Timing Games via Nash Equilibrium

Caller Number Five: Timing Games that Morph from One Form to Another
(Andreas Park and Lones Smith)

- Three or more players can stop at any point in time.
- Their action choice is the irrevocable stopping time \( t \geq 0 \). This is a nonnegative real number.
- Others’ actions are unobservable, and thus there is only one information set.
- Mixed strategy: measures the probability \( G(t) \) that a player has stopped by time \( t \).
- Strategies depend simply on calendar time.
We look for cdf’s $G$ over time $[0, \infty)$ such that all players are indifferent between entry at all moments in time.

This has two implications:

1. **Atomic entry** (i.e. when the chance that any two individuals enter at a moment is positive, since the cdf $G$ jumps): Entry at the moment before or after atom during slow play yields the same payoff.

2. **Slow play** (i.e. when the chance that any two individuals enter at a moment is zero, because $G$ has a density $g(t) = G'(t)$): Entry at consecutive moments in time has the same payoff.

We now apply the solution methodology in an example.
1. How Big is the Atom?

We guess there is smooth entry until an “atomic” entry.

With a common entry chance $G$, the chances that 0 or 1 or 2 others have entered is $(1 - G)^2$, $2G(1 - G)$, and $G^2$

⇒ My expected payoff before an atom is $\phi(G) = 2G(1 - G)$

⇒ My expected payoff in an atom is $(1 - G)^2 \cdot \frac{1}{3} + 2G(1 - G) \cdot \frac{1}{2}$

Equating $2G(1 - G) = (1 - G)2/3 + G(1 - G) \Rightarrow G = 1/4$
2. Solving for the War of Attrition Phase

- Assume a delay cost $c(t) = t$ if one waits till time $t$.
- Equate marginal benefits and costs of waiting $dt$:

$$dt = MC(\text{wait}) = MB(\text{wait}) = \phi'(G(t))dG$$

$$\Rightarrow 1 = \phi'(G(t))\dot{G} = (2 - 4G(t))\dot{G}(t)$$

- If we do not start with an atom, then $G(0) = 0$.

$$\Rightarrow G(t) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 2t}.$$ 

- This is well-defined until $t = 1/2$, when $G = 1/2$.
- Know atom occurs when $G = 1/4$

$$\Rightarrow \text{smooth play until } G(t) = 1/4 \text{ at time } t = 3/8.$$ 

- the game ends with a complete atom.
Players want to be first or last.

Urge to go for being first $\Rightarrow$ expect time-0 atom.

Atom size $G \Rightarrow$ get $1, 1/2, 2/3$ with chance $(1 - G)^2, 2G(1 - G)$ and $G^2$ respectively.
Example 2: How big is the atom?

⇒ expected payoff *in* an atom is
\[(1 - G)^2 + G(1 - G) + 2G^2/3.\]

Expected payoff *after* atom is \(\phi(g) = (1 - G)^2 \cdot 1 + G^2 \cdot 1.\)

Equating \((1 - G)^2 + G(1 - G) + 2G^2/3 = (1 - G)^2 + G^2\)
⇒ \(G = 3/4.\)
2. Solving for the War of Attrition Phase

- As above assume a delay cost $c(t) = t$ if one waits till time $t$.
- Equate marginal benefits and costs of waiting $dt$:

$$0 = -1 - \dot{G}(t)(2 - 4G(t))$$

- Since we start If we do not start with an atom, then $G(0) = 3/4$.

$$\Rightarrow G(t) = \frac{1}{2} + \frac{1}{2}\sqrt{1/4 + 2t}.$$  

- This is well-defined for all $G > 1/2$.
- Know atom $G = 3/4$ occurs at $t = 0 \Rightarrow$ smooth play from $t = 0$ until time $t = 3/8$ when $G(3/8) = 1$. 
Equilibrium Play with U-Shape

Expected payoffs $\phi$ and payoff from atom

Equilibrium strategy (blue) and possible range for $G$