CHAPTER 8: INDEX MODELS AND THE ARBITRAGE PRICING THEORY

1. a. To optimize this portfolio one would need:

\[
\begin{align*}
  n & = 60 \text{ estimates of means} \\
  n & = 60 \text{ estimates of variances} \\
  \frac{n^2 - n}{2} & = 1770 \text{ estimates of covariances} \\
  \frac{n^2 + 3n}{2} & = 1890 \text{ estimates}
\end{align*}
\]

b. In a single index model: 

\[
 r_i - r_f = \alpha_i + \beta_i (r_M - r_f) + e_i
\]

the variance of the rate of return on each stock can be decomposed into the components:

1. \( \beta_i^2 \sigma^2_M \) The variance due to the common market factor
2. \( \sigma^2(e_i) \) The variance due to firm specific unanticipated events

In this model \( \text{Cov}(r_i, r_j) = \beta_i \beta_j \sigma^2_M \). The number of parameter estimates required would be:

- n = 60 estimates of the mean \( E(r_i) \),
- n = 60 estimates of the sensitivity coefficient \( \beta_i \),
- n = 60 estimates of the firm-specific variance \( \sigma^2(e_i) \), and
- 1 estimate of the market mean \( E(r_M) \)
- 1 estimate for the market variance \( \sigma^2_M \)

Thus, the single index model reduces the total number of required parameter estimates from 1,890 to 182, and in general from \((n^2 + 3n)/2\) to \(3n + 2\).

2. a. The standard deviation of each individual stock is given by:

\[
\sigma_i = \left[ \beta_i^2 \sigma^2_M + \sigma^2(e_i) \right]^{1/2}
\]
Since $\beta_A = .8$, $\beta_B = 1.2$, $\sigma(e_A) = 30\%$, $\sigma(e_B) = 40\%$, and $\sigma_M = 22\%$ we get:

$$\sigma_A = (.8^2 \times 22^2 + 30^2)^{1/2} = 34.78\%$$

$$\sigma_B = (1.2^2 \times 22^2 + 40^2)^{1/2} = 47.93\%$$

b. The expected rate of return on a portfolio is the weighted average of the expected returns of the individual securities:

$$E(r_p) = w_A E(r_A) + w_B E(r_B) + w_f r_f$$

where $w_A$, $w_B$, and $w_f$ are the portfolio weights of stock A, stock B, and T-bills, respectively.

Substituting in the formula we get:

$$E(r_p) = .30 \times 13 + .45 \times 18 + .25 \times 8 = 14\%$$

The beta of a portfolio is similarly a weighted average of the betas of the individual securities:

$$\beta_p = w_A \beta_A + w_B \beta_B + w_f \beta_f$$

The beta of T-bills ($\beta_f$) is zero. The beta of the portfolio is therefore:

$$\beta_p = .30 \times .8 + .45 \times 1.2 + 0 = .78$$

The variance of this portfolio is:

$$\sigma^2_p = \beta_p^2 \sigma^2_M + \sigma^2(e_p)$$

where $\beta_p^2 \sigma^2_M$ is the systematic component and $\sigma^2(e_p)$ is the nonsystematic component. Since the residuals, $e_i$ are uncorrelated, the non-systematic variance is:

$$\sigma^2(e_p) = w_A^2 \sigma^2(e_A) + w_B^2 \sigma^2(e_B) + w_f^2 \sigma^2(e_f)$$

$$= .30^2 \times 30^2 + .45^2 \times 40^2 + .25^2 \times 0 = 405$$

where $\sigma^2(e_A)$ and $\sigma^2(e_B)$ are the firm-specific (nonsystematic) variances of stocks A and B, and $\sigma^2(e_f)$, the nonsystematic variance of T-bills, is zero. The residual standard deviation of the portfolio is thus:
\[ \sigma(c_p) = (405)^{1/2} = 20.12\% \]

The total variance of the portfolio is then:

\[ \sigma^2_p = 0.78^2 \times 22^2 + 405 = 699.47 \]

and the standard deviation is 26.45%.

5. The standard deviation of each stock can be derived from the following equation for \( R^2 \):

\[ R^2_i = \frac{\beta_i^2 \sigma^2_M}{\sigma^2_i} = \frac{\text{Explained variance}}{\text{Total variance}} \]

Therefore,

\[ \sigma^2_A = \frac{\beta_A^2 \sigma^2_M}{R_A^2} = \frac{0.70^2 \times 20^2}{0.20} = 980 \]

\[ \sigma_A = 31.30\% \]

For stock B

\[ \sigma^2_B = \frac{1.22^2 \times 20^2}{0.12} = 4800 \]

\[ \sigma_B = 69.28\% \]

6. The systematic risk for A is

\[ \beta_A^2 \sigma^2_M = 0.70^2 \times 20^2 = 196 \]

and the firm-specific risk of A (the residual variance) is the difference between A's total risk and its systematic risk,

\[ 980 - 196 = 784 \]

B's systematic risk is:

\[ \beta_B^2 \sigma^2_M = 1.22^2 \times 20^2 = 576 \]
and B's firm-specific risk (residual variance) is:

\[ 4800 - 576 = 4224 \]

7. The covariance between the returns of A and B is (since the residuals are assumed to be uncorrelated):

\[ \text{Cov}(r_A, r_B) = \beta_A \beta_B \sigma_M^2 = .70 \times 1.2 \times 400 = 336 \]

The correlation coefficient between the returns of A and B is:

\[ \rho_{AB} = \frac{\text{Cov}(r_A, r_B)}{\sigma_A \sigma_B} = \frac{336}{31.30 \times 69.28} = .155 \]

8. Note that the correlation coefficient is the square root of \( R^2 \): \( \rho = \sqrt{R^2} \)

\[ \text{Cov}(r_A, r_M) = \rho \sigma_A \sigma_M = .20^{1/2} \times 31.30 \times 20 = 280 \]

\[ \text{Cov}(r_B, r_M) = \rho \sigma_B \sigma_M = .12^{1/2} \times 69.28 \times 20 = 480 \]

9. The non-zero alphas from the regressions are inconsistent with the CAPM. The question is whether the alpha estimates reflect sampling errors or real mispricing. To test the hypothesis of whether the intercepts (3% for A, and –2% for B) are significantly different from zero, we would need to compute t-values for each intercept.

17. As a first step, convert the scenario rates of return to dollar payoffs per share, as shown in the following table:

<table>
<thead>
<tr>
<th>Scenarios</th>
<th>Price</th>
<th>Scenario 1</th>
<th>Scenario 2</th>
<th>Scenario 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$10</td>
<td>10(1 – .15) = $8.50</td>
<td>10(1 + .20) = $12.00</td>
<td>10(1 + .30) = $13.00</td>
</tr>
<tr>
<td>B</td>
<td>$15</td>
<td>15(1 + .25) = $18.75</td>
<td>15(1 + .10) = $16.50</td>
<td>15(1 – .10) = $13.50</td>
</tr>
<tr>
<td>C</td>
<td>$50</td>
<td>50(1 + .12) = $56.00</td>
<td>50(1 + .15) = $57.50</td>
<td>50(1 + .12) = $56.00</td>
</tr>
</tbody>
</table>

Identifying an arbitrage opportunity always involves constructing a zero investment portfolio. This portfolio must show non-negative payoffs in all scenarios.
For example, the proceeds from selling short two shares of A and two shares of B will be sufficient to buy one share of C.

\((-2)10 + (-2)15 + 50 = 0\)

The payoff table for this zero investment portfolio in each scenario is:

<table>
<thead>
<tr>
<th>Price</th>
<th># of shares</th>
<th>Investment</th>
<th>Scenarios</th>
</tr>
</thead>
<tbody>
<tr>
<td>A $10</td>
<td>-2</td>
<td>-20</td>
<td>1 -17 -24 -26</td>
</tr>
<tr>
<td>B $15</td>
<td>-2</td>
<td>-30</td>
<td>2 -37.5 -33 -27</td>
</tr>
<tr>
<td>C $50</td>
<td>+1</td>
<td>50</td>
<td>3 56 57.5 56</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0</td>
<td>+1.5</td>
<td>+.5</td>
<td>+3</td>
</tr>
</tbody>
</table>

This portfolio qualifies as an arbitrage portfolio because it is both a zero investment portfolio and has positive returns in all scenarios.

18. Substituting the portfolio return and the betas in the expected return-beta relationship, we obtain two equations in the unknowns, the risk-free rate and the factor risk premium, RP.

\[12 = r_f + 1.2 \times RP\]
\[9 = r_f + 0.8 \times RP\]

Solving these equations, we obtain

\[r_f = 3\% \text{ and } RP = 7.5\%\]

19. a. Shorting equally the 10 negative-alpha stocks and investing the proceeds equally in the 10 positive-alpha stocks eliminates the market exposure and creates a zero-investment portfolio. Denoting the systematic market factor as \(R_M\), the expected dollar return is (noting that the expectation of non-systematic risk, \(e\), is zero):

\[1,000,000 \times [.03 + 1.0 \times R_M] - 1,000,000 \times [-.03 + 1.0 \times R_M] = 1,000,000 \times .06 = 60,000\]

The sensitivity of the payoff of this portfolio to the market factor is zero because the exposures of the positive alpha and negative alpha stocks cancel out. (Notice that the terms involving \(R_M\) sum to zero.) Thus, the systematic component of total risk also is zero. The variance of the analyst's profit is not zero, however, since this portfolio is not well diversified.
For $n = 20$ stocks (i.e., long 10 stocks and short 10 stocks) the investor will have a $100,000 position (either long or short) in each stock. Net market exposure is zero, but firm-specific risk has not been fully diversified. The variance of dollar returns from the positions in the 20 firms is

$$20 \times [(100,000 \times .30)^2] = 18,000,000,000$$

and the standard deviation of dollar returns is $134,164.$

b. If $n = 50$ stocks (25 long and 25 short), $40,000 is placed in each position, and the variance of dollar returns is

$$50 \times [(40,000 \times .30)^2] = 7,200,000,000$$

The standard deviation of dollar returns is $84,853.$

Similarly, if $n = 100$ stocks (50 long and 50 short), $20,000 is placed in each position, and the variance of dollar returns is

$$100 \times [(20,000 \times .30)^2] = 3,600,000,000$$

The standard deviation of dollar returns is $60,000.$

Notice that when the number of stocks increased by a factor of 5, from 20 to 100, standard deviation fell by a factor of $\sqrt{5} = 2.236$, from $134,164$ to $60,000.$

20 a. $\sigma^2 = \beta^2 \sigma^2_M + \sigma^2(e)$

The standard deviations are:

<table>
<thead>
<tr>
<th></th>
<th>$\sigma(e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>25</td>
</tr>
<tr>
<td>B</td>
<td>10</td>
</tr>
<tr>
<td>C</td>
<td>20</td>
</tr>
</tbody>
</table>

and $\sigma_M = 20$. Thus,

$$\sigma_A^2 = .8^2 \times 20^2 + 25^2 = 881$$
$$\sigma_B^2 = 1.0^2 \times 20^2 + 10^2 = 500$$
$$\sigma_C^2 = 1.2^2 \times 20^2 + 20^2 = 976$$

b. If there are an infinite number of assets with identical characteristics, a well-diversified portfolio of each type will have only systematic risk since the non-systematic risk will approach zero with large $n$. The mean will equal that of the individual (identical) stocks.
c. There is no arbitrage opportunity because the well-diversified portfolios all plot on the security market line (SML). Because they are fairly priced, there is no arbitrage.