

# D'Alembert's Principle: The Original Formulation and Application in Jean d'Alembert's *Traité de Dynamique* (1743)

by

CRAIG FRASER\*

## *Part Two*

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### *Introduction*

In 1743 Jean d'Alembert published his work "Treatise on Dynamics, in which the Laws of Equilibrium and Motion of Bodies are reduced to the smallest possible number and demonstrated in a new manner, and where a general Principle is given for finding the Motion of several Bodies which act on one another in any way". In Part One of this study I showed how d'Alembert formulates and applies his "general Principle" in the *Traité de Dynamique* (*Centaurus* 28 (1985) pp. 31–61). In the present article I complete the study by examining several special uses d'Alembert makes of his principle in Chapters Three and Four of Part Two of the *Traité*.

\* Institute for the History and Philosophy of Science and Technology, University of Toronto, Room 36, Victoria College, Toronto, Canada M5S 1K7.

*a. Small Vibrations: Problem V*

In Problems I, V and VI of Chapter Three d'Alembert analyzes the small vibrations of special systems. Vibration theory is a part of 18th century mechanics that has received considerable historical attention (see Cannon and Dostrovsky [1981] and Truesdell [1960]). I shall show how d'Alembert applies his principle to vibration phenomena by describing his solution to Problem V.

In Problem V d'Alembert considers two masses  $m$  and  $M$  attached to a massless string that hangs from a point  $C$  (Figure 1). The string is displaced "infinitely little" from the vertical. The problem is to derive

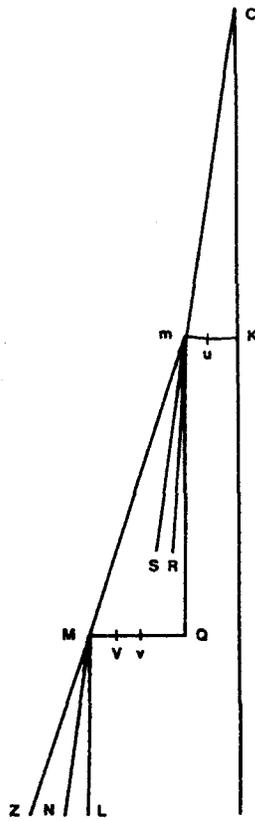


Figure 1. (D'Alembert's Figure 31 (*Traité* (1743))).

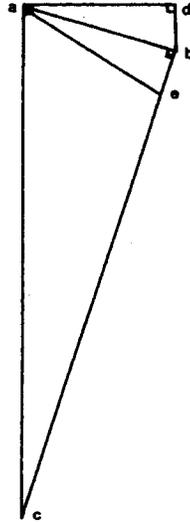


Figure 2.

differential equations which describe the small oscillations of  $m$  and  $M$  in terms of their displacements from the vertical.

In his solution d'Alembert neglects small quantities of the third order and higher. He makes use of the following result. Consider the right triangle  $abc$  in which angle  $acb$  is small (Figure 2). Project  $ab$  on the perpendicular to  $ac$  to form the right triangle  $adb$ . Construct the triangle  $abe$ , where  $e$  lies on the line  $bc$  and the angle  $bae$  is small. Then the lines  $ab$ ,  $ad$  and  $ae$  are equal up to second order. (If two quantities differ by a small quantity of the third order or higher I shall say they are equal, "up to second order".)

D'Alembert uses the symbols  $p$  and  $P$  to denote the weight ("pesanteur") of  $m$  and  $M$ . The gravitational forces acting on these masses equal  $mp$  and  $MP$ . D'Alembert refers to these forces as the "absolute efforts" of the weights  $p$  and  $P$ . He decomposes the force  $mp$  acting on  $m$  along  $mQ$  into two forces: a force which acts along the line perpendicular to  $Cm$  to produce the acceleration  $\varphi$ , indicated by the line  $mu$ ; a force which acts along the line (to be determined)  $mR$  which is "destroyed". The gravitational force acting on  $M$  is decomposed into two forces: a force which acts in a direction perpendicular to  $Cm$  to pro-

duce the same acceleration  $\varphi$  experienced by  $m$ , indicated for  $M$  by the horizontal line  $Mv$ ; a force directed along the line  $MN$ . The force along  $MN$  is in turn decomposed into two forces: a force which acts along the perpendicular to  $mM$  to produce the additional horizontal acceleration  $\Phi$ , indicated by the line  $Vv$ ; a force directed along  $MZ$  ( $mM$  extended) which is destroyed. The direction of the line  $mR$  must be such that the sum of the "lost" forces along  $MZ$  and  $mR$  converges on line  $mS$  ( $Cm$  extended); that is, the lost forces, viewed as impressed forces applied to the system, must give rise to tensions in the string which produce equilibrium. This fact provides the following condition:

$$(\text{Lost force along } MZ) : (\text{Lost force along } mR) = \sphericalangle SmR : \sphericalangle MmS. \quad (1)$$

Because the angles  $RmQ$  and  $ZML$  are small the lost forces along  $mR$  and  $MZ$  differ from  $mp$  and  $MP$  by small quantities of the second order. Since angles  $MmS$  and  $SmR$  are small, equation (1) becomes, up to second order:

$$(MP)(\sphericalangle MmS) = (mp)(\sphericalangle SmR). \quad (2)$$

D'Alembert sets  $Cm=l$ ,  $Mm=L$ ,  $mK=x$  and  $MQ=y$ . Hence  $x$  and  $x+y$  are the horizontal displacements of  $m$  and  $M$  from the vertical. From the way in which angles  $RmQ$  and  $NML$  were constructed it is clear that  $\sphericalangle RmQ = \varphi/p$  and  $\sphericalangle NML = \varphi/P$ . Thus we obtain the relations

$$\sphericalangle SmR = \frac{x}{l} - \frac{\varphi}{P}, \quad (3)$$

$$\sphericalangle ZMN = \frac{y}{L} - \frac{\varphi}{P}. \quad (4)$$

In addition, we have the trigonometric relation:

$$\sphericalangle MmS = \frac{y}{L} - \frac{x}{l}. \quad (5)$$

Combining (2), (3) and (5) we now solve for  $\varphi$ :

$$\varphi = \frac{px}{l} - \frac{MP}{m} \left( \frac{y}{L} - \frac{x}{l} \right). \quad (6)$$

The acceleration  $\Phi$  equals  $P(\angle ZMN) = P(y/L - \varphi/P)$ . Hence

$$\Phi = \frac{Py}{L} - \frac{px}{l} + \frac{MP}{m} \left( \frac{y}{L} - \frac{x}{l} \right). \quad (7)$$

D'Alembert proceeds to write down the time-differential equations obtained by equating  $-ddx$  and  $-ddy$  to  $\varphi dt^2$  and  $\Phi dt^2$ :

$$\begin{aligned} -ddx &= \left[ \frac{px}{l} - \frac{MP}{m} \left( \frac{y}{L} - \frac{x}{l} \right) \right] dt^2, \\ -ddy &= \left[ \frac{yP}{L} \left( \frac{M+m}{m} \right) - \frac{x}{l} \left( p + \frac{MP}{m} \right) \right] dt^2. \end{aligned} \quad (8)$$

The linear system (8) provides the desired description of the small oscillations of  $m$  and  $M$ .

D'Alembert's derivation of equations (8) constitutes an important advance over earlier analyses of the loaded hanging string (see Truesdell [1960] for details). The method he presents for integrating (8) (described in Hawkins [1975]) introduced many of the ideas (eigenvalue, symmetric matrix etc.) that were later developed and incorporated into linear algebra. In a series of corollaries to Problem V d'Alembert extends his analysis from two to arbitrarily many masses and considers the passage to the limit, in which the discretely weighted string becomes a continuous cord. In this last result he obtains one of the first partial differential equations of 18th century applied analysis.

D'Alembert's use of his principle in Problem V is illustrated by his treatment of the motion of the mass  $m$ . The decomposition given by this principle is for this mass a decomposition of forces. The external force due to gravity acting on  $m$  is decomposed into two forces: an accelerative force corresponding to the actual motion and a force "de-

stroyed" by the constraints. A similar decomposition holds for the mass  $M$ . By d'Alembert's principle the lost forces, viewed as impressed forces applied to the system, must produce equilibrium. This condition permits us to calculate values for the accelerative forces. By expressing these forces in terms of the second derivatives of the displacements  $x$  and  $y$  we obtain the final equations for the problem.

The interpretation of d'Alembert's principle outlined in the preceding paragraph is the one accepted by such historians as Mach [1883], Truesdell [1960] and Szabo [1979]. Although the interpretation describes d'Alembert's procedure in problems involving small vibrations, the latter must be regarded as a special case, not truly representative of how he conceived and applies his principle throughout the *Traité*. (The widespread acceptance of this interpretation probably stems from its appearance in the second edition of Lagrange's *Mécanique Analytique* [1811, 256, 264–267]. See also note 3 of Part One of this study.)

The special character of d'Alembert's principle in Problem V is illustrated by comparing his analysis to his solutions of Problems II and X. In the latter he examined the system at three successive infinitesimally close instants in order to arrive at a value for the second differential. In Problem V, by contrast, he simply invokes the definition of accelerative force to obtain equations (8). In particular, no decomposition of line segments is required; the segments  $mu$ ,  $MV$  and  $Vv$  play a heuristic rather than an essential role in the solution. (This last point is connected to the order analysis d'Alembert employs to arrive at (8). It is unclear whether he is considering the horizontal projections of the force components perpendicular to  $Cm$  and  $mM$ , or whether he is equating the second differential of the arc lengths of  $m$  and  $M$  to their horizontal projections. Even in this second case it would be unnecessary to actually introduce the lines  $mu$ ,  $MV$  and  $Vv$ .)

The introduction of the lines  $mu$ ,  $MV$  and  $Vv$ , although unnecessary, helps to explain how d'Alembert arrived at the special use of his principle in Problem V. D'Alembert states that the mass  $m$  travels the arc  $mu$  "during the first instant". He is treating the system as though it were released from rest and were being analyzed in its first instant or motion. Such a viewpoint, in which the masses are for the sake of analysis assumed to possess no residual velocities, leads naturally to an approach involving force decomposition.

Note finally that d'Alembert's treatment of small quantities in Problem V differs from his earlier use of infinitesimals in Problem II. His statement that the string is displaced "infinitely little" from the vertical should not be taken literally; he means only that this displacement is small, in the sense that the cube of quantities like  $x/l$ ,  $y/L$  may be neglected. By contrast, the entities  $AB$ ,  $\alpha$ ,  $FE$  etc. which appeared in Problem II were actually infinitely small: the product of the infinitesimal  $AB$  and any finite quantity is an infinitesimal. In Problem V equations (8) provide an *approximate* description of the small oscillations of the system; in Problem II equation (24) furnished an *exact* description of the ring-rod system throughout its motion.

#### *b. Some Results on Collision*

In Problems XI to XIV of Chapter Three of Part Two of the *Traité* d'Alembert applies his principle to several examples of hard or perfectly inelastic collision to obtain the post-impact velocities. (We saw how he proceeds here in our discussion of Problem IX in part One section a.) In addition, he advances two models to analyze the collision of (perfectly) elastic bodies: the first deduces their properties from those of a corresponding system of hard bodies; the second reduces them to a system of mass points separated by springs, springs whose tensions are governed by specifiable force laws. He closes his study of collision with an example in which he introduces a third type of body, a "soft" body, and briefly compares its collision properties to those of hard bodies.

D'Alembert's first model for elastic collision assumes that the compression and restitution experienced by the bodies occurs in an instant. Suppose the perfectly elastic body  $m$  obliquely strikes a fixed impenetrable wall with velocity  $u$  (Figure 3). Suppose now for the sake of analysis that the collision is one in which  $m$  is hard. Decompose the velocity  $u$  into the components  $v$  and  $w$  parallel and perpendicular to the wall. By d'Alembert's principle the component  $w$  is lost in the hard collision and  $m$  moves with the post-impact velocity  $v$ . In the elastic collision, by contrast, "the effect of the spring is to restore to each body in the opposite direction the motion it has lost by the action of the others". Hence in the elastic collision  $m$  moves with a

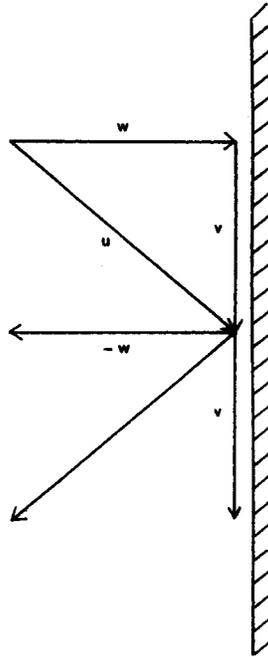


Figure 3.

post-impact velocity composed of  $v$  and  $-w$ . A knowledge of the post-impact and lost velocity in the hard collision therefore permits us to calculate the post-impact velocity in the elastic collision.

In Problems XI to XIV d'Alembert uses a hard body comparison system to analyze elastic impact. In a remark following his presentation of these problems he confronts a difficulty which can arise in the application of his method. In hard body impact the changes in motion occur instantaneously. In modelling elastic collision by hard bodies we are abstracting from the elapsed period of compression and restitution. In the collision of more than two bodies it may be the case that the different elastic interactions are completed in different times. Such differences can lead to results which vary quite significantly from the situation in which it is supposed all interactions are instantaneous. D'Alembert therefore proposes an alternate model for elastic collision.

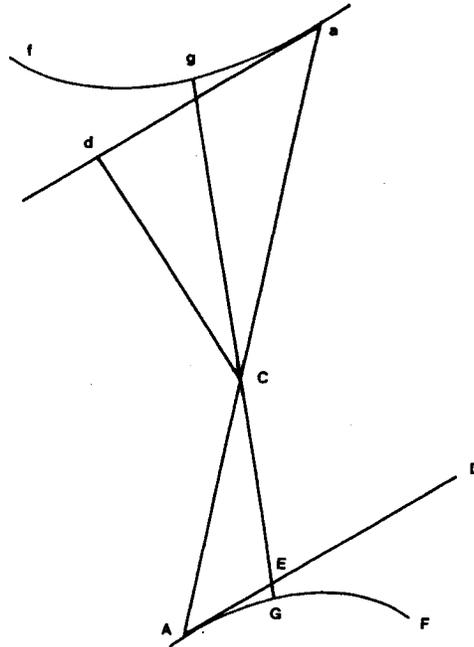


Figure 4. (D'Alembert's Figure 59 (*Traité* (1743))).

The alternate model consists in replacing the bodies by their centers of mass and assuming the changes of motion arise from springs placed between these centers; during the collision each spring contracts and dilates exerting a force  $\varphi$  on the bodies that is a function of the distance between these centers. This model, which d'Alembert attributes to "several authors", had appeared in 1724 in a treatise composed by John Bernoulli. It is evidently limited to collisions in which the force of impact at the point of contact exerts no moment about the mass centers of the bodies.

To illustrate d'Alembert's spring model I shall describe the example he presents at the close of his discussion of impact. Let us represent two bodies in collision by their mass centers  $a$  and  $A$  (see figure 4). Assume that their collective center of mass  $C$  remains at rest during the collision. The directions of the initial velocities of the bodies are represented by the lines  $ad$  and  $AD$ . The force of collision arises from

a spring (“ressort”) which passes through  $C$  and remains constantly positioned between  $a$  and  $A$ . As the collision proceeds these bodies describe similar paths  $agf$  and  $AGF$  (the paths are similar because the center of mass  $C$  remains at rest). During the first half of the collision the spring contracts until it has reached its minimum length; its position at this instant is represented by the line  $gG$ , which is the unique line passing through  $C$  that is perpendicular to the two curves. The spring then begins to expand and the bodies continue to describe the curves  $gf$  and  $GF$ , similar to  $ag$  and  $AG$ , until they reach the points  $f$  and  $F$ , at which instant the spring will have returned to its original length and the collision will be completed.

D’Alembert concludes his presentation of the example with a result concerning the impact of soft bodies. The latter constitute a third type of body in addition to those hard and those elastic: soft bodies are deformable (like elastic bodies) but perfectly inelastic (like hard bodies). In the soft collision between  $a$  and  $A$  the interaction is complete when the spring has contracted to its shortest length  $Gg$ . Thus in the case of soft impact, as in hard impact the bodies have no relative velocity along their line of interaction at the completion of the collision. D’Alembert shows, however, that the post-impact speeds in the two cases differ. Consider the following designations:

$$\begin{aligned}(v)^i &= \text{initial speed of } a \text{ (same for both cases)} \\ (v_s)^f &= \text{final speed of } a \text{ for the soft collision} \\ (v_h)^f &= \text{final speed of } a \text{ for the hard collision.}\end{aligned}$$

i. “Soft” Case. In figure 4 the lines  $Cd$  and  $Cg$  are the perpendiculars from the point  $C$  to the tangents to the curve  $agf$  at the points  $a$  and  $g$ . D’Alembert presents without explanation the following relation

$$\frac{(v_s)^f}{(v)^i} = \frac{Cd}{Cg}.$$

This relation expresses the fact that the angular momentum of  $a$  about  $C$  is constant.

ii. “Hard” Case. The magnitude of the post-impact velocity is simply the component of the initial velocity perpendicular to the line  $Ca$ . Be-

cause the initial velocity is directed along the line  $ad$  we have by similar triangles

$$\frac{(v_h)^f}{(v)^i} = \frac{Cd}{Ca}$$

Comparing now the soft and hard case we see that  $(v_h)^f > (v_h)^i$ , which establishes the result.

### c. The Chapter on Live Force

In the last chapter of the *Traité* d'Alembert takes up the study of the principle of the conservation of live forces. (For a discussion of the famous controversy over this principle see Hankins [1970, 204–213].) The live force of a system is the sum over all the masses of the product of each mass and the square of its speed; thus the live force equals twice the kinetic energy. The principle was reformulated in the 19th century to become the law of conservation of mechanical energy. D'Alembert's purpose is to give for it "if not a general demonstration for all cases, at least the Principles sufficient to find the demonstration in each particular case".

D'Alembert opens the chapter by considering two mass points  $A$  and  $B$  joined by a massless rigid rod (Figure 5). At a given instant velocities  $\vec{u}_A$  and  $\vec{u}_B$  are "impressed" on  $A$  and  $B$ . (I shall in this section use modern vector notation to facilitate the description of d'Alembert's procedure.) Because of the rigidity of the rod the masses actu-

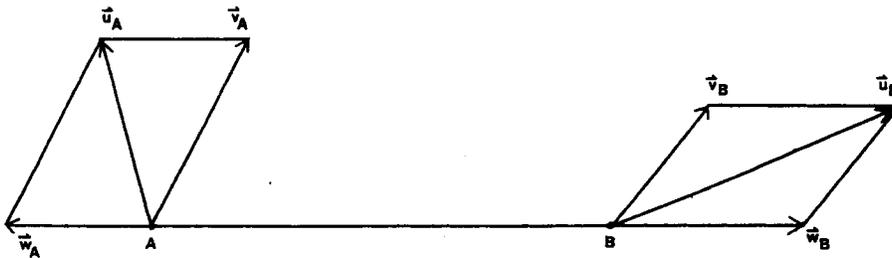


Figure 5. (Based on d'Alembert's Figure 60 (*Traité* (1743))).

ally follow  $\vec{v}_A$  and  $\vec{v}_B$ . By d'Alembert's principle we have the decompositions

$$\begin{aligned}\vec{u}_A &= \vec{v}_A + \vec{w}_A \\ \vec{u}_B &= \vec{v}_B + \vec{w}_B,\end{aligned}$$

in which  $\vec{w}_A$ ,  $\vec{w}_B$  designate the lost motions. Squaring these equations and multiplying by the masses we obtain the relations

$$\begin{aligned}Au_A^2 &= Av_A^2 + Aw_A^2 + 2A\vec{v}_A \cdot \vec{w}_A \\ Bu_B^2 &= Bv_B^2 + Bw_B^2 + 2B\vec{v}_B \cdot \vec{w}_B.\end{aligned}\quad (1)$$

According to his principle equilibrium would subsist if the masses possessed the lost velocities alone. Hence  $A\vec{w}_A = -B\vec{w}_B$ . Also, because the rod is rigid, the projections of  $\vec{v}_A$  and  $\vec{v}_B$  along the rod are equal. A combination of these two facts furnishes the relation

$$A\vec{v}_A \cdot \vec{w}_A + B\vec{v}_B \cdot \vec{w}_B = 0. \quad (2)$$

By adding the equations in (1) we obtain

$$Au_A^2 + Bu_B^2 = Av_A^2 + Bv_B^2 + Aw_A^2 + Bw_B^2. \quad (3)$$

Suppose now  $\vec{u}_A$  and  $\vec{u}_B$  are the instantaneous velocities of  $A$  and  $B$ , the motion changes continuously and no external forces act on the system. An equation similar to (3) holds at each time  $t$ . The velocities  $\vec{v}_A$  and  $\vec{v}_B$  are the velocities of  $A$  and  $B$  in the next instant; the lost velocities  $\vec{w}_A$  and  $\vec{w}_B$  are infinitesimal. In this situation d'Alembert presents the following consequence of (3):

$$Au_A^2 + Bu_B^2 = \text{constant}. \quad (4)$$

D'Alembert is using the following fact: If the difference of the total live force at each two successive instants is a second order infinitesimal then the first derivative of this total is zero and the total itself is therefore constant. He does not explicitly introduce or justify this fact.

D'Alembert proceeds to the case in which external forces  $\vec{\Phi}_A$ ,  $\vec{\Phi}_B$  act

on  $A$  and  $B$ . Suppose at a given instant these forces impart to  $A$  and  $B$  the incremental velocities  $\vec{\Phi}_A dt$  and  $\vec{\Phi}_B dt$ . The impressed velocities of the masses are now  $\vec{u}_A + \vec{\Phi}_A dt$  and  $\vec{u}_B + \vec{\Phi}_B dt$ . These velocities differ only infinitesimally from the velocities  $\vec{v}_A$  and  $\vec{v}_B$  in the next instant. Hence

$$A[\vec{u}_A + \vec{\Phi}_A dt]^2 + B[\vec{u}_B + \vec{\Phi}_B dt]^2 = Av_A^2 + Bv_B^2. \quad (5)$$

If we expand (5), ignore second order differentials and set  $d\vec{r}_A = \vec{u}_A dt$  and  $d\vec{r}_B = \vec{u}_B dt$  we obtain

$$Av_A^2 + Bv_B^2 - Au_A^2 - Bu_B^2 = 2A\vec{\Phi}_A \cdot d\vec{r}_A + 2B\vec{\Phi}_B \cdot d\vec{r}_B. \quad (6)$$

The left side of (6) is  $d(Au_A^2 + Bu_B^2)$ , the differential of the live force. An integration of (6) therefore yields

$$Au_A^2 + Bu_B^2 = 2\int A\vec{\Phi}_A \cdot d\vec{r}_A + 2\int B\vec{\Phi}_B \cdot d\vec{r}_B. \quad (7)$$

(D'Alembert is assuming the system begins its motion from rest. If the masses possess initial velocities we must add a constant equal to the initial live force to the right side of (7).) Equation (7) is the analytical expression for the mass-rod system of the principle of conservation of live forces.

D'Alembert provides a curious interpretation for the integrals in (7). He says  $2\int A\vec{\Phi}_A \cdot d\vec{r}_A$  will be equal to the "effect" the force would have on  $A$  if  $A$  were to move freely under the action of  $\vec{\Phi}$  along the trajectory it describes in its actual motion. (By "effect" he means the total live force imparted to  $A$  by  $\vec{\Phi}$ .) Unfortunately, it will not in general be the case that  $A$ , if moving freely under  $\vec{\Phi}$ , would describe such a trajectory; that is, the postulated hypothetical motion may not be dynamically possible. (This criticism is made by Arthur Korn [1899,207] in his German translation of the *Traité*.) This defect seems to me to be more one of formulation than of substance. Thus in the few examples d'Alembert presents the integrals in (7) are treated as functions of the positions of the bodies and equation (7) is used as a condition on the motion.

The key result used by d'Alembert to obtain (4) and (7) is equation (2):

$$A\vec{v}_A \cdot \vec{w}_A + B\vec{v}_B \cdot \vec{w}_B = 0. \quad (2)$$

In each of the examples he considers in the remainder of the chapter he establishes an equation similar to (2):

$$\Sigma m\vec{v} \cdot \vec{w} = 0. \quad (8)$$

In a general scholium at the end of the chapter d'Alembert writes:

It follows from all we have said until now that in general the conservation of live forces depends on this principle, that when two powers [puissances] are in equilibrium, the velocities of the points where they are applied estimated in the directions of these powers are in inverse ratio to these same powers. This fact has long been recognized by geometers as the fundamental principle of equilibrium; but no one that I know has yet demonstrated the principle or shown that the conservation of live forces necessarily results from it.

([1743,182]), ([1758,267])

The “fundamental principle” referred to in this passage later became known as the principle of virtual work. Equation (8) is its analytical expression: the “lost” forces  $m\vec{w}$  are in equilibrium and hence do zero virtual work with respect to the virtual displacements  $\vec{v}dt$ .

D'Alembert states that his general principle for equilibrium may be demonstrated in “any case, for example, in that of Problem X, where it is easy to discover”. The principle is for this problem equivalent to the condition for equilibrium used by d'Alembert in his solution to X. The reader may as an exercise wish to verify this fact using elementary geometry.

D'Alembert's treatment of the conservation of live forces in the final chapter of the *Traité* is of historical interest for its influence on two later geometers: J. L. Lagrange and Lazare Carnot. The fundamental axiom of Lagrange's *Mécanique Analytique* (1788), the general principle of virtual work, first appeared in a memoir Lagrange composed in 1764 on the libration of the moon; in this memoir Lagrange refers specifically to d'Alembert's remarks at the end of the *Traité*. (See Fraser [1983] for details and references.) Although Lagrange derived inspiration from his older contemporary, his own formal and algebraic style stands in contrast to the latter's physical approach to dynamics. A work that bears the direct imprint, both in content and style, of d'Alembert's influence is Lazare Carnot's *Essai sur les machines en*

*général* (1783). In this treatise Carnot establishes as the basis of mechanics two fundamental equations: the first is simply equation (8), and the second is the generalization of (8) obtained when the actual velocities are replaced by arbitrary virtual (what Carnot calls “geometric”) velocities. Unlike d’Alembert, Carnot was strongly motivated by considerations of engineering practice; the *Essai* is significant for its role in facilitating the introduction of energy concepts derived from engineering into mathematical mechanics. (Further discussion of this subject is contained in Fraser [1981].)

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