) > 0$ for all $r > r^*$ (otherwise, w^* would not be strictly decreasing).

Step 3. Consider $r > r^*$. Then the optimal policies $(y(r), r'(r))$ are increasing in r. Moreover,

$r'(r) < r$.

In Step 1 we have shown that the value function is differentiable in a neighborhood of r^*. Moreover, concavity of w^* implies that w^* is differentiable almost everywhere. We can then take first-order conditions to characterize the optimal choices of (y, r') :

$$\begin{aligned} y &: z'(y) - \mu + \lambda f'(y) = 0, \\ r' &: \beta w^{*\prime}(r') + \mu\beta + \lambda\alpha = 0, \\ \mu &: \beta r' - y \geq 0 \ (= 0 \text{ if } \mu > 0), \\ \lambda &: f(y) + \alpha r' - r \geq 0 \ (= 0 \text{ if } \lambda > 0) \end{aligned}$$

where μ is a Lagrange multiplier associated with the producer's constraint, λ is the Lagrange multiplier associated with the return constraint, and both multipliers are functions of r. In Step 2 we showed that $\lambda(r) > 0$ for all $r > r^*$. We divide the analysis into two cases depending on whether the producer constraint binds or not.

Case 1: $\mu(r) = 0$. Then, the FOC can be combined to obtain (substituting away λ) $\beta w^{*\prime}(\frac{r-f(y)}{\alpha}) = \alpha\frac{z'(y)}{f'(y)}$, or $\beta w^{*\prime}(\frac{r-f(y)}{\alpha}) = \alpha\frac{u'(y)-1}{u'(y)+1}$. Notice that the left-hand side is a decreasing function of y while the right-hand side is an increasing function of y. Both expressions are negative. They define two curves as a function of y. The intersection of the two curves defines the optimal policy $y(r)$. Notice that the first expression shifts up when r increases, implying that the intersection point and thus $y(r)$, shifts up with a change in r. Since the value of $\beta w^{*\prime}(\frac{r-f(y)}{\alpha})$ shifts up with the increase in r (becomes less negative), we also obtain that $r' = \frac{r-f(y(r))}{\alpha}$ increases with r. Moreover, using the FOC for r' we obtain

$$\lambda(r') = \mu(r) + (1 - \frac{1}{N})\lambda(r) = (1 - \frac{1}{N})\lambda(r),$$

where we use $w^{*\prime}(r') = -\lambda(r')$ and the assumption $\mu(r) = 0$. It follows that $\lambda(r') < \lambda(r)$ when

$\mu(r) = 0$. We thus have $w^{*\prime}(r') = -\lambda(r') > -\lambda(r) = w^{*\prime}(r)$, which by the concavity of w^* implies $r' < r$.

Case 2: $\mu(r) > 0$. Since the producer's constraint binds, we know that $y = \beta r'$. Since the return constraint binds, we know that $r' = \frac{r-f(y)}{\beta}$. Combining these expressions we obtain $\frac{r-f(y)}{\beta} = \frac{y}{\beta}$, which implies that y and r' are increasing functions of r. To establish that $r'(r) < r$, consider the function $g(r') = f(\beta r') + \alpha r'$, which is strictly positive, concave, and satisfies $g(r) \leq r$ for all $r \in R$ and with equality only when $r = \bar{r}$. Then, the inverse function of g is well defined and satisfies $r' = g^{-1}(r) < r$ for all $r < \bar{r}$. Since g^{-1} gives tomorrow's return as a function of today's return when the producer's constraint binds, we conclude that the level of return decreases for all $r \in int(R)$ when the producer's constraint binds ($\mu(r) > 0$)

The analysis of Case 1 and Case 2 tells us that the policy functions for output and return are strictly increasing and that $r'(r) < r$. As a result, the optimal policy function r' implies that the sequence $\{r_n = r'(r_{n-1})\}$ converges to r^* for any initial $r_0 \in (r^*, \bar{r})$. ■

4.2 Aggregate uncertainty

In this section, we show that when the economy features aggregate uncertainty, the planner may gain from changes in the return on money so that optimal allocations may become history dependent. The idea is that when the economy is hit by a high shock and the planner is facing a binding producer constraint, the planner may want to increase the return on money. This is done by requesting in the future more production in the states where the producer constraint is not binding (states with low realization of the preference shocks). In this case, return will follow non-trivial dynamics: return will increase in periods of high shocks and decrease in periods of low shocks. Even with aggregate uncertainty, however, we need certain parameter restrictions to insure that the planner gains by changing the return on money over time. Essentially, the planner wants to increase the return on money only in economies in which the discount factor is not too high. Moreover, the planner can only promise to increase returns in economies in which the discount factor is not too low.

Proposition 8 *Assume that* $y_h^* < \beta r^*$, *where* $r^* = \frac{1}{1-\alpha}[\pi_h f_h(y_h^*) + \pi_l f_l(y_l^*)]$. *Then, the optimum is the first-best allocation.*

Proof. Note that $y_h^* < \beta r^*$ iff $y_h^* < \beta \frac{1}{1-\beta(1-\frac{1}{N})}[\pi_h f_h(y_h^*) + \pi_l f_l(y_l^*)]$ (using $\alpha = \beta(1 - \frac{1}{N})$). The last inequality is equivalent to $\beta > \beta^* = \frac{y_h^*}{\pi_h f_h(y_h^*)+\pi_l f_l(y_l^*)+(1-\frac{1}{N})y_h^*}$, where the denominator of the last term can be written as $\frac{1}{N}\{\pi_h[(1-m)u_h(y_h^*) + my_h^*] + \pi_l[(1-m)u_l(y_l^*) + my_l^*]\} + (1 - \frac{1}{N})y_h^*$. A sufficient condition for $\beta^* < 1$ is thus $(1-m)u_l(y_l^*) + my_l^* > y_h^*$, which holds if shocks are sufficiently small.

The proof follows by showing that when $1 > \beta > \beta^*$, the first best is attainable. Let $r_s^* = f_s(y_s^*) + \alpha r^*$ for $s_t \in \{l, h\}$. Consider the sequence

$$\left(y(s^t), r_h(s^{t+1}), r_l(s^{t+1})\right) = \begin{cases} (y_l^*, r_l^*, r_h^*) \text{ if } s_t = l \\ (y_h^*, r_l^*, r_h^*) \text{ if } s_t = h \end{cases} \text{ for all } s^t \in S.$$

The definition of r_s^* ensures that the sequence satisfies the return constraint for all histories. Since $y_h^* < \beta r^*$ when $\beta > \beta^*$, the producer's constraint is satisfied whenever the current realization of the shock is high. Since $y_l^* < y_h^*$, the producer constraint is satisfied whenever the current realization of the shock is low. It follows that the return constraint is satisfied for all possible histories. We thus conclude that the first best solution is attainable. ■

Next, we consider local dynamics for the economy in the previous proposition. The interesting (non-trivial) dynamics occur when $r_s > r_s^*$ (otherwise, the economy converges immediately to the first best).

Proposition 9 *Assume* $y_h^* < \beta r^*$. *Let* $(r_l'(s,r), r_h'(s,r))$ *and* $y(s,r)$ *denote the optimal policy functions for return and output. Consider an initial value of return* r_s^0 *and an initial realization of* *the shock s. Then, for $r_s^0 > r_s^*$, but in a neighborhood of r_s^*, the stochastic sequences defined by the recursion $(r_l^{n+1}, r_h^{n+1}) = (r_l'(s, r_l^n), r_h'(s, r_h^n))$ and $y_n = y(s, r_s^n)$ converge to (r_l^*, r_h^*) and $y_s^*$with probability* 1.

Proof. Consider r^* and r_s^* as defined in the statement and proof of the previous proposition. When parameters are such that $y_l^* < y_h^* < \beta r^*$, the previous proposition implies that (r_l^*, r_h^*) and y_s^* is an optimal policy. Under this policy, moreover, the producer constraint has some slack, and the return constraint is satisfied with equality. As a result, if r_s^0 is sufficiently close to r^* we know that under the optimal policy the producer constraint will be satisfied with slack, and the return constraint will be binding. We can then write $\mu(s, r_s^0) = 0$ and $\lambda(s, r_s^0) > 0$. From FOC we obtain that the optimal values of return for the following period satisfy

$$\lambda(l, r_l^1) = \lambda(h, r_h^1) < \lambda(s, r_s^0), \tag{15}$$

where the first equality reveals that tomorrow's return is selected to smooth the marginal burden of returns across states of nature, and the inequality uses the fact that $\mu(s, r_s^0) = 0$. Thus, the marginal burden of return decreases from today to tomorrow, no matter the realization of tomorrow's state. Since $\lambda(s, r)$ gives the derivative of $w(s, r)$, concavity of the value function implies that $r_s^1 < r_s^0$. If we denote by $\bar{s}$ the state that is not s, we know from the continuity of the policy function that $r_{\bar{s}}^1$ is close to $r_{\bar{s}}^*$. As a result, whether the next period's state is s or $\bar{s}$, we know that the producer constraint will not bind so that $\mu(s, r_s^1) = \mu(\bar{s}, r_{\bar{s}}^1) = 0$. So, if the realization of the shock is s_1 in period 1, FOC imply that the optimal choice of return satisfies

$$\lambda(l, r_l^2) = \lambda(h, r_h^2) < \lambda(s_1, r_{s_1}^1). \tag{16}$$

Combining (15) and (16) we obtain

$$\lambda(l, r_l^2) = \lambda(h, r_h^2) < \lambda(l, r_l^1) = \lambda(h, r_h^1)$$

so that the concavity of w implies $r_l^2 < r_l^1$ and $r_h^2 < r_h^1$. Note that this result does not depend on the realization of s_1. By induction we can then show that the sequence $\{r_l'(s, r_s^n), r_h'(s, r_s^n)\}$ converges to (r_l^*, r_h^*) and $\{y(s, r_s^n)\}$ converges to y_s^*. ■

In the next proposition, r^* and $\bar{r}$ are defined as above.

Proposition 10 *There exists β such that the following holds: The values of r^* and $\bar{r}$ satisfy $y_l^* < \beta r^* < y_h^* < \beta \bar{r}$. Moreover, the optimum is history dependent.*

Proof. We first note that for sufficiently low values of return, the return constraint does not bind. As a result, there exists $(\widehat{r}_l, \widehat{r}_h)$ such that w_s is flat for all $r \leq \widehat{r}_s$, and w_s is strictly decreasing for all $r > \widehat{r}_s$.

Step 1. We show that when $s = l$ and $r \leq \widehat{r}_l$, the optimal choice of output is y_l^*.

We start considering economies that differ in β. In particular, we note that r^* is an increasing function of β and that there exists a threshold value β^* for which $\beta^* r^*(\beta^*) \geq y_h^*$ for all $\beta \geq \beta^*$ (and with equality when $\beta = \beta^*$). We consider economies in which β is in a left neighborhood of β^*. When $\beta = \beta^*$ output is at its first-best level and w_s is flat for all $r \leq r_s^*$. In this economy, moreover, the producer constraint does not bind in the low state since $y_l^* < \beta^* r^*$. By continuity, when β is in a left neighborhood of β^*, we have that $\widehat{r} = \pi_l \widehat{r}_l + \pi_h \widehat{r}_h$ is close to $r^*(\beta^*)$ so that $y_l^* < \beta \widehat{r}$, which establishes that the producer constraint does not bind in the low state. Then, the first-best level of output y_l^* is a feasible choice when $s = l$ and $r_l \leq \widehat{r}_l$. We thus conclude that $y(l, r) = y_l^*$ for all $r_l \leq \widehat{r}_l$.

Step 2. We show that when $s = l$ and $r > \widehat{r}_l$, output and new return are increasing functions of r_l and that (r_l^n, r_h^n) converges to $(\widehat{r}_l, \widehat{r}_h)$ when r_0 is in a right neighborhood of $\widehat{r}_l$

Since w_l is strictly decreasing for $r > \widehat{r}_l$, we know that the return constraint binds for all $r > \widehat{r}_l$. Fix r_l^0 close to, but in the right of, $\widehat{r}_l$. Since the producer constraint does not bind on the right of $\widehat{r}_l$, we can set $\mu(l, r_l^0) = 0$. Then, FOC promptly imply that output and new return are increasing functions of r. Moreover, optimality conditions imply that $\lambda(l, r_l^1) = \lambda(h, r_h^1) = -w_l'(r_l^1) = -w_h'(r_h^1) < \lambda(l, r_l^0) = -w_l'(r_l^0)$. Concavity of w_l then implies $r_l^1 < r_l^0$ so that the producer constraint does not bind when $s = l$ and $r = r_l^1$. We can then write $\mu_l(l, r_l^1) = 0$ so that FOC imply $\lambda(l, r_l^2) = \lambda(h, r_h^2) = -w_l'(r_l^2) = -w_h'(r_h^2) < \lambda(l, r_l^1) = -w_l'(r_l^1) = -w_h'(r_h^1)$. Concavity of w_s implies that $r_l^2 < r_l^1$ and $r_h^2 < r_h^1$. By induction we can then show that the sequence $(r_l^{n+1}, r_l^{n+1}) = (r_l'(l, r_l^n), r_h'(l, r_l^n)$ converges monotonically to $(\widehat{r}_l, \widehat{r}_h)$.

Step 3. We show that $\lambda(h, \widehat{r}_h) = 0$ and $\mu(h, \widehat{r}_h) > 0$.

That $\lambda(h, \widehat{r}_h) = 0$ follows from the fact that the value function is flat on $[0, \widehat{r}_h]$. As a way of finding a contradiction, assume that $\mu(h, \widehat{r}_h) = 0$. Then, FOC w.r.t. y and r' imply that $y(h, \widehat{r}_h) = y_h^*, r'(h, \widehat{r}_h) = \widehat{r}_h$ and $r'(l, \widehat{r}_h) = \widehat{r}_l$. This characterization of decisions when $s = h$ together with $y(l, \widehat{r}_l) = y_l^*, r'(l, \widehat{r}_l) = \widehat{r}_h$ and $r'(l, \widehat{r}_l) = \widehat{r}_l$ implies that the first-best allocation is attained. But, this result contradicts the assumption that $\beta r^* < y_h^*$. We thus conclude that $\mu(h, \widehat{r}_h) > 0$.

Step 4. We show that $(r_l'(h, \widehat{r}_h), r_h'(h, \widehat{r}_h)) > (\widehat{r}_l, \widehat{r}_h)$.

Using $\mu(h, \widehat{r}_h) > 0$ and $\lambda(h, \widehat{r}_h) = 0$ and the FOC we obtain

$$\lambda(l, r_l') = \lambda(h, r_h') > \lambda(h, \widehat{r}_h) = 0,$$

which implies $\lambda(l, r_l') = \lambda(h, r_h') = -w_l'(r_l') = -w_h'(r_h') < \lambda(h, \widehat{r}_h) = -w_h'(\widehat{r}_h) = -w_l'(\widehat{r}_l) = 0$. Concavity of w_s' implies that $r_s' > \widehat{r}_s$. ■

Proposition 11 *There exists β_0 so that, when $\beta \leq \beta_0$, the optimum is state dependent, with constant output. Moreover, output equals y_l^* only if $\beta = \beta_0$.*

Proof. Let $g(r; \beta) = \frac{1}{1-\beta(1-\frac{1}{N})}[\pi_h f_h(\beta r) + \pi_l f_l(\beta r)]$. Notice that g is a strictly increasing and concave function of r and has a positive fixed point $\bar{r} = g(\bar{r}; \beta) > 0$. It can be shown that the fixed point $\bar{r}$ gives the maximum sustainable expected return for an economy characterized by β ($\bar{r} = \pi_l \bar{r}_l + \pi_h \bar{r}_h$, where $\bar{r}_s = f_s(y) + \alpha \bar{r}$). Since $g(r, \beta)$ is an increasing function of β, it follows that $\bar{r}$ increases with β.It is thus possible to find β_0 such that $y_l^* = \beta_0 \bar{r}(\beta_0)$ and $y_l^* > \beta \bar{r}(\beta)$ for $\beta < \beta_0$. It follows that any feasible allocation should satisfy $y(s^t) < y_{s_t}^*$ when $\beta < \beta_0$, which implies that $z'_{s_t}(y(s^t)) > 1$ for all s^t. As a result, in any feasible allocation the planner would be better off if output could be increased in any node of the history tree. The optimal allocation is thus the one that gives maximum (feasible) production in all nodes of the history tree. Since production cannot be bigger than $\beta \bar{r}$ (otherwise the producers constraint is violated), the optimal allocation is given by $\left(y(s^t), r_h(s^{t+1}), r_l(s^{t+1})\right) = (\beta \bar{r}, \bar{r}_h, \bar{r}_l)$,with $y(s^t) = y_l^*$ only if $\beta = \beta_0$. ■

5 Comments on monitoring and indivisibilities

As we have shown, our monetary model with low or high discount factors is consistent with a constant sequence of promises. That feature no longer holds in an intermediate range of preferences because of an envelope-theorem argument. While it is feasible to hold output at the first-best level in one state, regardless of the history of past realizations, it is not wise to do so because small increases in output have zero welfare costs on the margin, in that state, but a positive one in the constrained state. Moreover, the impact on the return on money that such a perturbation produces has a decaying effect over time due to discounting at rate $1 - \frac{1}{N}$. In the next subsection we introduce monitoring into the previous model and ask whether history-dependence is a natural implication of the theory of the second best. We find that the answer to this question is "no".

5.1 Introducing imperfect monitoring

We now study an economic environment in which the envelope-theorem argument does not imply history-dependence. In this environment, the lack of (ex-post) heterogeneity between agents implies that the maximization of welfare and the provision of incentives are aligned. As a result, the planner cannot alleviate participation constraints in binding states by increasing output above the first-best level in states with non-binding constraints. We conclude that heterogeneity is needed for history-dependence to improve welfare. Interestingly, we note that heterogeneity naturally arises in environments in which money is essential.

We consider a gift-giving version of the model in [[3]]. In our example, the planner has an imperfect monitoring technology in that he only knows whether an individual has cooperated in the past or not. Individuals who defect from the recommended allocation in the present are effectively removed from the economy (punished with autarky). Since their identity is known, others can be instructed never to produce for them. Never producing for a defector is obviously incentive-feasible.

Because we consider a version of [[3]] in which all individuals are monitored, there is no role for money. We introduce *iid*, aggregate preference shocks as in the monetary model that we discussed above. Participation constraints are still important in this economy because individuals give gifts in exchange for promises of receiving gifts in the future.

Due to the lack of individual states, the pair (s, v_s), describing the shock realization, s, as well as the contingent utility promised to a typical individual in that state, v_s, can be used as a state variable for the planner problem. Then w_s, the planner's value function, must solve

$$\begin{aligned} w_s(v) &= \max z_s(y) + \beta[\pi_l w_l(v_l') + \pi_h w_h(v_h')] \\ &\quad \text{s.t.} \\ y &\leq \beta(\pi_l v_l' + \pi_h v_h'), \\ v &\leq z_s(y) + \beta(\pi_l v_l' + \pi_h v_h'), \\ (v_l', v_h') &\in R_l \times R_h \end{aligned} \tag{17}$$

where the logic of the Bellman equation (10) has been followed. The set of sustainable promises R_s is an interval whose derivation we leave for the proposition below. The inequality on the return constraint has been used without loss of generality. As before, the first constraint is the producer constraint, while the consumer constraint has been ignored without loss of generality. It should be noticed that the return variable is now v, which is not a difference in values as before, due to the new form of the producer constraint. For the producer, the reward from cooperation $-y+\beta(\pi_l v_l'+\pi_h v_h')$, while the reward from defection is zero, due to autarky. Since v is just a scalar, the dimensionality of the problem is essentially the same as that of the monetary case. Moreover, for v in the ergodic set defined by the optimal policy function, the return constraint holds with equality, so that the identity $w_s(v) = v$ holds in this set.

Note that, as our next Proposition shows, there is an important difference between this credit economy and the monetary economy analyzed before. In this credit economy the provision of incentives and welfare are aligned. This follows from the fact that an increase in output increases utility $z_s(y)$ and return repayment v by the same amount (see the Bellman equation (17)). Thus, $y_s = y_s^*$ should be used whenever feasible. An increase in output above y_s^* reduces welfare and the punishment to individuals that deviate. On the contrary, in the monetary economy, output affects utility according to $z_s(y)$ and return repayment according to $f_s(y)$ (see Bellman equation

(10)). This misalignment of welfare and producers' incentives is driven by the heterogeneity among agents in the monetary economy. An increase in the promised value of money affects differently consumers and producers because they hold different amounts of money after trading. We show now that, for some values of β, the optimum in the credit economy is not history dependent, even when the economy satisfies the envelope condition.

Proposition 12 *The optimum of the credit economy with imperfect monitoring is state dependent. There is, moreover, a range of β values such that the optimum has output y_l^* at $s = l$, and output lower than y_h^* at $s = h$.*

Proof. The proof demonstrates that the optimum can be described according to cut-off values β^* and β^{**} as follows: (i) If $\beta > \beta^*$ then output at state s is y_s^*; (ii) If $\beta^* > \beta > \beta^{**}$ then output at $s = l$ is y_l^* while output at $s = h$ is a constant lower than y_h^*; (iii) If $\beta < \beta^{**}$ then output is constant.

Part (i). We first solve the Bellman equation ignoring the return constraint and the participation constraint. In this case, the solution requires maximizing output period by period, which gives $y_s = y_s^*$. The solution to the Bellman equation is then represented by two scalars, according to $w_s^* = z_s(y_s^*) + \frac{1}{1-\beta}[\pi_l z_l(y_l^*) + \pi_h z_h(y_h^*)]$. We now need to find conditions for which the participation and return constraints are satisfied. To this end, note that $y_l^* < y_h^* \leq \beta \left\{ z_s(y_s^*) + \frac{1}{1-\beta}[\pi_l z_l(y_l^*) + \pi_h z_h(y_h^*)] \right\}$ implies that the producer constraint is satisfied for β sufficiently close to 1, a condition defining β^*. The return constrain is satisfied as long as $v_s \leq w_s^*$. Since higher levels of utility cannot be achieved, the problem is only well defined for $v_s \leq w_s^*$.

Parts (ii) and (iii). We now solve the Bellman equation ignoring the return constraint and the participation constraint of the low state. To this end, we guess that the solution has the form $y(h, v) = \hat{y} < y_h^*$, $y(l, v) = y_l^*$ and $w_s^*(v) = z_s(y_s) + \frac{1}{1-\beta}[\pi_l z_l(y_l^*) + (\pi_h) z_h(\hat{y})]$ and solve for $\hat{y}$

satisfying

$$\hat{y} = \beta\pi_l[z_l(y_l^*) + \frac{1}{1-\beta}(\pi_l z_l(y_l^*) + \pi_h z_h(\hat{y}))] + \beta\pi_h[z_h(\hat{y}) + \frac{1}{1-\beta}(\pi_l z_l(y_l^*) + \pi_h z_h(\hat{y}))]$$

Note that if $\hat{y} > y_l^*$, then the producer constraint does not bind in the low state. Then, we can compute w_s^* and obtain the values of v for which the Bellman equation is well defined. If $\hat{y} < y_l^*$, the producer constraint is not satisfied in the low state and we have a type-*iii* solution. ■

5.2 The role of indivisibilities

In our monetary economy, differently from the credit economy described in the previous subsection, there is a role for deviating output away from the first-best level in order to relax incentive constraints. That deviation, moreover, should be updated along repetitions of the unconstrained state in a history-dependent manner. We have applied the term *envelope condition* to describe a property of state-dependent allocations featuring one state with binding constraints, and another with slacking constraints. But, it is not difficult to imagine a weaker form of the envelope condition in which all states have constraints binding with equality, although with different values for their multipliers. That is, it is not difficult to imagine environments with state-dependent allocations with states whose constraints bind more strongly than in others. In a sequel to this research we are able to show that such a weaker condition appears in cash-in-advance models where the binding constraint is precisely the cash-in-advance constraint. Based on that research we can assert that indivisibilities are not necessary to produce history-dependence. There are, however, comments that should be made about the role of over-production in the random-matching model with 0-1 holdings since elevating output above the first-best level in some state is not feasible in many models of divisible money, including the cash-in-advance model.

History-dependence in cash-in-advance economies must be supported by reductions in output that have the property of increasing the return on money. In our findings, these effects require a sufficient level of curvature in the utility functions. In the 0-1 random-matching model, by contrast, increases in the return on money can only be supported by increases in output. Indivisibilities are important to the extent that they allow the planner to implement allocations with over production. Since over production reduces the surplus of the pair in a bilateral meeting, there is a straightforward way to remove this option from the planner. If instead of adopting our criteria of implementability (one that only considers off-equlibrium paths associated with individual defections) we allow individuals in a bilateral match to jointly deviate, then over-production is not possible. One way that this joint deviation can take place is by individuals committing to trade by means of lotteries, where randomization is taken over the probability that money is (or is not) transferred to producers. In principle, this lottery allows the return on money to increase without the need for over production. We think that the argument for the dominance of one equilibrium concept over the other is debatable, and in the case of the use of lotteries, the assumption of commitment to the randomization device is particularly strong. However, the discussion provided by such an alternative does serve the purpose to illustrate that indivisibilities are indeed being used by the planner to support over production..

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