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Choice under Limited Uncertainty

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Abstract

This paper considers the problem of an agent's choice under uncertainty in a new framework. The agent does not know the true probability distribution over the state space but is objectively informed that it belongs to a specified set of probabilities. Maintaining the hypothesis that this agent is a subjective expected utility maximizer, we address the question of how the objective information influences her subjective prior.

Three plausible rules are proposed. The first, named state independence, states that the subjective probability should not depend on how the uncertain states are 'labeled'. Location-consistency, the second property, assumes that 'similar' objective sets of probabilities result in 'similar' subjective priors. The third rule is an 'update-consistency' rule. Suppose the agent selects some probability p. She is then told that the likelihood assigned by p to some event A is in fact correct; then this should not cause her to revise her choice of p.

Another property, alternative to update-consistency, is also proposed. When an agent forms her subjective prior assigning subjective probabilities to events in some ordered sequence, this property requires that the resulting prior be independent of that order. This last property, named order independence, is shown to be equivalent to update-consistency.

A class of sets of probabilities is found on which state independence, location-consistency and update consistency (order independence) uniquely determine a selection rule. Some intuition is given regarding why these properties work in this collection of problems.

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1 Introduction

Two polar settings have been considered in the literature on choice under uncertainty. At one extreme, the von Neumann-Morgenstern model assumes that the likelihood of all events is given. This is a framework with objective risk and no uncertainty. At the opposite extreme, the Savage (1954) setting (and its Anscombe-Aumann (1963) reformulation) is one of 'complete uncertainty'. The agent is completely ignorant about the probability of events. This paper introduces an intermediate setting where the agent, while still uncertain about the probability of some events, has information which leads her to exclude some of the possible values that these probabilities might take. We can think of the agent as being informed that the true probability distribution over the state space belongs to some, objectively known, set of probabilities. This set could be derived, for example, from her ability to assess the minimum and maximum possible values for the likelihood of any event.

How does the agent make decisions given such set of probabilities? As in the standard model, we will think of the agent as forming a subjective prior and acting as if that probability were correct, that is, behave like a Subjective Expected Utility (SEU) maximizer with respect to this unique prior. Given a set of objectively possible candidates for the true probability, it is now meaningful to ask which one the agent should select as her subjective prior.

In the standard models of choice under uncertainty, nothing precludes us from interpreting the subjective probability as being influenced by what the decision maker knows objectively. Nevertheless, because this information is indistinguishable from the preference ordering, its effect on the decision maker's final beliefs cannot be seen. This work formalizes the intuitive idea that, receiving objective information (even if imperfect) about the probability of unsure events, affects the agent's beliefs. More importantly, an answer is proposed to the question of how information on uncertainty affects the decision maker's choices.

The importance of this question becomes clear when comparing an agent who makes choices after being given an objective probability distribution, to one who, in an uncertain environment, forms an identical subjective prior. In the classical model, there is no distinction between the two. A subjective expected utility maximizer is, indeed, 'probabilistically sophisticated'; she treats uncertainty as if it were risk.

Our model of choice gives uncertainty a direct role in the process of beliefs' formation. This results in a decision maker who still acts as an expected utility maximizer with respect to a subjective prior, but who differentiates between an objectively given probability and a subjective prior. In a dynamic setting, this agent will not update her subjective prior as if it were an objective probability. On the contrary, she will update her objective knowledge to obtain a new set of possible probabilities. She will then form new beliefs taking into account this updated set of objectively possible probabilities.

In the formal model used, we consider the case when the relevant uncertain state space is of finite dimension. Probability distributions are therefore points in the simplex. The agent's objective knowledge, allows her to recognize that the true probability distribution lies in some subset of the simplex. The problem of forming a subjective prior in this framework, is equivalent to selecting a probability from among the convex hull of this objectively given set of candidates (objective uncertainty set). Thus, the act of an agent forming beliefs for any uncertainty set intrinsically defines a selection rule. That is, a function from sets of probabilities to a single prior.

Are there properties one might think desirable for this selection rule? Do those properties identify a process, possibly unique, for the selection of a probability? We begin by suggesting three possible properties. The first, named state independence, states that the selection rule should not depend on how the uncertain states are labeled. The second, referred to as location-consistency, is respected when the application of a selection rule to two sets of priors with the same shape but different locations in the simplex results in selected probabilities that differ by exactly the 'difference' between the two sets.

The third property is the most interesting. Suppose an agent forms some subjective prior p. She is then told that the objective probability of some event A is in fact $p(A)$. In other words, she discovers that her subjective prior was correct at least on the likelihood assignment on A. A selection rule is update-consistent if the agent does not have to revise her subjective prior.

Alternatively, one could also imagine an agent who selects a unique probability sequentially. Starting from some event, she assigns to this event a likelihood value coherent with her objective information. She then picks some other event and repeats the exercise. When assigning a probability to successive events, though, she chooses values that respect her objective information as well as the previous subjective assignments. This procedure, referred to as a sequential selection mechanism, obviously results in the selection of a unique probability.

For this type of selection rule, another property is proposed as an alternative to update-consistency. This last property, named order independence, requires that the selected prior does not depend on the order in which the events are assigned a subjective probability.

Interestingly, it turns out that, given state independence and location consistency, update-consistency and order independence are equivalent. A selection rule that satisfies the first three properties exists if and only if one can find a sequential rule that satisfies the first two plus order-independence. Moreover, the two rules select the same subjective prior.

It is not possible to find a selection rule that respects the desired properties and is defined for any convex closed set of probabilities. This motivates our focus on a smaller class of uncertainty sets. We proceed by restricting our attention to sets of probabilities that are derived from knowledge of the minimum and maximum probabilities of all events. In Section 3, we show how this class of sets can be conveniently described using capacities, where capacities are functions describing the minimum probability of each event. This formalization illustrates some links between the problem at hand and that of selecting an allocation in a cooperative transferable utility game. We can then fruitfully borrow and adapt to our purposes some tools and results from that literature.

We succeed in characterizing a large collection of sets, for which there exists a unique selection rule that satisfies state independence, location-consistency and update-consistency (order independence). We show that this selection rule 'coincides', on those sets, with the nucleolus, a value operator for cooperative games first defined by Schmeidler (1969).

Some intuition is given on why these properties result in a unique selection rule for the specified collection of uncertainty sets. Interestingly, the structure of the objective information in this collection makes the update-consistency requirement most appealing on these sets.

The remainder of the paper is organized into four sections. Section 2 formulates the problem of belief formation in an objective uncertainty setting in terms of selecting a probability from among a set. Two types of selection rules are formally described and four possible properties are defined. In Section 3 we proceed by restricting attention to the class of uncertainty sets that are derived from knowledge of the minimum and maximum probabilities of all events. A convenient representation of these sets is given. In Section 4, we illustrate the relation between selection rules and value operators for cooperative games with transferable utility (TU). Finally, in the last section of the paper, we characterize a class of sets of probabilities for which there exists a unique selection rule that satisfies state independence, location-consistency and update-consistency.

2 SELECTION MECHANISMS

An agent is faced with a problem of choice in an uncertain setting. As in the Anscombe-Aumann setting, uncertainty denotes the lack of an objectively known probability distribution over a finite state space. Unlike that framework, the agent at the moment of choice has some information that objectively restricts the set of possible probability distributions.

As mentioned in the introduction we will consider the case of a finite state space. Ω will denote the set of possible state realizations and $\Delta^{|\Omega|-1}$ the space of all probability distributions over Ω. The agent knows that the true probability distribution on Ω lies in some closed convex subset of $\Delta^{|\Omega|-1}$. Unless otherwise specified, D will denote a generic convex closed subset of $\Delta^{|\Omega|-1}$. $\mathcal{D}$ will be used to denote the collection of all possible D's.

A set of probabilities D is a representation of both the information objectively available to the decision maker (she knows that the true probability measure on the state space is within that set), as well as of the residual uncertainty (she does not know which of the probabilities in the set is the true one). We will refer to such a set as an objective uncertainty set.

How does the agent make decisions given this uncertainty set? As mentioned in the introduction, a possible behavioral rule for the agent is to select a probability measure from among the uncertainty set and act as if that probability were correct. The question then becomes which probability should the agent select. Can one describe any meaningful properties for the selection criterion used by the decision maker?

The act of an agent selecting a probability from among a set, defines a function, mapping from sets of probabilities to a single probability distribution over Ω. We will call this function a selection mechanism.

Formally, given a finite state space Ω and $\mathcal{D}$ being the collection of all convex and closed subsets of $\Delta^{|\Omega|-1}$, a selection mechanism is a mapping $\phi : \mathcal{D} \to \Delta^{|\Omega|-1}$, such that:

$$\phi(D) \in D \quad \text{for all} \quad D \in \mathcal{D}. \text{ [1]} \tag{1}$$

The definition of selection mechanisms above encompasses a large class of selection procedures an agent might follow. Next we propose some properties one might think desirable for a selection mechanism to respect. Only some selection mechanisms will respect one or more of those properties.

Later on in this section, we will also define a particular type of selection mechanisms, which we call sequential selection mechanisms. We claim that they can be interpreted as describing a particular procedure followed by the agent to construct a subjective probability. For this particular class of mechanisms an additional property will be defined.

2.1 STATE INDEPENDENCE

The first property we introduce will be referred to as state independence. Loosely, such axiom requires that the selected probability be independent of the 'name of the states'. Given an uncertainty set $D \subset \Delta^{|\Omega|-1}$, one could permute the name of the states (i.e. call ω_1, ω_2 and vice versa). As a result D is 'transformed' into a new uncertainty set D'. A selection mechanism is state independent if, when transforming D into D', the selected probability changes in a corresponding way (i.e. $p(\omega_1) = \hat{p}(\omega_2)$ where p and $\hat{p}$ are the selected prior before and after the change of name occurred).

[1] Alternatively, one could give a definition of selection mechanism without imposing this requirement. (1) would then have the interpretation of a rationality property for ϕ.

Such property is most appealing when there is no information attached to the name or nature of a particular state. Indeed, this is the underlying assumption of this paper. The agent is always thought of as not having any further knowledge regarding the true probability measure other than the knowledge that it belongs to the uncertainty set.

Formally, the state independence requirement can be expressed as follows:

Definition 1 (state independence) *Let $\phi : \mathcal{D} \rightarrow \Delta^{|\Omega|-1}$ be a selection mechanism. We say that ϕ is state independent if for any bijection of Ω into itself, τ, and any $D \in \mathcal{D}$,*

$$\phi(\tau(D)) = \tau(\phi(D)) \tag{2}$$

where

$$\tau(D) = \left\{p \in \Delta^{|\Omega|-1} | p = \tau(d) \quad \text{for some} \quad d \in D\right\} \tag{3}$$

and, $\tau(x)_i = x_{\tau(i)}$ for all $x \in \mathbf{R}^{|\Omega|}$.

2.2 LOCATION-CONSISTENCY

Other than denoting what are the admissible probability distributions over Ω, a set of probabilities D, could be used as a measure of the amount of uncertainty the decision maker faces when making her choices. In the extreme case, where D is a singleton, all uncertainty is resolved. The agent knows the true probability distribution over Ω. The opposite case when $D = \Delta^{|\Omega|-1}$, indicates a situation where none of the probability measures over Ω are excluded, a situation of maximal uncertainty.

Given two uncertainty sets, D and D', which one denotes a 'more uncertain' setting? If $D' \subseteq D$ (or vice versa), the answer is not controversial. D indicates more uncertainty. When D' is the uncertainty set, the agent's objective knowledge allows her to further refine, with respect to D, the set of possible probability distributions.

Less trivial comparisons require some concept of uncertainty similarity among elements of $\mathcal{D}$. The following definition provides a benchmark for the comparison of sets of probabilities for which a set inclusion relation cannot be found. The intuition behind this definition also provide the motivation for our second axiom.

Definition 2 (uncertainty similar) *Let D and D' be two sets of probability measures on Ω. They are said to be uncertainty similar if there is a vector $\beta \in \mathbf{R}^{|\mathbf{\Omega}|}$ such that: $D = D' + \beta$.*[2]

[2] The sum on the RHS is intended as the Minkowski sum. That is, for any $A \subset \mathbf{R}^{|\mathbf{\Omega}|}$ and $b \in \mathbf{R}^{|\mathbf{\Omega}|}$, $A + b = \left\{x \in \mathbf{R}^{|\mathbf{\Omega}|} | \exists a \in A \text{ s.t. } x = a + b\right\}$.

Uncertainty for an agent is the inability to assign objective likelihood values to events. A framework of limited uncertainty is one where the agent can assign 'some' objective probability to events. For example she might know that an event E has at least a 0.3 chance of occurring. However, her knowledge is insufficient to uniquely determine the exact likelihood of all events. When selecting a subjective prior from the uncertainty set, the agent is complementing the 'objective likelihood assignments', implicit in that set of probabilities, with subjective assessments that results in a unique probability measure. For example, if her selected prior assigns a likelihood of 0.4 to the event E above, it means that other than the 0.3 minimum objective chance, the agent has subjectively attributed to that event an additional 0.1 chance of occurrence.

If one looks at the process of beliefs formation as one of assigning likelihood values to all events, then, selecting a subjective prior from among a given set of candidates, D, is equivalent to assigning likelihood values which are consistent with the restrictions imposed by D. In fact, a set of probabilities D, can be seen as a set of objective 'instructions' on how to assign likelihood values to uncertain events. Moreover, two different kinds of 'instructions' can be distinguished. First, given a set D of probabilities, the agent knows, for each possible state realization its least chance of occurring. The agent also knows that one of the ω's in Ω must be realized; that is, she knows that Ω is a sure event. The uncertainty derives from the fact that the minimum probabilities of all states do not sum up to 1. D contains further instructions on how to assign the remaining probability. For example, for some event $B = \{\omega_1 \cup \omega_2\}$ it might be that all p in D attribute to B a chance of happening greater that the sum of the minimum assignments for ω_1 and ω_2. That is, the uncertainty set gives instruction to distribute some additional probability between ω_1 and ω_2 so that the likelihood of B reaches at least the minimum objectively known value.

Notice that this second type of instructions are imprecise. In general there are several ways to assign some additional likelihood to the two states so that the chance of B happening reaches the critical threshold. A closed convex set of priors carries a rich set of instructions on how to assign the 'residual probability' after the all states have been assigned their minimum chance of occurring.

Two uncertainty sets D and D', which respect the condition of definition 2, carry very similar sets of restrictions to the likelihood of uncertain events. More precisely, they differ only in the way they assign minimum probabilities to single states. The similarity between D and D' appears when one consider that not only the 'residual probabilities' are the same, but the two sets also deliver the exact same additional restrictions regarding how to distribute it among events.

The problem of selecting an effective prior from among a set of possible candidates, is equivalent to subjectively assigning the 'residual probability' among events while respecting the restriction imposed by such set of candidates. We have explained how this problem is identical when facing D or D'. It seems therefore a desirable property that the agent reaches the same conclusion in the two circumstances. We will refer to this requirement as location-consistency with reference to the fact that D and D' differ from each other only because of their 'location' in the simplex.

Formally, a location-consistent selection mechanism respects the following definition:

Definition 3 (location-consistency) *Let $\phi : \mathcal{D} \to \Delta^{|\Omega|-1}$ be a selection mechanism. We say that ϕ is location consistent if, for any $D, D' \in \mathcal{D}$ and any $\beta \in \mathbf{R}^{|\Omega|}$:*

$$\{D' = D + \beta\} \Rightarrow \{\phi(D') = \phi(D) + \beta\}. \tag{4}$$

2.3 UPDATE-CONSISTENCY

The third property we define, concerns the way the selected probability changes after the agent learns more about the objective probability of some events. Suppose that, given some initial set of priors D, the agent selects some probability measure $\hat{p} \in D$ and acts as if this $\hat{p}$ were the true probability measure on Ω. Subsequently she learns (she is told), that the true probability of some event E is exactly $\hat{p}(E)$. She processes this information using some updating rule R to modify the uncertainty set. So doing, a new set of priors D' is obtained. The agent will in general choose a new effective prior $\tilde{p}$ from among the probabilities in D'.

The idea of the update-consistency requirement is that, whenever the new information confirms the decision maker 'prediction', the selected probability remains the same. In the example above, this would imply $\tilde{p} = \hat{p}$.

To define update-consistency, we first need to know how the agent updates an uncertainty set. An updated set of probabilities, naturally depends on three objects: the initial uncertainty set, a collection of events whose objective probability is discovered and a probability measure that specify the value for the discovered objective probability of these events.

We will use P to represent a generic collection of events in Ω (the events whose objective probability is told to the agent), $\mathcal{P}$ will denote the collection of all possible P.

When the exact likelihood of some event E is known to the agent, so is the likelihood of its complement E^C. Similarly, when the likelihood of two disjoint events is known, the same is true for their union. When discovering the true likelihood of the events in some collection P, the agent might infer the likelihood values of other events. Given an uncertainty set D, and a collection of events P, $\mathcal{B}(P, D)$ will denote the smallest collection of events in Ω, closed under complement and under disjoint union, such that $\mathcal{B}(P, D)$ contains all events in P as well as all events whose likelihood can be inferred from knowing the probability of events in P and knowing that the true probability measure on Ω lies in D. $\mathcal{B}(P,D)$ will contain all the events in a collection of partitions of Ω and is closed under union of disjoint events.

$\mathcal{B}(P,D)$ is the effective collection of events whose objective probability is learned once the agent is told the true likelihood of the events in P. For ease of notation, whenever it does not generate any confusion, we will drop the reference to D and to P. $\mathcal{B}$ or $\mathcal{B}(P)$ will denote $\mathcal{B}(P,D)$ when the reference to the appropriate D and/or P is obvious.

Given a $D \in \mathcal{D}$, a $P \in \mathcal{P}$ and a $\delta \in \Delta^{|\Omega|-1}$, the updating rule will generate a new set of probabilities D'. An updating rule, is a function

$$L : \mathcal{D} \times \mathcal{P} \times \Delta^{|\Omega|-1} \to \mathcal{D}, \tag{5}$$

such that, if $p \in D' = L(D,P,\delta)$, then

$$p(B) = \delta(B) \quad \text{for all} \quad B \in \mathcal{B}(P,D). \tag{6}$$

The update-consistency property depends on the updating rule used by the agent. A mechanism might be update-consistent with respect to some updating rule but not to all updating rules. In what follows we will use the following simple updating rule:

$$R(D,P,\delta) = \{p \in D \,|p(B) = \delta(B) \quad \text{for all} \quad B \in \mathcal{B}(P)\}. \tag{7}$$

R specifies that, if the agent discovers the objective probability δ of a collection of events P, then the objective uncertainty set reduces to the set of probabilities in D which assign that exact probability to P.[3]

For any updating rule, one can formalize the concept of update-consistency through the following definition:

Definition 4 (update-consistency) *Let ϕ be a selection mechanism, and L an updating rule. We say that ϕ is L-update-consistent if, for any $D \in \mathcal{D}$, and any $P \in \mathcal{P}$:*

$$\phi(D) = \phi(L(D,P,\phi(D))). \tag{8}$$

Unless otherwise specified, when we mention a generic updating rule we mean the R rule defined above. Similarly the term update-consistency should read as R-update-consistency.

[3]While the updating rule might result in an empty set, we are not interested in such instances since they only occur if the new information contradicts the initial information as represented by D (i.e. $\delta \notin D$.)

2.4 A SEQUENTIAL RULE FOR PROBABILITY SELECTION

We have defined a selection mechanism ϕ as a mapping from the space of the possible uncertainty sets to the space of probability distributions on Ω. One could think of this kind of selection rule as a one stage mechanism. Given any set of priors D, ϕ selects a unique probability measure $\phi(D)$.

When faced with a set of multiple priors on Ω, the agent may find it difficult to select a probability in one stage trying at the same time to be somehow consistent in her choice, especially when the uncertainty set has a complex structure. Alternatively, she might find simpler ways to arrive to the selection of a single probability. We want to describe and then formalize what could be another process used by the agent to select a probability.

The problem of selecting a unique probability distribution is equivalent to assigning a unique probabilistic value to all events. In fact, one could reformulate the problem as one of assigning likelihood values, consistent with the rules of probability, to all events. The 'one step' mechanism above, is one that makes these assignments in a single move. A different technique could proceed by assigning probabilities to events sequentially.

A simple method could be the following. Start from any event E. The agent objectively knows the range of possible value for the likelihood of E. Selecting a value among this range is a much simpler task than selecting the entire probability distribution over Ω. The agent can do this using some simple rule of her choice. The next step is to pick some other event A, and repeat the same exercise. The probability value assigned to A should now be consistent with both the information contained in the uncertainty set and the likelihood value assigned to E.

A procedure like the one just described, defines what we call a sequential selection mechanism. One might consider a sequential mechanism simpler because it requires only a rule to assign to one event a probability compatible with a given set of probabilities. We will call this type of rule a simple assignment rule.

Definition 5 (simple assignment rule) *A function $\chi : \mathcal{D} \times 2^{\Omega} \to \mathcal{D}$ is said to be a simple assignment rule if, for all $D \in \mathcal{D}$ and any $E \subset \Omega$:*

$$\chi(D, E) = \left\{ p \in D \middle| p(E) = k, \quad k \in \left[\min_{p \in D} p(E), \max_{p \in D} p(E) \right] \right\}. \qquad (9)$$

For a given set of probabilities D and an event E, a simple assignment rule selects a value k for the likelihood of E coherent with the set of probabilities (there must be at least a $p \in D$ such that $p(E) = k$). It then returns the set of probabilities in D, which assign to E a likelihood value of k.

Let Θ be the collection of all possible orderings of the elements of 2^{Ω}. A sequential selection rule is a mechanism for the selection of a probability, which for a given ordering of the elements in 2^{Ω}, θ, uses iteratively a simple assignment rule. More formally:

Definition 6 (sequential selection mechanism) *A function $\psi : \mathcal{D} \times \Theta \to \Delta^{|\Omega|-1}$ is called a sequential selection mechanism if, there exists a simple assignment rule χ such that: for any $D \in \mathcal{D}$ and any $\theta \in \Theta$,*

$$\psi(D, \theta) = \chi(\chi(\chi(...), \theta_{k-1}), \theta_k). \tag{10}$$

A sequential mechanism is a function of both a set of probabilities and an ordering of 2^Ω. This means that for a given set of priors D, the probability selected using a sequential mechanism will depend, in general, on the order θ, in which the uncertainty is resolved. This allows us to distinguish a special class of sequential selection mechanisms that, for a given multiple prior set, select the same probability $p \in D$ independently of θ.

Definition 7 (order independence) *A sequential selection mechanism $\psi : \mathcal{D} \times \Theta \to \Delta^{|\Omega|-1}$, is said to be order independent, if*

$$\psi(D, \theta) = \psi(D, \theta')$$

for all $D \in \mathcal{D}$ and any $\theta, \theta' \in \Theta$.

The order-independence property is defined only for sequential mechanisms. Also, both the state independence property and the location consistency property can be redefined in a natural way to apply to simple assignment rules and sequential selection mechanisms. The following Lemma recognizes a direct relation between the state independence and location-consistency properties in simple assignment rules and sequential selection mechanisms.

Lemma 1 *A sequential selection mechanism ψ respects state independence and/or location-consistency if and only if the corresponding simple assignment rule χ, satisfies state independence and/or location-consistency.*

Proof. By definition. ■

It turns out, as the following proposition show, that update-consistency and order independence are equivalent.

Proposition 2 (order-independence update-consistency equivalence) *Let $\mathcal{D}$ be a collection of subsets of $\Delta^{|\Omega|-1}$, closed under updating. A selection mechanism ϕ exists which satisfies anonymity, location-consistency and update-consistency if and only if there is a sequential selection mechanism ψ, state independent, location-consistent and order-independent such that $\psi = \phi$.*

Proof. See Appendix. ■

Remark 1 *Let $D = \left\{y \in \Delta^{|\Omega|-1} | y = \alpha x + (1-\alpha) x', \quad \alpha \in [0,1]\right\}$ for some $x, x' \in \Delta^{|\Omega|-1}$, and $x(i) = x'(i)$ for all $i \neq j, k$. If ϕ is an SI and LC selection mechanism defined on $\mathcal{D}$, then*

$$\phi(D) = \frac{1}{2}x + \frac{1}{2}x' \tag{11}$$

The prove of (11) involves taking two transformations of D. First re-label the states using a bijection τ such that $\tau(j) = k$, $\tau(k) = j$ and $\tau(i) = i$ for all $i \neq j, k$. Then add to the set obtained a vector β with $\beta(i) = x(i) - x'(j)$, $\beta(j) = x'(j) - x(i)$ and $\beta(i) = 0$ for all other states. This two transformations yield the initial set D, that is $\tau(D) + \beta = D$. By SI and LC it should be

$$\phi(D) = \phi(\tau(D) + \beta) = \tau(\phi(D)) + \beta,$$

which is not true unless (11) holds.

Proposition 3 *Let ϕ be a selection mechanism defined on $\mathcal{D}$, satisfying SI, LC and UC. For every $D \in \mathcal{D}$ and for any pair $j, k \in \Omega$, $\phi(D)$, is the middle point of the segment $A(j, k, D)$, defined by:*

$$A(j, k, D) = D \cap \left\{ y \in \Delta^{|\Omega|-1} | y(i) = \phi(D)[i] \, \forall i \in \Omega, i \neq j, k \right\}.$$

Proof. Take a partition of the state space $P(j, k) = \left(j \cup k, \{i\}_{i \in \Omega, i \neq j,k} \right)$. By UC

$$\phi(D) = \phi(R(D, P(j, k), \phi(D)))$$

must hold. Given the definition of updating rule, $R(D, P(j, k), \phi(D)) = A(j, k, D)$ (see (7)). By remark (11), if ϕ is SI and LC, it selects the middle of $A(j, k, D)$. ∎

Remark 2 *There is no selection mechanism ϕ defined for all $D \in \mathcal{D}$, which is state independent, location-consistent and update-consistent.*

A simple geometric example will help understand the previous remark. Let $|\Omega| = 3$ and $D = \left\{ x \in \Delta^2 | \left(x(1) - \frac{1}{3}\right)^2 + \left(x(2) - \frac{1}{3}\right)^2 + \left(x(3) - \frac{1}{3}\right)^2 \leq \alpha \right\}$. For α small enough, D is a circle centrally located in the two dimensional simplex. Let $x = \phi(D)$. Given our choice of D, for any $x \in D$, the midpoint of $A(i, j, D)$ is given by:

$$x(j) = x(k) = \frac{1 - x(i)}{2} \tag{12}$$

By Proposition (3), (12) must hold for any $j, k \in \{1, 2, 3\}$, $j \neq k$ and $i = \Omega \setminus \{j, k\}$. Therefore, it must be $\phi(D) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Now for some small $\epsilon > 0$ consider:

$$D' = D \cap \left\{ x \in \Delta^2 | x(2) < x(3) \Rightarrow \left(x(2) - \frac{1}{3}\right)^2 + \left(x(3) - \frac{1}{3}\right)^2 \leq \alpha - \epsilon \right\}.$$

D' is obtained taking away from D, asymmetrically, a small region in the area where $\left(x(1) - \frac{1}{3}\right)^2 < \epsilon$ and $x(2) < x(3)$. This transformation leaves D' a convex closed set. The center of the simplex does not satisfy (12) for the uncertainty set D' when $i = 1$ and $j, k = 2, 3$. Still $x = \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is the only point that satisfy (12) for $i = 2$ and $i = 3$.

The final remark of this section is a direct consequence of the previous and Proposition (2).

Remark 3 *There is no sequential selection mechanism, defined on $\mathcal{D}$, which respects state independence, location-consistency and order independence.*

3 INFORMATION STRUCTURE AND UNCERTAINTY SETS

Proposition 2 states that it is not possible to find a selection procedure that satisfies the three properties described and is defined for all closed convex uncertainty sets. State independence, location-consistency and update-consistency are properties too strong to be satisfied together by a selection mechanism defined for any set.

Even if we recognize that an interesting research topic would be to look for weaker properties that result in a selection rule defined on the class of all uncertainty sets, in this paper we will proceed in a different direction. Instead of looking for weaker properties, we will focus on selection rules that are defined only on subsets of $\mathcal{D}$. More specifically, we restrict attention to the class of uncertainty sets obtained by the agent's awareness of the lowest and highest possible values for the likelihood of each event.

This restriction has two desirable properties. First, it allows a more convenient way to represent uncertainty sets. Moreover, it will be shown how the update-consistency property, as well as other assumptions to be introduced, have a stronger appeal when the uncertainty sets have the structure used here.

When the information available to the agent concerns only the bounds on the likelihood of each event, an easy way to represent it is through a capacity on Ω.

Definition 8 (capacity) *A capacity on Ω is a mapping $v : 2^{\Omega} \cup \emptyset \rightarrow [0, 1]$ respecting the following:*

1. $v(\emptyset) = 0$, *and* $v(\Omega) = 1$;
2. *if* $A \subseteq B \subseteq \Omega$ *then,* $v(A) \leq v(B)$.

Moreover, if for any $A, B \subseteq \Omega$, $\quad v(A \cup B) \geq v(A) + v(B) - v(A \cap B)$, v is is said to be a *convex capacity.*

We use a capacity only as a convenient means of representing the information objectively available to the decision maker. A very simple example may help describe how a capacity on Ω suffices to represent information of the kind described above.

Example 1 Imagine having an urn with 100 balls of three possible colors: black (B), red (R) and yellow (Y). The exact number of B, R and Y balls in the urn is unknown but the following information is given:

1. *There are at least 20 B balls, at least 0 R balls and at least 15 Y balls.*
2. *There are at least 20 R or Y balls, at least 25 B or Y balls and at least 40 B or R balls.*

One ball is extracted from the urn. The state space is the collection of all possible realizations of this draw, so $\Omega = \{B, R, Y\}$. Let B be the event: "a black ball is drawn". It is not possible to assess the precise probability of this event but, given the information above, we are aware that the lowest value for $p(B)$ is 0.2, so $v(B) = 0.2$. Similarly we have $v(R) = 0$ and $v(Y) = 0.25$. We can also set $v(B \cup R) = 0.4$, $v(R \cup Y) = 0.2$, and so on.

Suppose an agent is aware of the minimum probability of all events, and let a capacity v describe the agent's knowledge. The objective information available to the agent restricts the set of admissible probability distributions on Ω in an obvious way. The only probability measures on Ω consistent with what the agent knows are those $p \in \Delta^{|\Omega|-1}$ such that:

$$p(E) \geq v(E) \quad \text{for all} \quad E \subseteq \Omega. \tag{13}$$

Equivalently one could interpret the pair (Ω, v) as a transferable utility cooperative game. The space of all events, 2^{Ω}, is the space of all coalitions and v is the characteristic function. The uncertainty set is the set of probabilities $p \in \Delta^{|\Omega|-1}$ that satisfy (13), which is also the definition of the core of (Ω, v), $c(\Omega, v)$. Therefore, we can equivalently denote any uncertainty set, as the core of the TU game defined by the pair (Ω, v), where v is the capacity representing the objective information available to the agent.

3.1 CONSISTENCY

It is possible for $c(\Omega, v)$ to be empty. Whenever this is the case, the information available to the agent is contradictory. The information on the bounds on the likelihood of events must be incorrect. Either the agent is able to recognize the mistake and obtain a new capacity v' such that $c(\Omega, v') \neq \emptyset$, or she should reject all of the information. This paper considers only those capacities which generate a non-empty core.

In terms of uncertainty sets, we restrict attention to a subset of $\mathcal{D}$. Let us denote with $\mathcal{V}$ the collection of capacities on Ω such that $c(\Omega, v) \neq \emptyset$[4].

[4] For TU games, balancedness is a necessary and sufficient condition for the existence of a core (Scarf, 1967). Therefore $\mathcal{V}$ is class of all balanced capacities on Ω.

The collection of uncertainty sets resulting from coherent information on the minimum and maximum values for the probability of all events is therefore:

$$\mathcal{D}_{\mathcal{V}} = \{D \in \mathcal{D} \,|D = c(\Omega, v) \quad \text{for some} \quad v \in \mathcal{V}\}. \tag{14}$$

One might notice that there is not a one to one correspondence between $\mathcal{D}_{\mathcal{V}}$ and $\mathcal{V}$. In fact, there might be two different capacities $u, v \in \mathcal{V}$, such that $c(\Omega, v) = c(\Omega, u)$. This happens when the information represented by a capacity is not 'tight'. Suppose that, for some $E \subset \Omega$,

$$\min_{p \in c(\Omega, v)} p(E) > v(E). \tag{15}$$

In this case, the agent, originally informed that E has a probability of at least $v(E)$, can infer a higher lower bound for the likelihood of this event from what she knows about the minimum probabilities of other events. Allowing the agent to draw this type of conclusion and update her knowledge of the minimum probability of E results into a capacity u different from v but that delivers the same amount of information. In other words, v and u generates the same uncertainty set.

Whenever the information expressed in terms of bounds on the probability of events, is tight the capacity v representing it will respect the following condition:

$$\min_{p \in c(\Omega, v)} p(E) = v(E) \quad \textit{for all} \quad E \subset \Omega. \tag{16}$$

We will refer to capacities that satisfy equation (16) as binding[5].

We will treat two capacities with the same core as equivalent. That is, we assume that the agent can always draw the minimal type of inference described above.

3.2 UPDATING THE UNCERTAINTY SET

In section 2.3 we considered an agent who discovers additional objective information, and updates her uncertainty set. We defined an updating rule as a function mapping from a triplet (D, P, δ) to a set of probabilities D'. D and D' are the initial and the updated uncertainty sets respectively. P represent the collection of events whose objective probabilities given by δ is discovered.

In the previous section we have shown a convenient way to denote uncertainty sets derived by the agent's ability to assess minimum and maximum probability of any event through a capacity on Ω. One can then define an updating rule for uncertainty set in $\mathcal{D}_{\mathcal{V}}$ through a function $U : \mathcal{V} \times \mathcal{P} \times \Delta^{|\Omega|-1} \rightarrow \mathcal{V}$.[6]

[5] The class of binding capacities is a subset of the class of totally balanced capacities.

[6] One needs only make sure that U is such that $c(\Omega, u) = c(\Omega, v) \Rightarrow c(\Omega, U(u, \cdot, \cdot)) = c(\Omega, U(v, \cdot, \cdot))$.

In section 2.3, we also defined the simple updating rule that we want to use. In this section we will define an intuitive updating rule for capacities. We will show that this capacity updating rule results into an updating rule for uncertainty sets in $\mathcal{D}_{\mathcal{V}}$ that is equivalent to (7).

A capacity represents the objective information available to the agent. It describes for each event the objectively known minimum probability. The new information received by the agent concerns the objective probabilities of a collection of events. How does the discovery of the objective probability of some event affects the agent knowledge of the minimum probability of some other event?

This question is answered by our capacity-updating rule. Before giving a formal definition, an example will be useful to show the rationale behind that rule.

Example 2 Take the usual urn with 100 balls. This time the balls can be of four different colors, black (B), red (R), yellow (Y) and green (G). The information given about the number of balls of each color can be represented by the following capacity v

$$\begin{array}{rclrcl} v(B) & = & 10 & v(R) & = & 10 \\ v(Y) & = & 10 & v(G) & = & 10 \\ v(B \cup Y) & = & 20 & v(B \cup R) & = & 20 \\ v(B \cup G) & = & 20 & v(Y \cup R) & = & 20 \\ v(Y \cup G) & = & 20 & v(R \cup G) & = & 20 \\ v(B \cup Y \cup R) & = & 60 & v(B \cup Y \cup G) & = & 60 \\ v(B \cup R \cup G) & = & 60 & v(R \cup Y \cup G) & = & 60 \end{array}$$

New information is acquired, revealing that there are exactly 40 "black or red balls". How do we update the capacity describing the uncertainty set? The new capacity should reflect the fact that there is no uncertainty regarding the number of 'black or red balls' anymore (0.4 is now the objective probability of the event 'either a black or a red ball is drawn'). If $\tilde{v}$ denotes the update capacity it must be $\tilde{v}(B \cup R) = 40$ and $\tilde{v}(G \cup Y) = 60$.

The fact that there are exactly 40 red or black balls also tells the agent something more about the minimum number of yellow balls. In fact, she knows $v(B \cup R \cup Y) = 60$ or that there are at least 60 non green balls. From knowing that the exact number of 'red or black' balls is 40 she can infer that there are at least 20 yellow balls (the probability of the event 'a yellow ball is drawn' is objectively greater or equal than 0.2). The agent can therefore update the value of $v(Y) = 10$ to $\tilde{v}(Y) = 20$.

Similar reasoning for the number of green balls results in an updated minimum value of 20 green balls. The same procedure does not change the minimum known number of red balls or black balls. Having updated the value of $(Y \cup G)$ to 60 and knowing that there are at least 10 red balls, the agent can also easily infer that there must be at least 70 non black balls, which is more than she knew before. Proceeding with this line of reasoning the new capacity $\tilde{v}$ will be as follows:

$$\begin{array}{rclrcl} \tilde{v}(B) &=& 10 & \tilde{v}(R) &=& 10 \\ \tilde{v}(Y) &=& 20 & \tilde{v}(G) &=& 20 \\ \tilde{v}(B\cup Y) &=& 30 & \tilde{v}(B\cup R) &=& 40 \\ \tilde{v}(B\cup G) &=& 30 & \tilde{v}(Y\cup R) &=& 30 \\ \tilde{v}(Y\cup G) &=& 60 & \tilde{v}(R\cup G) &=& 30 \\ \tilde{v}(B\cup Y\cup R) &=& 70 & \tilde{v}(B\cup Y\cup G) &=& 70 \\ \tilde{v}(B\cup R\cup G) &=& 60 & \tilde{v}(R\cup Y\cup G) &=& 70 \end{array}$$

The new capacity $\tilde{v}$ in the example above, is obtained applying the definition of capacity-updating rule given below. The capacity-updating rule is a function from $\mathcal{V}\times\mathcal{P}\times\Delta^{|\Omega|-1}$ to $\mathcal{V}$. The definition given is general and allows the updating of a capacity v on Ω when the probability of any collection of events P is discovered. P always contains all the events whose probability is already known.

The following additional notation is used in the definition: For any $C\subset\Omega$,

$$\mathcal{B}_C=\{B\in\mathcal{B}(P)\,|B\cap C\neq\emptyset\}\,. \tag{17}$$

$\mathcal{B}_C$ is the collection of all events of objectively known probability with a non empty intersection with C. Similarly,

$$\mathcal{B}_{-C}=\{B\in\mathcal{B}(P)\,|B\cap C=\emptyset\} \tag{18}$$

is the collection of all events of known probability with an empty intersection with C. Finally, $\overline{C}$ is the union of all sets in $\mathcal{B}_C$ and τ_C is defined as the collection of all the partitions of $\overline{C}$ that use only sets in $\mathcal{B}_C$.

$$\tau_C=\left\{\tau\subset\mathcal{B}_C|\cup_{A\in\tau}A=\overline{C},\quad A,B\in\tau\Rightarrow A\cap B=\emptyset\right\} \tag{19}$$

Definition 9 (capacity-updating rule) *A capacity updating rule is a function $U:\mathcal{V}\times\mathcal{P}\times\Delta^{|\Omega|-1}\rightarrow\mathcal{V}$ such that: for all $v\in\mathcal{V}$, all $\delta\in c(\Omega,v)$ and all $P\in\mathcal{P}$:*

$$\begin{array}{rcl} U(v,P,\delta)[B] &=& \delta(B)\quad \forall B\in\mathcal{B}(P) \\ U(v,P,\delta)[C] &=& \max\limits_{T\in\mathcal{B}_{-C}\cup\emptyset}\{v(C\cup T)-\delta(T)\}\quad \forall C\ s.t.\ B\in\mathcal{B}_C\Rightarrow B\supseteq C \\ U(v,P,\delta)[C] &=& \max\left\{\max\limits_{T\in\mathcal{B}_{-C}\cup\emptyset}\{v(C\cup T)-\delta(T)\},\right. \\ && \left.\max\limits_{\tau\in\tau_C}\sum\limits_{A\in\tau}U(v,P,\delta)[C\cap A]\right\}\ \textit{otherwise.} \end{array}$$

The principle through which the function formally defined above updates a capacity is the same as in the example. The complication arises from the fact that the definition allows the objective probability of a generic collection of events to be learned at once. If P denotes such collection, then $\mathcal{B}(P)$ lists all event whose likelihood is objective after that discovery.

There are three components in the definition of capacity updating rule. First, the updated capacity assign to any event in $\mathcal{B}(P)$ a value equal to that event's (now) objective likelihood.

To understand the meaning of the second line in the definition of capacity updating rule one needs to consider the following fact. $\mathcal{B}(P)$ is made of all the events in some collection of partitions of Ω and is closed under union of disjoint events. That means, there is some collection of partitions of Ω, $\{A_1, A_2, ..., A_k\}$ such that for any event C the following holds:

$$C \in A_i \text{ for some } i \Leftrightarrow C \in \mathcal{B}(P) \tag{20}$$

The second line of the definition updates an event C if C is a subset of an event in each of the partitions $\{A_1, A_2, ..., A_k\}$. In other words, there is some collection of events $\{a_1, a_2, ..., a_k\}$ with $a_i \in A_i$ for all $i = 1, .., k$, such that:

$$C \subseteq \cap_i a_i. \tag{21}$$

For each event of this type, $v(C)$ is reevaluated, as in the example, looking at the minimum probabilities of the union of C and events of known probability.

The third line of the definition deals with all other events. The reason why the updating formula for those is different, is to guarantee superadditivity (without superadditivity the update of a capacity would not be a capacity). Suppose C has a non-empty intersection with more than one event, say a_i and b_i in A_i. Were A_i is one of the partitions of Ω of known probability. Then it might happen that:

$$\max_{T \in \mathcal{B}_{-C} \cup \emptyset} \{v(C \cup T) - \delta(T)\} < U(v, P, \delta)[C \cap a_i] + U(v, P, \delta)[C \cap b_i]. \tag{22}$$

In this situation, as in the example, C is assigned the higher minimum probability derived by the sum of the minimum probabilities of its parts. Because this last step requires $U(v, P, \delta)[C \cap a_i]$ and $U(v, P, \delta)[C \cap b_i]$ to be known, the updated capacity needs to be computed iteratively starting from the smallest sets.

The capacity updating rule U defines an updating function $R_U : \mathcal{D}_\mathcal{V} \times \mathcal{P} \times \Delta^{|\Omega|-1} \to \mathcal{D}_\mathcal{V}$, such that:

$$R_U(D_v, P, \delta) = c(\Omega, U(v, P, \delta)). \tag{23}$$

The next proposition shows that R_U is indeed a function and is equivalent to the updating rule R defined by (7).

Proposition 4 *(Ω, v) be a generic TU game, and $\Delta = c(\Omega, v)$. Let P be a collection of subsets of Ω and $q^* \in c(\Omega, v)$. Then,*

$$\{p \in \Delta | p(B) = q^*(B) \quad \textit{for all} \quad B \in \mathcal{B}(P)\} \equiv \Delta' = \tilde{\Delta} \equiv c(\Omega, \tilde{v})$$

where $\tilde{v} = U(v, P, q^*)$.

Proof. See Appendix. ■

4 SELECTION MECHANISMS AND VALUE OPERATORS

When we restrict attention to the class of uncertainty sets in $\mathcal{D}_{\mathcal{V}}$, we know that we can either interpret a selection mechanism as a function from $\mathcal{D}$ to $\Delta^{|\Omega|-1}$ or as a function from $\mathcal{V}$ to $\Delta^{|\Omega|-1}$. That is, a selection mechanism ϕ, defines a value operator for the class of all TU games defined by a balanced capacity on Ω. Given a grand coalition (state space) Ω, a value operator is a mapping φ from $\mathbf{R}^{2^\Omega}$ into $\mathbf{R}^{|\Omega|}$. Since $\mathcal{V}$ is a subset of $\mathbf{R}^{2^\Omega}$, then for any $v \in \mathcal{V}$ a selection mechanism ϕ on $\mathcal{D}_{\mathcal{V}}$ defines a value operator φ such that $\varphi(v) = \phi(D_v)$. Moreover φ will have the two following properties:

$$\varphi(\Omega, v) \in c(\Omega, v)^{7}; \tag{24}$$

$$c(\Omega, v) = c(\Omega, u) \Rightarrow \varphi(v) = \varphi(u)^{8}. \tag{25}$$

We can compare the selection mechanism properties defined above with properties defined for value operators of cooperative games. It turns out that the definition of state independence coincides with that of anonymity and that there is a strong similarity between the location-consistency requirement and the following property, usually referred to as *zero-independence.*

Definition 10 (zero-independence) *A value operator φ is said to be **zero independent** if for all Ω, all $u, v \in \mathbf{R}^{2^\Omega}$, and all $\beta \in \mathbf{R}^{|\Omega|}$, we have*

$$\left\{ u(S) = v(S) + \sum_{i \in S} \beta_i \quad \textit{for all} \quad S \right\} \Rightarrow \{\varphi(\Omega, u) = \varphi(\Omega, v) + \beta\}$$

Remark 4 *Let ϕ be a selection mechanism on $\mathcal{D}_{\mathcal{V}}$. Define $\varphi : \mathcal{V} \to \mathcal{R}^{|\Omega|}$ so that $\varphi(v) = \phi(c(\Omega, v))$. If ϕ is location-consistent then φ is zero-independent on $\mathcal{V}$.*

Since the capacity-updating rule U is equivalent to our uncertainty set updating rule R, we can reformulate the update-consistency property into a similar requirement for value operators. We call this property capacity-update-consistency.

[7] φ is a core selection value operator

[8] φ has a fixed location in the core.

Definition 11 (capacity-update-consistency) *Let φ be a value operator defined for any $v \in \mathcal{V}$. $\varphi(\cdot)$ is capacity-update-consistent if: for any collection of subsets of Ω, P,*

$$\varphi(U(v, P, \varphi(v))) = \varphi(v)$$

where U is the capacity-updating rule defined in section 3.2.

Remark 5 *Let ϕ be a selection mechanism on $\mathcal{D}_{\mathcal{V}}$. Define $\varphi : \mathcal{V} \to \mathcal{R}^N$ so that $\varphi(v) = \phi(c(\Omega, v))$. $If \phi$ is update-consistent, then φ is capacity-update-consistent on $\mathcal{V}$.*

By Remark 5 and Remark 4, we know that if we find a selection mechanism that is state independent, location-consistent and update-consistent on $\mathcal{D}_{\mathcal{V}}$, then we can immediately define a value operator that satisfy anonymity, zero-independence and capacity-update-consistency on $\mathcal{V}$. Next we look at one well known value operator, the nucleolus, and check if it respects these properties.

4.1 NUCLEOLUS

Several value operators have been proposed in the cooperative game theoretic literature. In this section we will briefly describe one of them, the nucleolus. It is well known that the nucleolus is both anonymous and zero-independent. The main result of this section is to show that the nucleolus is also capacity-update-consistent on $\mathcal{V}$.

Definition 12 (Nucleolus[9], Schmeidler[1969]) *Consider a TU game (Ω, v). Denote by B the set of efficient allocations $\left\{x \in \mathbf{R}^{|\Omega|} \left| \sum_{\omega_i \in \Omega} x[\omega_i] = v(\Omega) \right.\right\}$. To every vector in B, associate the following vector $e(x) \in \mathbf{R}^{2^\Omega}$:*

$$e(x, S) = \sum_{\omega_i \in S} x[\omega_i] - v(S) \quad \textit{for all} \quad S \subset \Omega$$

There is a unique allocation $\gamma \in B$ such that for every other $x \in B$, the lexmin[10] ordering prefers $e(\gamma)$ to $e(x)$. It is called the nucleolus of the game (N, v).

Given a characteristic function v for a TU game (Ω, v), $e(x, \cdot)$ determines, for any allocation x, the excess with respect to the opportunity surplus (as determined by the characteristic function) for each coalition. The nucleolus treats each coalition in an egalitarian spirit, selecting the allocation that

[10]The original definition of Schmeidler required individual rationality. This definition corresponds to what is usually referred to in the literature as the prenucleolus. Whenever the core is non-empty, the nucleolus and prenucleolus coincide. For our purposes (we always consider balanced games) the two are indistinguishable and the definition of the prenucleolus is more convenient.

[10]Lexicographic minimum.

maximizes the smallest of the coalitions' excess surpluses. If more than one allocation solves the maxmin problem, the nucleolus selects among them the one which maximizes the second smallest excess surplus and so on. We can now state the main theorem of this section.

Proposition 5 *In every balanced game the nucleolus is capacity-update consistent.*

Proof. See Appendix. ■

It is useful, to have a better understanding of the result of Proposition 5, to compare the capacity-update consistency property to another property for value operators known in the literature as reduced game property. The definition of reduced game and that of reduced game property, were originally formulated in Davis and Maschler (1965). The version given below, borrowed from Moulin (1988), reflects the reformulation of these definitions in Peleg (1986).

Definition 13 (Reduced Game, Reduced Game Property) *Given a TU game (Ω, v), an efficient allocation x, and a proper coalition $S \subset \Omega$, the* ***reduced game on*** S ***at*** x *is the following game* $\left(S, \tilde{v}_x^S\right)$:

$$\begin{aligned} \tilde{v}_x^S(S) &= \sum_{i \in S} x_i \\ \tilde{v}_x^S(T) &= \max_{\emptyset \subset R \subset N \setminus S} \left\{ v(T \cup R) - \sum_{i \in R} x_i \right\} \text{ for all } \quad T \subset S \end{aligned}$$

Consider a value operator φ defined for the TU games of all sizes. We say that φ satisfies the reduced game property if for any game (Ω, v) and any proper coalition $S \subset \Omega$, we have

$$x = \varphi(\Omega, v) \Rightarrow \varphi\left(S, v_x^S\right) = \pi_S(x) \tag{26}$$

where π_S is the projection of $\mathbf{R}^{|\Omega|}$ on the space of allocations for S.

The Reduced Game for a subset of agents E is the TU game which would obtain after the value of all agents not in E has been fixed at the level determined by the allocation x under the assumption that any member, or coalition of members in E, can buy out any member or coalition of members in $\Omega \setminus E$ at exactly the price set by x.

Even if the motivation for the reduced game is very different from that of the capacity-updating rule, the two definitions are similar. In fact, if one takes $P = \left\{S, \{\omega_i\}_{\omega_i \notin S}\right\}$, then, $U(v, P, \delta)[E] = v_\delta^S(E)$ for any $E \subseteq S$ and any δ.

There are two main differences between the capacity updating rule and the definition of reduced game. First, the updated capacity defines a TU game on Ω while the reduced game is necessarily a game of smaller size. Moreover there is no reduced game analogous to the game obtained when the collection P of discovered events does not contains all $\omega_i \notin S$. Finally, the similarity between the two, points to another interpretation of the reduced game, which seems, to us, to be quite natural, even if it is very different from the idea that originally motivated its definition.

One could slightly change the definition of reduced game, so that the reduced game is a game with the same number of players as the original game adding to Definition 13 the following:

$$\begin{array}{l} \tilde{v}_x^S(T) = \sum_{i \in T} x_i \quad \forall T \subset \Omega \setminus S \\ \tilde{v}_x^S(T) = \tilde{v}_x^S(T \cap (\Omega \setminus S)) + \tilde{v}_x^S(T \cap S) \quad \text{otherwise.} \end{array}$$

Using this extended definition of reduced game (ERG) we have the following remark.

Remark 6 *The nucleolus satisfies the ERG property on $\mathcal{V}$.*

To prove the above remark one needs to notice three facts. 1) The nucleolus of any balanced game is a point in the core. 2) The ERG with respect to an allocation in the core is a balanced game ($\mathcal{V}$ is closed under reduction with respect to the nucleolus). 3) The nucleolus of the ERG is a vector $x^* \in R^{|\Omega|}$ with the property that its projection on S is exactly the nucleolus of the reduced game, and $x^*(i) = x(i)$ for all $i \in \Omega \setminus S$.

Remark 7 *Let ϕ be any core selection value operator defined on $\mathcal{V}$. If ϕ is update consistent than ϕ satisfies the ERG property.*

The definition of capacity updating rule encompasses that of a ERG.

4.2 SELECTION MECHANISMS

It is known in the literature that the nucleolus respects both anonimity and zero-independence. Proposition (5) establishes that it is also capacity-update consistent. It seems therefore a very good candidate for our selection mechanism. The nucleolus though does not have a fixed location in the core. In general, capacities in $\mathcal{V}$ with the same core might have different nucleoli[11]. A selection mechanism has instead, by definition, a fixed location in the core. In other words, one cannot define $\phi : \mathcal{D}_{\mathcal{V}} \to \Delta^{|\Omega|-1}$ from

$$\phi(D_v) = \eta(\Omega, v) \quad \text{for all} \quad v \in \mathcal{V}$$

where $\eta(\Omega, v)$ denotes the nucleolus of the game (Ω, v) because ϕ would assign to some $D_v \in \mathcal{D}_{\mathcal{V}}$ more than one value.

[11] see for example Maschler, Peleg and Shapley 1977

Alternatively, one could define a selection mechanism on $\mathcal{D}_{\mathcal{V}}$ from the nucleolus restricted to the class of balanced and binding capacities $\mathcal{V}_B$, that is:

$$\phi(D_{v_B}) = \eta(\Omega, v_B) \quad \text{for all} \quad v_B \in \mathcal{V}_B. \tag{27}$$

This would be sufficient to define a selection mechanism on $\mathcal{D}_{\mathcal{V}}$ because $\mathcal{D}_{\mathcal{V}_B} = \mathcal{D}_{\mathcal{V}}$. The following example shows that the resulting selection mechanism would not be update-consistent.

Example 3 Let $\Omega = \{1,2,3,4,5\}$. For sake of exposition we will use an example with $v(\Omega) = 3$. v is defined in the following way: $v(i) = 0$ for all $i \in \Omega$. $v(1,2) = v(1,4) = v(2,5) = v(3,5) = v(4,5) = 1$. $v(1,5) = v(2,4) = 0$. $v(1,3) = v(3,4) = \frac{1}{2}$. $v(2,3,5) = v(3,4,5) = 2$. $v(i,j,k) = 1$ for all $(i,j,k) \neq (2,3,5), (3,4,5)$. $v(1,2,3,4) = \frac{3}{2}$, $v(i,j,k,l) = 2$ for all $(i,j,k,l) \neq (1,2,3,4)$.

One can verify that v is a binding capacity with a non empty core. The nucleolus of (Ω, v) is given by $\eta = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right)$. Calculating the reduced game $\tilde{v}_{\tilde{\eta}}^S$ for the coalition $S = \{1,2,3,4\}$, one obtains: $\tilde{v}(1) = 0$ for all $i \in S$. $\tilde{v}(2,4) = 0$, $\tilde{v}(1,3) = \frac{1}{2}$ and $\tilde{v}(i,j) = 1$ for all other pairs of players. $\tilde{v}(i,j.k) = 1$ for all $(i,j,k) \subset A$ and finally $\tilde{v}(A) = 2$. For any other coalitions T, $\tilde{v}(T) = \tilde{v}(T \cap A) + 1$.

While η is still the nucleolus of $\tilde{v}$, this updated capacity not binding. The binding capacity $\overline{v}$, with the same core as $\tilde{v}$ would differ from $\tilde{v}$ with respect to the following coalitions: $\overline{v}(1) = \overline{v}(3) = \frac{1}{4}$, $\overline{v}(1,2,3) = \overline{v}(1,3,4) = \frac{5}{4}$. $\eta(\Omega, \overline{v}) = \left\{\frac{5}{8}, \frac{3}{8}, \frac{5}{8}, \frac{3}{8}, 1\right\} \neq \eta(\Omega, \tilde{v})$.

If ϕ is defined from (27) we have :

$$\phi(R(D_v, (A, \{5\}), \phi(D_v))) = \phi(D_{\overline{v}}) = \eta(\Omega, \overline{v}) \neq \eta(\Omega, v) = \phi(D_v), \tag{28}$$

and ϕ does not respect update consistency.

Example (3) illustrates the following fact. $\mathcal{V}_B$ is not a rich enough class of capacities in the sense that it does not contain all the capacities obtained from updating any $v_B \in \mathcal{V}_B$ at the nucleolus allocation. When the update from a capacity in $\mathcal{V}_B$ fails to be in $\mathcal{V}_B$, (27) uses some different capacity in $\mathcal{V}_B$. The latter might not have the same nucleolus as the former yielding a failure of update consistency for ϕ.

5 UPDATE CONSISTENCY

In this last section, we further investigates the implications of the update consistency property. This leads to identify a class of uncertainty sets on which the properties of state independence, location-consistency and update-consistency characterize a unique selection rule.

Take any set $D_v \in \mathcal{D}_{\mathcal{V}}$ for any allocation $x \in D_v$ and any pair of states $i, j \in \Omega$ define the following:

$$R_{i,j}(x, D_v) = \{y \in D_v | y(k) = x(k) \quad \forall k \neq i, j\}. \tag{29}$$

The set $R_{i,j}(x, D_v)$ is a line segment contained in D_v in the $i-j$ direction. That is, all allocations in $R_{i,j}(x, D_v)$ differ only in the values they assign to the i'th and j'th components. Let $\overline{R}_{i,j}(x, D_v)$ and $\underline{R}_{i,j}(x, D_v)$ denote the endpoints of $R_{i,j}(x, D_v)$. We know, from Proposition (3), that a selection mechanism ϕ on $\mathcal{D}_{\mathcal{V}}$ satisfying SI, LC and UC must satisfy

$$\phi(D_v) = \frac{1}{2}\underline{R}_{i,j}(\phi(D_v), D_v) + \frac{1}{2}\overline{R}_{i,j}(\phi(D_v), D_v) \tag{30}$$

for all pairs $i, j \in \Omega$. Next, let's define

$$B_{i,j}(D_v) \equiv \left\{ y \in D_v \middle| y = \frac{1}{2}\underline{R}_{i,j}(x, D_v) + \frac{1}{2}\overline{R}_{i,j}(x, D_v) \quad \text{for some} \quad x \in D_v \right\}.$$

The set $B_{i,j}(D_v)$ is the bisecting hypersurface obtained by taking the mid-points of all the segments $R_{i,j}(x, D_v)$ through points in D_v.

Remark 8 *If ϕ is a selection mechanism on $\mathcal{D}_{\mathcal{V}}$ satisfying SI, LC and UC then:*

$$\phi(D_v) \in \cap_{i,j \in \Omega} B_{i,j}(D_v). \tag{31}$$

Condition (31) in the above remark, is sufficient to guarantee that SI and LC are satisfied. Moreover it is necessary for SI, LC and UC to be satisfied together. Condition (31) is obtained by using only some of the restrictions imposed by the update consistency requirement. Specifically the restrictions of the form

$$\phi(D_v) = \phi(R(D_v, P, \phi(D_v))), \forall P = \left(i \cup j, \{k\}_{k \in \Omega, k \neq i,j}\right), \quad i, j \in \Omega. \tag{32}$$

In what follows, we show that these restrictions are sufficient for UC. That is, a selection mechanism that satisfies (32) is update consistent. We can then conclude that (31) is both necessary and sufficient for SI, LC and UC to hold together.

Similarly to what we have above for pairs of states, for a collection of three different states $S = \{i, j, k\}$ and any $x \in D_v$ let's define

$$D_v(x, S) \equiv \{y \in D_v | y(l) = x(l) \forall l \in \Omega, l \notin S\}.$$

The set $D_v(x, S)$[12], is the set of allocations in D_v that assign to every agent not in S the same value assigned by x. We can now define two additional operators

$$R_S(x, D_v) \equiv \cap_{i,j \in S} B_{i,j}(D_v(x, S)); \text{ and} \tag{33}$$

[12] $D_v(x, S)$ is also equal to $R\left(D_v, \left(S, \{l\}_{\Omega \ni l \notin S}\right), x\right)$.

$$B_S(D_v) \equiv \{y \in D_v \,|y \in R_S(x, D_v) \quad \text{for some } x \in D_v\} \tag{34}$$

If a selection mechanism ϕ satisfies SI, LC and UC, then it must be that

$$\phi(D_v) \in \cap_{i,j,k\in\Omega} B_{i,j,k}(D_v). \tag{35}$$

To understand (35) first notice that any $D_v(x, S)$ belongs to the collection $\mathcal{D}_\mathcal{V}$. Therefore, because of (31), $\phi(D_v(x, S)) \in R_S(x, D_v)$[13] If ϕ is update consistent, the following must hold:

$$\phi(D_v) = \phi\left(R\left(D_v, \left(S, \{l\}_{\Omega\ni l\notin S}\right), \phi(D_v)\right)\right). \tag{36}$$

Moreover, since $R\left(D_v, \left(S, \{l\}_{\Omega\ni l\notin S}\right), \phi(D_v)\right) = D_v(\phi(D_v), S)$, from (31) we have

$$\phi(D_v) = \phi(D_v(\phi(D_v), S)) \in R_S(\phi(D_v), D_v). \tag{37}$$

The above implies

$$\phi(D_v) \in B_S(D_v). \tag{38}$$

Since (38) must hold for any coalition of three players in Ω, (35) obtains.

We can define, iteratively, $B_S(D_v)$ for coalitions of 4 players and and so on up to the coalition of all players and obtain restrictions analogous of (35). Putting all those restrictions together we have:

$$\phi(D_v) \in \cap_{S\subset\Omega} B_S(D_v). \tag{39}$$

Finally, for any collection of events in Ω, P, and any $x \in D_v$, we can define:

$$D_v(x, P) = \{y \in D_v \,|y(S) = x(S) \ \ \forall S \in P\}, \tag{40}$$

and

$$R_P(x, D_v) = R_\Omega(x, D_v(x, P)). \tag{41}$$

Defining $B_P(D_v)$ from (34), we can include these last restrictions imposed by update consistency into (39) and obtain:

$$\phi(D_v) \in \cap_{P\in\mathcal{P}} B_P(D_v), \tag{42}$$

with the convention that $P = \left\{S, \{i\}_{i\notin S}\right\}$ corresponds to S in (39).

The next proposition establishes that any restriction imposed by update consistency beyond those given by (32) are superfluous. That is, they are satisfied by any allocation that has the bisection property with respect to any $i - j$ directions.

Proposition 6 *For any $D_v \in \mathcal{D}_V$,*

$$\cap_{i,j\in\Omega} B_{i,j}(D_v) = \cap_{P\in\mathcal{P}} B_P(D_v). \tag{43}$$

[13] In the definition of $R_S(x, D_v)$, the intersection of the $B_{i,j}$'s is taken only for $i, j \in S$. This is sufficient since, for any other pair of agents, $B_{k,l}(D_v(x, S)) = x$.

Proof. $\cap_{P\in\mathcal{P}} B_P(D_v) \subset \cap_{i,j\in\Omega} B_{i,j}(D_v)$ holds by definition as:

$$\cap_{P\in\mathcal{P}} B_P(D_v) = \left(\cap_{i,j\in\Omega} B_{i,j}(D_v)\right) \cap \left(\cap_{P\in\mathcal{P},P\neq\{i,j\}} B_P(D_v)\right) \qquad (44)$$

We need to show that

$$x \in \cap_{i,j\in\Omega} B_{i,j}(D_v) \Rightarrow x \in \cap_{P\in\mathcal{P}} B_P(D_v).$$

Take $x^* \in \cap_{i,j\in\Omega} B_{i,j}(D_v)$. For any coalitions of three agents S the following holds for any $i,j \in S$:

$$x^* \in B_{i,j}(D_v) \Rightarrow x^* \in B_{i,j}(D_v(x^*, S)).$$

Therefore:

$$R_S(x^*, D_v) = \cap_{i,j\in S} B_{i,j}(D_v(x^*, S)) \ni x^*.$$

$x^* \in R_S(x^*, D_v)$ implies $x^* \in B_S(D_v)$. As the latter holds for any coalition of three agents, x^* belongs to the intersection of all $B_S(D_v)$.

Now we can show the inductive step. Let T be any coalition of k players. And let $x^* \in B_C(D_v)$ for any coalition C s.t. $|C| < k$. The following holds for any subcoalition of players $S \subset T$:

$$x^* \in B_S(D_v) \Rightarrow x^* \in B_S(D_v(x^*, T)). \qquad (45)$$

The LHS of (45) is true by assumption, therefore:

$$R_T(x^*, D_v) = \cap_{S\subset T} B_S(D_v(x^*, S)) \ni x^*. \qquad (46)$$

(46) holds for any T with $|T| = k$, and $x^* \in \cap_{C,|C|\leq k} B_C(D_v)$ follows. This concludes the proof of the inductive step, so we can conclude that:

$$x^* \in \cap_{S\subseteq\Omega} B_S(D_v). \qquad (47)$$

To complete the proof of the proposition we need to show that:

$$x^* \in \cap_{i,j\in\Omega} B_{i,j}(D_v) \Rightarrow x^* \in \cap_{i,j\in\Omega} B_{i,j}(D_v(x^*, P)) \;\; \forall P \in \mathcal{P}. \qquad (48)$$

Take any $i,j \in \Omega$ $i \neq j$. We need to distinguish between two cases. If there are two distinct elements of P, P_i and P_j, such that $i \in P_i \not\ni j$ and $j \in P_j \not\ni i$ then:

$$R_{i,j}(x^*, D_v(x^*, P)) = x^*. \qquad (49)$$

$x^* \in B_{i,j}(D_v(x^*, P))$ follows from the definition of $B_{ij}(\cdot)$.

Otherwise:

$$R_{i,j}(x^*, D_v) = R_{i,j}(x^*, D_v(x^*, P)). \qquad (50)$$

$x^* \in B_{i,j}(D_v(x^*, P))$ follows from $x^* \in B_{i,j}(D_v)$ and (50). This concludes the proof of the proposition. ■

5.1 CORE BISECTION PROPERTY AND THE KERNEL

Proposition 6 establishes that all the restrictions imposed by the update consistency requirement on ϕ reduce to (31). For an arbitrary uncertainty set D in the collection$\mathcal{D}$, the intersection $\cap_{i,j\in\Omega}B_{i,j}(D)$, might be empty. This explains Remark 2. However, for any uncertainty set D_v in the smaller collection $\mathcal{D}_{\mathcal{V}}$, we have that $\cap_{i,j\in\Omega}B_{i,j}(D_v) \neq \emptyset$. This result, due to Maschler, Peleg and Shapley (1979), was obtained while investigating the geometric properties of another solution concept for TU games known as the kernel. The kernel has been extensively studied in the literature and it has many interesting properties. For completeness of exposition, we will state the definition of kernel. For brevity of exposition though, rather than discussing the properties of the kernel and its justification as a solution concept, we will limit ourselves to recall the properties which will be of use for this paper.

5.1.1 KERNEL

Let (Ω, v) be a TU game. For $i, j \in \Omega$, $i \neq j$, $\mathcal{I}_{i,j}$ denotes the set of coalitions containing i but not j. $\mathcal{X}(\Omega, v)$ denotes the set of all individually rational and efficient allocations[14]. For an allocation x, and any coalition S, $e(S, x) = v(S) - \sum_{i\in S} x(i)$. The maximum surplus of i over j at x is defined as:

$$s_{i,j}(x) = \max_{S\in\mathcal{I}_{i,j}} e(S, x) \tag{51}$$

i is said to outweigh j at x if and only if:

$$s_{i,j}(x) > s_{j,i}(x) \quad \text{and} \quad x(j) > v(j). \tag{52}$$

Definition 14 (kernel) *The kernel (for the grand coalition) of* (Ω, v), $\mathcal{K}(\Omega, v)$ *is the set of all* $x \in \mathcal{X}(\Omega, v)$ *such that for no* $i, j \in \Omega$, i *outweighs* j *at* x.

Maschler, Peleg and Shapley (1979) establish that the intersection of the core and the kernel of a game (Ω, v) coincides with $\cap_{i,j\in\Omega}B_{i,j}(c(\Omega, v))$. Restated in terms of sets in $\mathcal{D}_{\mathcal{V}}$ this condition becomes:

$$\cap_{i,j\in\Omega}B_{i,j}(D_v) = \mathcal{K}(\Omega, v) \cap D_v \ \forall D_v \in \mathcal{D}_{\mathcal{V}}{}^{15}. \tag{53}$$

Moreover the set $\mathcal{K}(\Omega, v) \cap c(\Omega, v)$ is known to be non empty whenever the core is non empty (Maschler and Peleg (1966)). One last thing to notice is that the nucleolus is a point in the kernel and lies in the intersection of the core and the kernel.

(53) and (31) justify the following remark.

Remark 9 *Any selection mechanism on* $\mathcal{D}_{\mathcal{V}}$ *that is SI, LC and UC must satisfy the following condition:*

$$\phi(D_v) \in \mathcal{K}(\Omega, v) \cap c(\Omega, v) \ \ \forall D_v \in \mathcal{D}_{\mathcal{V}}.$$

[14] $\mathcal{X}(\Omega, v) = \left\{x \in \mathbf{R}^{|\Omega|} \,\middle|\, \sum_{i\in\Omega} x(i) = v(\Omega), x(i) \geq v(i) \ \ \forall i \in \Omega \right\}$

[15] Whenever $D_v = D_{\tilde{v}}$, $\mathcal{K}(\Omega, v) \cap D_v = \mathcal{K}(\Omega, \tilde{v}) \cap D_{\tilde{v}}$ (Maschler, Peleg and Shapley (1979)).

5.2 EXISTENCE

The kernel is not a single valued solution so it does not define a selection mechanism. However, Maschler, Peleg and Shapley (1972) show that, if v is convex than $K(\Omega, v)$ contains a single allocation. Moreover that allocation is the nucleolus of (Ω, v).

The class of convex capacities is not rich enough as it is not closed under updating as the capacity updating of a convex capacity might not be convex. We can simply extend the class including the capacities obtained from the updating of a convex capacity with respect to the nucleolus.

Denote the collection of all convex capacities on Ω as $\mathcal{V}_C$. We can then define the following collection of capacities $\overline{\mathcal{V}} \subset \mathcal{V}$

$$\overline{\mathcal{V}} = \{ \ v \in \mathcal{V} \,|v \in \mathcal{V}_C \quad \text{or} \quad \exists v' \in \mathcal{V}_C, P \in \mathcal{P} \quad \text{s.t.} \quad v = U\left(v', P, \eta\left(\Omega, v'\right)\right) \ \} \tag{54}$$

$\overline{\mathcal{V}}$ is the collection of all capacities that are either convex or are the result of the application of the capacity-updating rule to some convex capacity, with respect to the nucleolus.

Proposition 7 (existence) *There exists a selection mechanism ϕ, which is state independent, location-consistent and update-consistent on $\mathcal{D}_{\overline{\mathcal{V}}}$. Moreover, $\phi\left(D_{\overline{v}}\right) = \eta\left(\Omega, \overline{v}\right)$.*

Proof. For any convex capacity v,

$$\mathcal{K}\left(\Omega, v\right) = \eta\left(\Omega, v\right) = \cap_{i,j \in \Omega} B_{i,j}\left(D_v\right). \tag{55}$$

We need to show that, for all $P \in \mathcal{P}$:

$$\cap_{i,j \in \Omega} B_{i,j}\left(D_v\left(\eta\left(\Omega, v\right), P\right)\right) \ni \cap_{i,j \in \Omega} B_{i,j}\left(D_v\right) = \eta\left(\Omega, v\right). \tag{56}$$

Take any P and any pair of agents i, j. If there are two events in P such that $i \in P_i \ni j$ and $j \in P_j \ni i$, than for any $x \in D_v\left(\eta\left(\Omega, v\right), P\right)$,

$$R_{i,j}\left(x, D_v\left(\eta\left(\Omega, v\right), P\right)\right) = x. \tag{57}$$

(57) holds because any $x \in D_v\left(\eta\left(\Omega, v\right), P\right)$ must assign to P_i a value $\eta\left(P_i\right)$ and to P_j a value $\eta\left(P_j\right)$. It is not possible to increase the i'th component of x and just decrease the j'th component, without increasing the total value of P_i above $\eta\left(P_i\right)$. From (57) it follows that:

$$B_{i,j}\left(D_v\left(\eta\left(\Omega, v\right), P\right)\right) = D_v\left(\eta\left(\Omega, v\right), P\right). \tag{58}$$

For any other i, j:

$$R_{i,j}\left(x, D_v\left(\eta\left(\Omega, v\right), P\right)\right) = R_{i,j}\left(x, D_v\right),$$

and therefore:

$$B_{i,j}\left(D_v\left(\eta\left(\Omega, v\right), P\right)\right) = B_{i,j}\left(D_v\right). \tag{59}$$

(58), (59) and (55) prove the claim. ■

5.3 SOME PROPERTIES OF $\mathcal{D}_{\overline{\mathcal{V}}}$

Proposition 7 distinguishes a class of capacities for which the capacity updating rule always generates binding capacities. Are there characteristics of this class of capacities that help explain why that happens? The following example is one where the capacity-updating rule applied to a binding capacity generates a non-binding capacity. It provides some intuition of why this never happens with convex capacities.

Example 4 Let $\Omega = \{1, 2, 3, 4\}$ and v be the following:

$$\begin{array}{rcl} v(1) = v(2) & = & 1/8 \\ v(3) = v(4) & = & 0 \\ v(1,2) & = & 1/4 \\ v(1,3) = v(2,3) & = & 3/8 \\ v(1,4) = v(2,4) & = & 1/8 \\ v(3,4) & = & 1/4 \\ v(1,2,3) & = & 1/2 \\ v(1,2,4) & = & 1/4 \\ v(1,3,4) = v(2,3,4) & = & 3/8 \\ v(\Omega) = 1 & & \end{array}$$

It easy to check that v is binding even thought it is not convex[16]. Take the partition $P = \{(1,2), (3,4)\}$ and any allocation $\delta \in c(\Omega, v)$ such that $\delta(1 \cup 2) = 1/2$. The updated capacity $\tilde{v} = U(v, P, \delta)$ is reported below. It is easy to see that it is not binding

$$\begin{array}{rcl} \tilde{v}(1) = \tilde{v}(2) & = & 1/8 \\ \tilde{v}(3) = \tilde{v}(4) & = & 0 \\ \tilde{v}(1,2) = \tilde{v}(3,4) & = & 1/2 \\ \tilde{v}(1,3) = \tilde{v}(2,3) & = & 3/8 \\ \tilde{v}(1,4) = \tilde{v}(2,4) & = & 1/8 \\ \tilde{v}(1,2,3) = \tilde{v}(1,2,4) & = & 1/2 \\ \tilde{v}(1,3,4) = \tilde{v}(2,3,4) & = & 5/8 \end{array}$$

Once a value of $1/2$ has been assigned to the event $(1 \cup 2)$, the lower bounds on the probability of $(1 \cup 3)$ and $(2 \cup 3)$, constrain state 3 to a minimum chance of $1/8$. v shows some 'correlation' between the probabilities of states 1 and 3, in the sense that they cannot be both 'too small'. The same holds for states 2 and 3. Because of this correlation, the lowest bound for the probability of the event $(1, 2, 3)$, can be achieved only when the event $(1, 2)$ has a low probability $(1/4)$. Discovering that $(1, 2)$ has a $1/2$ probability, implies that the likelihood of the event $(1, 2, 3)$ must be at least $5/8$. Let us compare this last type of deduction with the standard deductions made by the capacity updating rule.

[16] $v(e_1 \cup e_2 \cup e_3) < v(e_1 \cup e_3) + v(e_2 \cup e_3) - v(e_3)$.

When the minimum probability of an event is updated by the capacity-updating rule, it is always because of two statements like the following type:

'$E \cup T$ has at least a k chance of occurring';

'T has exactly a l chance of occurring',

These two induce the agent to conclude that 'E has at least a $k - l$ chance of happening'. This kind of reasoning uses information that the agent already had ($v(E \cup T) \geq k$ together with new information (the objective probability of T). In contrast, the statement 'the chance of state 3 occurring is at least 1/8', does not come from any of these deductions. This is really further 'new' information. This new information is not picked by the capacity-updating rule. Therefore, as in the example above, it might fail to fully update the bounds on the probability of some events. Convex capacities show no 'correlation' structure of the type above. Therefore, discovering the objective probability of some event does not carry any information other than the probability of that event. The updates come from using this new piece of information together with what is already known. One can say that the only new information is the discovered objective probability of the specified event. Notice that changing $v(1, 2, 3)$ from $1/2$ to $3/4$ would make v a convex capacity. It is easy to check that the capacity-updating rule in such case, would generate a binding capacity.

Is the update-consistency property appealing when v is not convex? The motivation for this property was that, if the agent receives information that confirms her prediction, then she does not change her beliefs. But we have just shown an example where given that v is not convex, learning the objective probability of an event carries more information. If it is true that the agent's prior assignments on some events were correct, there is more information in the news. It is plausible that this additional information might induce the agent to form new beliefs.

The above discussion implies that, interestingly, $\mathcal{D}_{\overline{\mathcal{V}}}$ is exactly the class of uncertainty sets in which the update-consistency property has the most appeal. The fact that a selection mechanism with the three properties defined can be found on that class seems an interesting result.

6 APPENDIX

6.1 Proof of Proposition 2

Given any selection mechanism, ϕ, there is a natural way to define a simple assignment rule χ and from that a sequential mechanism ψ. For any event E, and for any $D \in \mathcal{D}$, define $\chi(D,E) = \{p \in D | p(E) = \phi(D)[E]\}$. It is obvious that if ϕ is update-consistent, then ψ is order-independent. Moreover, the update-consistency of ϕ implies $\psi(D,\theta) = \phi(D)$ for all D. So ψ is also state independent and location-consistent if ϕ is.

Suppose ψ is a sequential selection mechanism that is state independent, location-consistent and order-independent. Since it is order independent, ψ also defines a selection mechanism ϕ such that $\phi(D) = \psi(D,\cdot)$ (we can write ψ as function of D only). We need to show that ϕ is update-consistent.

Take any collection of events P and an ordering of 2^{Ω} with the events in P coming before any other event θ^P. Because of order independence and by definition of χ,

$$\chi\left(\chi\left(\chi\left(...\chi\left(D,\theta_1^P\right)...\right),\theta_{|P|-1}^P\right),\theta_{|P|}^P\right) = R(D,P,\phi(D)) \tag{60}$$

Again by order independence,

$$\psi(D,\cdot) = \psi(R(D,P,\phi(D)),\cdot) = \phi(R(D,P,\phi(D))). \tag{61}$$

This concludes the proof of the Proposition.■

6.2 Proof of Proposition 4

Given $v \in \mathcal{V}$, $P \in \mathcal{P}$ and any $\mu \in c(\Omega,v)$, let v_n denote a capacity on Ω such that:

$$v_n(T) = \begin{cases} \mu(T) & \text{if} \quad T \in \mathcal{B}(P) \\ 0 & \text{otherwise} \end{cases} \tag{62}$$

Now define another capacity on Ω, v_m, such that:

$$v_m(E) = \max\{v(E), v_n(E)\} \quad \text{for all} \quad E \subseteq \Omega. \tag{63}$$

It is obvious that $c(\Omega,v_m) = \{p \in c(\Omega,v) | p(B) = \mu(B) \quad \text{for all} \quad B \in \mathcal{B}(P)\}$. Then, to prove the claim of the proposition it will suffice to show that

$$c(\Omega,v_m) = c(\Omega,\tilde{v}), \tag{64}$$

where $\tilde{v} = U(v,P,\mu)$. We will first that any allocation in the core of (Ω,v_m) is also in the core of $(\Omega,\tilde{v})$.

Suppose $x \in c(\Omega,v_m)$. First notice that for any $B \in \mathcal{B}(P)$,

$$v_m(B) = \tilde{v}(B) = \left(1 - v_m\left(B^C\right)\right) = \left(1 - \tilde{v}\left(B^C\right)\right).$$

Therefore any allocation x in $c(\Omega, v_m)$ and/or in $c(\Omega, \tilde{v})$ must be such that $x(B) = \mu(B)$ for all $B \in \mathcal{B}(P)$.

Take now any $E \subset \Omega$ such that $E \notin \mathcal{B}(P)$.

If $\tilde{v}(E) = \max_{T \in \mathcal{B}_{-E} \cup \emptyset} [v(E \cup T) - \mu(T)]$ then, define:

$$T(E) = \arg \max_{T \in \mathcal{B}_{-E} \cup \emptyset} [v(E \cup T) - \mu(T)]. \tag{65}$$

By construction,

$$v_m(E \cup T(E)) = v(E \cup T(E)) = \tilde{v}(E \cup T(E)) \tag{66}$$

and,

$$\tilde{v}(E) = v(E \cup T(E)) - \mu(T(E)) = v(E \cup T(E)) - x(T(E)). \tag{67}$$

Since x is in $c(\Omega, v_m)$, it follows that:

$$\begin{aligned} x(E \cup T(E)) &\geq v_m(E \cup T(E)) \\ x(E \cup T(E)) &\geq \tilde{v}(E \cup T(E)) \\ x(E) + x(T(E)) &\geq \tilde{v}(E) + \tilde{v}(T(E)) \\ x(E) &\geq \tilde{v}(E) \end{aligned}$$

Suppose otherwise that $\tilde{v}(E) \neq \max_{T \in \mathcal{B}_{-E} \cup \emptyset} [v(E \cup T) - \mu(T)]$, then, one can find a collection of events $\{E_1, ..., E_k\}$ that partition E such that:

$$\tilde{v}(E) = \sum \tilde{v}(E_i), \tag{68}$$

and

$$\tilde{v}(E_i) = \max_{T \in \mathcal{B}_{-E_i} \cup \emptyset} [v(E_i \cup T) - \mu(T)]. \tag{69}$$

Using the same argument as above it is trivial to see that:

$$x \in c(N, v_m) \Rightarrow x(E) \geq \tilde{v}(E). \tag{70}$$

This concludes the proof that $c(N, v_m) \subseteq c(N, \tilde{v})$. The proof that $c(N, \tilde{v}) \subseteq c(N, v_m)$ is trivial once one notice that:

$$\tilde{v}(E) = v_m(E) \quad \text{for all} \quad E \in \mathcal{B}(P) \tag{71}$$

and

$$\tilde{v}(E) \geq v(E) = v_m(E) \quad \text{for all} \quad E \notin \mathcal{B}(P) \tag{72}$$

This concludes the proof of the proposition.■

6.3 Proof of Proposition 5

Let (Ω, v) be a balanced TU game, $P \in \mathcal{P}$ and $\eta^* = \eta(\Omega, v)$. Construct a new game $(\Omega, \tilde{v})$, where $\tilde{v}$ is the updated capacity, $\tilde{v} = U(v, P, \eta^*)$.

By definition of U, the following two conditions must hold for any $B \in \mathcal{B}(P)$:

$$\tilde{v}(B) + \tilde{v}\left(B^C\right) = \tilde{v}(\Omega) = v(\Omega) \tag{73}$$

$$\tilde{v}(B) = \eta^*(B) \quad \text{for all} \quad B \in \mathcal{B}(P) \tag{74}$$

Since v is balanced we know that $\eta^* \in c(\Omega, v)$.

Now define the following:

$$\mathcal{X} \equiv \left\{x \in \mathbf{R}^{|\Omega|} : x(B) = \eta^*(B) \quad \text{for all} \quad B \in \mathcal{B}(P)\right\}; \tag{75}$$

$\mathcal{X}$ is the collection of all allocations for Ω, which agree with the nucleolus of (Ω, v) for any $B \in \mathcal{B}(P)$. (73) and (74), Proposition 4 and the fact that $\eta^* \in c(\Omega, \tilde{v})$ imply:

$$\mathcal{X} \supseteq c(\Omega, \tilde{v}) \ni \eta(N, \tilde{v}) = \tilde{\eta}. \tag{76}$$

Since the nucleolus of $(\Omega, \tilde{v})$ must belong to $\mathcal{X}$, we can ignore any allocation outside $\mathcal{X}$.

For any proper subcoalition S, $\tilde{e}(S, x)$ will denote the excess function under the game $(\Omega, \tilde{v})$ for the coalition S at the allocation x; that is $\tilde{e}(S, x) = x(S) - \tilde{v}(S)$. For any $B \in \mathcal{B}(P)$, $\tilde{v}(B) = \eta^*(B)$ and, obviously, $\tilde{e}(B, x) = 0$ for all $x \in \mathcal{X}$. For all other subsets of Ω that are not in $\mathcal{B}(P)$ we can distinguish two cases.

Case 1.
$S \subset \Omega$ s.t. $B \in \mathcal{B}_S \Rightarrow B \supseteq S$

This is the case where for any $B \in \mathcal{B}(P)$, (that is, for any event of objectively known probability) it is either $B \supseteq S$, or $S \cap B = \emptyset$. Let T^* solve:

$$\max_{T \in \mathcal{B}_{-S} \cup \emptyset} \{v(S \cup T) - \eta^*(T)\}. \tag{77}$$

By definition, it follows that

$$\tilde{v}(S) = v(S \cup T^*) - \eta^*(T^*). \tag{78}$$

At any efficient allocation x, I

$$\begin{aligned}
\tilde{e}(S, x) &= x(S) - \tilde{v}(S) \\
&= x(S) - (v(S \cup T^*) - \eta^*(T^*)) \\
&= x(S \cup T^*) - v(S \cup T^*) + (\eta^*(T^*) - x(T^*)) \\
&= e(S \cup T^*) + (\eta^*(T^*) - x(T^*)).
\end{aligned}$$

Thus, by definition of $\mathcal{X}$, for any $x \in \mathcal{X}$:

$$\tilde{e}(S, x) = e(S \cup T^*). \tag{79}$$

Case 2.
$S \subset \Omega$ and $\exists B \in \mathcal{B}(P)$ s.t. $S \supset S \cap B \neq \emptyset$

Similarly to the case above, let T^* solve:

$$\max_{T \in \mathcal{B}_{-S} \cup \emptyset} \{v(S \cup T) - X(T)\}. \tag{80}$$

By definition of capacity-updating rule one can find a collection of events $\{S_1, .., S_k\}$, which partitions S, such that, for all i, S_i is a subset of Ω not contained in the Case 2 category and

$$\begin{aligned} \tilde{v}(S) &= \max\left\{v(S \cup T^*) - \eta^*(T^*), \textstyle\sum_{i \leq k} \tilde{v}(S_i)\right\} \\ &= \max\left\{v(S \cup T^*) - \eta^*(T^*), \textstyle\sum_{i \leq k} \left[v\left(S_i \cup T^*_{S_i}\right) - \eta^*\left(T^*_{S_i}\right)\right]\right\} \end{aligned}$$

Therefore, for any $x \in \mathcal{X}$,

$$\tilde{e}(S, x) = \min\left\{e(S \cup T^*), \sum e\left(S_i \cup T^*_{S_i}\right)\right\}. \tag{81}$$

Equations (81) and (79) imply that for any $S \notin \mathcal{B}(P)$, one can find a collection of subsets of Ω, $\mathcal{C}$, such that, for all $x \in \mathcal{X}$,

$$\tilde{e}(S, x) = \sum_{C \in \mathcal{C}} e(C, x). \tag{82}$$

Obviously, for any event $B \in \mathcal{B}(P)$, $\tilde{e}(B, x) = 0$ for all $x \in \mathcal{X}$.

Let $\tilde{\eta}$ be the nucleolus of $(\Omega, \tilde{v})$ and suppose that $\eta^* \neq \tilde{\eta}$. We will now proceed to prove that the contrahypothesis leads to a contradiction.

Let $\left\{A_1, A_2, ..., A_{|2^\Omega|}\right\}$ be the collection of all elements of 2^Ω in increasing order with respect to $\tilde{e}(\cdot, \eta^*)$[17]. Let k be an integer such that $\tilde{e}(A_i, \eta^*) = \tilde{e}(A_i, \tilde{\eta})$ for all $i \leq k$ and $\tilde{e}(A_{k+1}, \eta^*) < \tilde{e}(A_{k+1}, \tilde{\eta})$. It easy to see that such an integer always exists and is greater than 1. First, recall that $\tilde{\eta}$ must be in $\mathcal{X}$ and for all $B \in \mathcal{B}(P)$

$$0 = \tilde{e}(B, \tilde{\eta}) = \tilde{e}(B, \eta^*).$$

[17] $\tilde{e}(A_i, \eta^*) \leq \tilde{e}(A_j, \eta^*)$ for any $i < j$.

Moreover, if $\tilde{e}(A_i, \eta^*) = \tilde{e}(A_i, \tilde{\eta})$ for $i = 1, .., j$, there cannot be an $E \subset \Omega$, $E \notin \{A_1, ..., A_j\}$, such that $\tilde{e}(E, \tilde{\eta}) < \tilde{e}(A_j, \eta^*)$, because it would contradict the hypothesis that $\tilde{\eta}$ is the nucleolus of $(\Omega, \tilde{v})$. Therefore there cannot be a k such that $\tilde{e}(A_k, \eta^*) > \tilde{e}(A_k, \tilde{\eta})$ and $\tilde{e}(A_i, \eta^*) = \tilde{e}(A_i, \tilde{\eta})$ for all $i < k$.

By uniqueness of the nucleolus and the hypothesis that $\tilde{\eta} \neq \eta^*$, it cannot be the case that $\tilde{e}(A_i, \eta^*) = \tilde{e}(A_i, \tilde{\eta})$ for all i. Therefore there must exist some k that respects the hypothesis above.

By construction it must also be that

$$\tilde{e}(A_l, \eta^*) > \tilde{e}(A_{k+1}, \eta^*) \text{ for any } l > k+1 \tag{83}$$

$$\tilde{e}(A_l, \tilde{\eta}) > \tilde{e}(A_{k+1}, \eta^*) \text{ for any } l > k+1 \tag{84}$$

(83) holds because of the ordering, (84) holds because $\tilde{\eta}$ is the nucleolus of $(\Omega, \tilde{v})$. We know that for any A_i, it is either $0 = \tilde{e}(A_i, \tilde{\eta}) = \tilde{e}(A_i, \eta^*)$, or one can find a collection of coalitions $\mathcal{C}$ for which $\tilde{e}(A_i, x) = \sum_{C \in \mathcal{C}} e(C, x)$ for any $x \in \mathcal{X}$.

Let $\mathcal{G} = \{G_1, .., G_J\}$ be the collection of all sub-coalitions of Ω for which there is at least an $i \leq k$ such that:

$$\begin{array}{lllll} \tilde{e}(A_i, \tilde{\eta}) & = \tilde{e}(A_i, \eta^*) & = & e(G_j, \tilde{\eta}) & = e(G_j, \eta^*) \quad \textit{and,} \\ \tilde{e}(A_{k+1}, \tilde{\eta}) & = e(G_J, \tilde{\eta}) & > & \tilde{e}(A_{k+1}, \eta^*) & = e(G_J, \eta^*) \end{array}$$

By construction, on $\mathcal{G}$ the lexmin ordering prefers $e(\cdot, \tilde{\eta})$ to $e(\cdot, \eta^*)$. For any $S \subset \Omega$ such that $S \notin \mathcal{G}$, if $e(S, \eta^*) \geq e(S, \eta^*)$, then adding S to $\mathcal{G}$, the lexmin ordering still prefers $e(\cdot, \tilde{\eta})$ to $e(\cdot, \eta^*)$.

If $e(S, \eta^*) < e(G_J, \eta^*)$, then

$$\tilde{e}(S, \eta^*) \leq e(S, \eta^*) < e(G_J, \eta^*) = \tilde{e}(A_J, \eta^*). \tag{85}$$

(85) implies that $S \in \mathcal{A}$ and, therefore, $\tilde{e}(S, \eta^*) = \tilde{e}(S, \tilde{\eta})$. Consequently, also $e(S, \eta^*) = e(S, \tilde{\eta})$ and again, adding S to $\mathcal{G}$, the lexmin ordering still prefers $e(\cdot, \tilde{\eta})$ to $e(\cdot, \eta^*)$. We can then conclude that $e(\cdot, \tilde{\eta})$ is preferred to $e(\cdot, \eta^*)$ by the lexmin ordering on 2^Ω, a contradiction since η^* is the nucleolus of (Ω, v). ■

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