State Preference Theory, Pure Securities, Hedge Portfolios, & Risk Neutral Pricing

This Supplement presents some concepts that are both useful and preliminary to understanding option pricing theory. Some, but not all of these concepts, appear in Chapter 4 of our textbook that deals with what is known as State Preference Theory.

State preference Theory was created in the late 1950's and early 1960's by Kenneth Arrow and Gerard Debreu. (Both later won Nobel Prizes for their efforts). It begins with the theory of consumer choice under complete certainty, then generalizes it to allow for uncertainty.

Start with a world of perfect certainty in which there are any number of households and any number of time periods. I will simplify the analysis by restricting consideration to only two households (A and B) and only two time periods (the present, period 0, and the future, period 1.) The lifetime utility functions of the two households may differ and are described by the following:

\[ U_A = u_A(c_0^A) + \beta_A u_A(c_1^A) \]
\[ U_B = u_B(c_0^B) + \beta_B u_B(c_1^B) \]

where \( c_t^A \) denotes consumption of household A during time period \( t \) and \( \beta_A \) is a time preference parameter for household A, etc.

Each household begins with given endowments of the consumption goods. Household A begins with \( Y_0^A \) in the current time period and \( Y_1^A \) in the future. Similarly, household B begins with \( Y_0^B \) and \( Y_1^B \).

The households are assumed to exchange a risk-free asset. Think of this as a real discount bond that pays 1 unit of the consumption good during time period 1. The time period 0 price of this asset is \( \frac{1}{1+r} \), where \( r \) is the real rate of interest.

Let \( Q_A \) denote the quantity of this asset purchase by household A during time period 0. (Note that \( Q_A \) can be negative, which means that household A sells this asset). Similarly, let \( Q_B \) denote the quantity of this asset purchased by household B during time period 0.

Taking the value of \( r \) as a given, household A chooses \( Q_A \) so as to maximize \( U_A \), subject to the budget constraints

\[ c_0^A = Y_0^A - \frac{Q_A}{1+r} \text{ and } c_1^A = Y_1^A + Q_A. \]

Household B chooses \( Q_B \) so as to maximize \( U_B \), subject to corresponding budget constraints.

Now we add the following market clearing restriction:

\[ Q_A + Q_B = 0. \]
We now have a system of 7 equations consisting of 2 budget constraints for each household, 2 First Order Conditions, and the market-clearing restriction. These 7 equations uniquely determine equilibrium values for the following 7 variables: $c_0^A, c_1^A, c_0^B, c_1^B, Q_A, Q_B$, and $r$. The equilibrium is Pareto Optimal in the sense that no other allocation can make one household better off without making the other worse off.

A numerical example can help illustrate this. Suppose that both households have logarithmic utility functions and $\beta_A = \beta_B = 1$. Then

$$U_A = \ln(c_0^A) + \ln(c_1^A) \quad \text{and} \quad U_B = \ln(c_0^B) + \ln(c_1^B).$$

Suppose also that the endowments are

$$Y_0^A = 75 \quad Y_1^A = 100 \quad Y_0^B = 80 \quad Y_1^B = 60.$$

Substituting the budget constraints into the utility functions we have

$$U_A = \ln(75 - \frac{Q_A}{1 + r}) + \ln(100 + Q_A) \quad \text{and} \quad U_B = \ln(80 - \frac{Q_B}{1 + r}) + \ln(60 + Q_B).$$

The student can readily verify that the FOC's are:

$$Q_A = \frac{1}{2} (-25 + 75r) \quad \text{and} \quad Q_B = \frac{1}{2} (20 + 80r)$$

Applying the market clearing restriction that $Q_A + Q_B = 0$, yields the following equilibrium values:

$$r = 0.03226 \quad Q_A = -11.29 \quad Q_B = +11.29.$$

It is left to the student to compute equilibrium values for the consumption variables.

While we have examined this for only two households and two time periods, the analysis generalizes to any number of households and any number of time periods. Pareto Optimality obtains, provided only that there is one asset (a one-period bond) that permits households to exchange consumption goods across consecutive time periods.

State Preference Theory expands this setup to permit uncertainty regarding the future. A useful way of representing uncertainty about the future is to say that the present can evolve into any one of a number of possible future "states of the world". To keep things simple at this stage, we will expand the previous 2-period representation to allow for only 2 possible future states of the world. With probability $\pi_1$ we will end up in State 1 in time period 1; with probability $\pi_2$ we will end up in State 2. ($\pi_1 + \pi_2 = 1$).

Modify the notation of certainty setup in the following way:
Let \( c_1^A \) and \( c_1^B \) now denote consumptions of the 2 households in time period 1 if State 1 occurs. Let \( c_2^A \) and \( c_2^B \) denote consumptions of the 2 households in time period 1 if State 2 occurs. Let \( Y_1^A \) and \( Y_1^B \) now denote endowments of the 2 households in time period 1 if State 1 occurs. Finally, let \( Y_1^A \) and \( Y_1^B \) denote endowments of the 2 households in time period 1 if State 2 occurs.

We will for now disregard the risk-free asset and introduce a different kind of security, which is known as a pure security. (It is also sometimes called an "Arrow-Debreu" security or a "primitive security"). A pure security is a security that pays 1 unit of the consumption good if a particular state of nature occurs and pays 0 if any other state occurs. [In contrast, a complex security is defined as a security that has a non-zero payout in at least 2 states]. In our simple setup there are only 2 possible states of nature, so we must consider only 2 pure securities. Let \( p_1 \) denote the time period 0 price of the pure security associated with State 1. That is, \( p_1 \) is the current price of a security that will pay 1 unit of consumption in time period 1 if State 1 occurs and 0 in time period 1 if State 2 occurs. Similarly, let \( p_2 \) denote the time period 0 price of the pure security associated with State 2.

Now we can formulate the choice problems faced by each of our households. Let \( Q_1^A \) and \( Q_2^A \) denote the quantities of the 2 pure securities purchased during time period 0 by household \( A \). (Either of these quantities may be positive or negative). Taking \( p_1 \) and \( p_2 \) as givens, household \( A \) will choose \( Q_1^A \) and \( Q_2^A \) so as to maximize

\[
E[U_A] = u_A(c_0^A) + \pi_1 \beta_A u_A(c_1^A) + \pi_2 \beta_A u_A(c_2^A),
\]

subject to

\[
c_0^A = Y_0^A - p_1 Q_1^A - p_2 Q_2^A, \quad c_1^A = Y_1^A + Q_1^A, \quad c_2^A = Y_2^A + Q_2^A.
\]

Household \( B \) will solve a similar problem.

Then market clearing requires the following to be satisfied

\[
Q_1^A + Q_1^B = 0 \quad \text{and} \quad Q_2^A + Q_2^B = 0.
\]

Here there are 12 equations, consisting of 3 budget constraints and 2 First Order Equations for each household, plus the 2 market-clearing restrictions. The 12 equations uniquely determine equilibrium values for the 6 consumption variables, the 4 \( Q \) variables and \( p_1 \) and \( p_2 \). The equilibrium is Pareto Optimal.

This analysis generalizes to any number of households and any number of states of the world that can occur in each future time period. The equilibrium that obtains in the generalized model is Pareto Optimal, provided that there are as many "distinct" securities as there are future states of the world. A security is "distinct" if its payoffs in the various states are linearly independent of the payoffs of other securities. That is, its payoffs cannot be replicated by a linear combination of the payoffs of other securities. When the number of "distinct" securities is equal to the number of future states, the model is said to
have a *complete security market*. The security market is always *complete* when there exists one *pure security* corresponding to each future state. (Because the payoffs of any *pure security* are always linearly independent of the payoffs of other *pure securities*). However, it is not necessary that *pure securities* exist and be traded for the security market to be *complete*. If there are $K$ states and there exist $K$ linearly independent complex securities that are traded, then the security market will be *complete*. When there are $K$ states and fewer than $K$ linearly independent, traded securities, the security market is said to be *incomplete*. In an *incomplete* market there are not enough distinct securities to permit households to fully engage in risk sharing and Pareto Optimality does not obtain. The welfare of households in an *incomplete* security market can always be increased by introducing one or more new securities with payoffs that are linearly independent of those of existing securities. To state this in a different way: when the security market is *incomplete* there do not exist enough securities to permit economic agents to fully eliminate diversifiable risks. This point is demonstrated in the following (rather lengthy) numerical example.

**An Illustration of the Welfare Benefits of Complete Security Markets**

Consider a 2 time period, 2-state world in which there are two types of households that differ only in regard to their endowments in the different states of the world. "Judy" will be a representative household of the first type and "Tom" will be a representative household of the second type. The following table describes their respective endowments.

<table>
<thead>
<tr>
<th>Current Time Period 0</th>
<th>Future Time Period 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>State 1</td>
</tr>
<tr>
<td>Judy's Endowments</td>
<td>$10</td>
</tr>
<tr>
<td>Tom's Endowments</td>
<td>$10</td>
</tr>
</tbody>
</table>

Probability State 1 occurs = probability State 2 occurs = 1/2.

Without loss of generality we will make the following assumptions.

1. The price of a consumption good is $1 in both time period 0 and time period 1, no matter which state occurs in the latter time period. Thus, though they are denominated in dollars, the endowments are equivalent to units of consumption.

2. Both types of agent have the same preferences represented by the lifetime utility function

$$U = \ln (C_0) + \ln (C_1).$$

This kind of utility function exhibits both risk aversion and an implicit benefit from consumption smoothing. That is, maximum lifetime utility occurs when $C_0 = C_1$. It is clear from their respective endowments that the risks that Tom and Judy individually face
are fully diversifiable. That is, these individual risks could be totally eliminated if Judy would transfer $5 to Tom in the event State 1 occurs and Tom would transfer $5 to Judy if State 2 occurs. A complete security market would permit Judy and Tom to make such transfers. An incomplete security market will not permit total diversification to occur – as the following will demonstrate.

Suppose that there is no securities market at all, in which case the security market is clearly incomplete. In this case each agent will have no choice but to consume exactly her/his respective endowment in each time period -- a situation termed "autarky" in the jargon of Economists. Judy will consume 10 in time period 0 and either 15 or 5 in time period 1, depending upon which state occurs. Tom will consume 10 in time period 0 and either 5 or 15 in time period 1, depending on the state of the world. Under autarky each agent has an expected lifetime utility of

$$E[U] = \ln(10) + 0.5 \ln(15) + 0.5 \ln(5) = 4.4613.$$ 

Now let us introduce a securities market, but one that has only 1 security and is, therefore, incomplete in this 2-state world. For convenience I will let that security be a pure security that pays $1 if state 1 occurs and 0 if state 2 occurs. Let \( p_1 \) denote the price of this pure security during time period 0 and let \( q_J \) and \( q_T \) denote the quantities of this security purchased in time period 0 by Judy and Tom, respectively. (If one of the households sells the security, its \( q \) will be negative.) Each household will choose the value of \( q \) that maximizes the expected value of its lifetime utility. The choice problem faced by Judy is as follows.

Maximize \( E[U] = \ln(10 - p_1 q_J) + 0.5 \ln(15 + q_J) + 0.5 \ln(5) \)

To solve this problem, we take the derivative with respect to \( q_J \), set it equal to 0 and solve for the optimal value of \( q_J \).

$$\frac{dE[U]}{dq_J} = \frac{-p_1}{10 - p_1 q_J} + 0.5 \frac{1}{15 + q_J} = 0,$$

which yields

$$q_J = \frac{5 - 15 p_1}{1.5 p_1}.$$

Similarly, Tom's optimal holding of the pure security is

$$q_T = \frac{5 - 5 p_1}{1.5 p_1}.$$

Equilibrium in the securities market requires

$$q_J + q_T = 0.$$

Now we can solve these 3 equations for the unknown values of \( p_1, q_J, \) and \( q_T \):
\[ p_J = 0.50 \quad q_J = -3 \frac{1}{3} \quad q_T = +3 \frac{1}{3} \]

Judy will sell and Tom will buy 3 \( \frac{1}{3} \) units of the pure security at a price of $0.50 per unit. In time period 0 Judy's consumption expenditures will be $11.67 and Tom's consumption expenditures will be $8.33. In time period 1, Judy will consume $11.67 if state 1 occurs and 5 if state 2 occurs. Tom will consume 8.33 if state 1 occurs and 15 if state 2 occurs. The expected utilities are:

\[
\begin{align*}
\text{Judy} & \quad E[U] = \ln (11.67) + 0.5 \ln (11.67) + 0.5 \ln (5) = 4.4898. \\
\text{Tom} & \quad E[U] = \ln (8.33) + 0.5 \ln (8.33) + 0.5 \ln (15) = 4.5338. 
\end{align*}
\]

Observe that the utility of each household improves relative to the autarky result, though Tom's increases by a little bit more than Judy's. Even though the security market is incomplete, the introduction of 1 security does permit some diversification to occur.

Now let us add a second security. Provided that this security delivers a non-zero payoff in state 2, it will create a complete securities market and it does not matter what the second security is. More importantly, it will permit our two households to completely diversify the risks inherent in their respective endowments and thereby achieve maximum possible utilities by each consuming 10 in time period 0 and 10 in time period 1, irrespective of which state occurs.

To demonstrate this, let us suppose that the second security is a risk free discount bond that pays $1 in each of states 1 and 2 and sells in time period 0 for a price \( P_B \). Let \( B_J \) and \( B_T \) denote the quantities of this bond purchased in time period 1 by Judy and Tom, respectively. (Remember that the \( B \)'s can be negative). Then the expected utilities of the two households are given by

\[
\begin{align*}
\text{Judy} & \quad E[U] = \ln (10 - p_J q_J - P_B B_J) + 0.5 \ln (15 + q_J + B_J) + 0.5 \ln (5 + B_J). \\
\text{Tom} & \quad E[U] = \ln (10 - p_T q_T - P_B B_T) + 0.5 \ln (5 + q_T + B_T) + 0.5 \ln (15 + B_T).
\end{align*}
\]

Each household will choose values for its \( q \) and \( B \) so as to maximize expected utility. I will spare you the computational details, but these maximization decisions will generate 2 first-order conditions for each household -- a total of 4 equations. When the following market-clearing conditions are added, we will have a total of 6 equations that we can solve simultaneously for the 6 unknowns \( q_J, q_T, B_J, B_T, Q_1 \), and \( P_B \).

\[ q_J + q_T = 0 \quad \text{and} \quad B_J + B_T = 0. \]

The solution values are as follows:

\[ q_J = -10 \quad q_T = +10 \quad B_J = +5 \quad B_T = -5 \quad p_J = 0.50 \quad P_B = 1. \]

Since \( P_B \) is the price of a discount bond that pays $1 in time period 1, irregardless of which state occurs, it must be the case that \( P_B = 1 / (1 + r_f) \), where \( r_f \) is the risk free rate.
of interest. (Why? Because the price of a risk free asset is always equal to the present discounted value of its future payout using the risk-free rate of interest as the discount factor). Here \( P_B = $1 \) in equilibrium, so \( r_f \) is equal to zero. (The reason for this will become apparent shortly). Each household consumes 10 in time period 0 and 10 in time period 2, irrespective of which state occurs. Uncertainty about future consumption has been completely eliminated. Lifetime utility for each household is 
\[ 2 \ln (10) = 4.6052 \] with complete certainty, and this is the highest possible value. It is the absence of uncertainty about future consumption, coupled with the fact that households do not discount the future that generates the result: \( r_f = 0 \).

It is totally obvious that there is nothing to be gained in this economy by the addition of still more securities. Admittedly, this has been an extreme example. The endowments of the two households are mirror images across the two states of the world and this permits consumption uncertainty to be completely eliminated in the presence of a complete securities market. If the endowments had not been mirror images, two results would change. First, some consumption uncertainty would remain even with 2 securities. Second, the addition of the second security would have an impact on the price of the first security. That is, the value of \( p_1 \) when it is the only security would be different from the value of \( p_1 \) in the presence of a second security. The result that would remain the same even if the endowments had not been mirror images is the finding that adding more securities to an already complete securities market will have absolutely no impact on households’ expected utility or their inter-temporal consumption opportunities.

In this example the agents did not trade the pure security that is associated with State 2. But when the security market is complete, we can always determine what the equilibrium price of this asset would be. Let \( p_2 \) denote the time period 0 price of the pure security that pays $1 if State 2 occurs and 0 otherwise. Now consider a portfolio made up of the 2 assets that are traded. Let the portfolio consist of 1 unit of the risk-free bond and -1 units of pure security 1. Since this portfolio will deliver 0 if State 1 occurs and $1 if State 2 occurs, it replicates the payoffs of the non-traded pure security 2. It must, therefore, be the case that \( p_2 \) is equal to \( (1/(1+r_f) - p_1) \), which is the cost of this portfolio; otherwise there would exist a profitable arbitrage opportunity. [If, to the contrary, \( p_2 > (1/(1+r_f) - p_1) \), one could write and sell pure security 2 and use the proceeds to invest in the portfolio, thereby earning a certain profit with no risk and no equity investment – something we rule out with the no arbitrage condition.]

The preceding result generalizes. When securities markets are complete, we can use the no arbitrage condition to determine the price of any traded or non-traded security. There are alternative ways to do this, but the easiest is to simply use the prices of the pure securities. Let Security \( S \) be any security with (non-zero) payouts of \( s_1 \) and \( s_2 \) in States 1 and 2, respectively. Then the price of Security \( S \) is \( P_S = p_1s_1 + p_2s_2 \). In the special case that Security \( S \) is the risk-free asset, \( s_1 = s_2 = $1 \), which implies \( P_S = p_1 + p_2 \). But we already know that the price of the risk-free asset is \( 1/(1+r_f) \); consequently

\[ p_1 + p_2 = \frac{1}{1+r_f}. \]

........end of example..............
Alternative Arbitrage Pricing Methods

Consider the expanded setup for the uncertainty associated with the future one time period ahead. Here there are \( K \) possible states of the world and \( N+1 \) traded securities, one of which is a risk-free discount bond. Table 1 shows the payouts of each security in each state. It will be assumed that all of the securities are complex securities and that there are at least as many securities as states of the world; that is \( N+1 \geq K \).

<table>
<thead>
<tr>
<th>Probability</th>
<th>State 1</th>
<th>State 2</th>
<th>State 3</th>
<th>…</th>
<th>StateK</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_1 )</td>
<td>( \pi_2 )</td>
<td>( \pi_3 )</td>
<td>…</td>
<td>( \pi_K )</td>
<td></td>
</tr>
</tbody>
</table>

**Payoffs**

<table>
<thead>
<tr>
<th>Security A</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>…</th>
<th>( A_K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Security B</td>
<td>( B_1 )</td>
<td>( B_2 )</td>
<td>( B_3 )</td>
<td>…</td>
<td>( B_K )</td>
</tr>
<tr>
<td>Security N</td>
<td>( N_1 )</td>
<td>( N_2 )</td>
<td>( N_3 )</td>
<td>…</td>
<td>( N_K )</td>
</tr>
</tbody>
</table>

| Risk Free Security | \$1 | \$1 | \$1 | … | \$1 |

Here \( A_j \) denotes the $ payoff of Security \( A \) if State \( j \) occurs; \( B_j \) denotes the $ payoff if State \( j \) occurs, and so on. We will adopt the convention of viewing the risk free security as a discount bond that pays $1 in every possible state. Since these are existing securities, their current prices \( P_A, P_B, ..., P_N \) are known and observable. The price of the risk free security is \( \frac{\$1}{1 + r_f} \), so the value of \( r_f \) is also observable.

The question we wish to ask is whether it is possible to determine the price of a non-traded security or of a hypothetical security that does not yet exist. (For example, we may be owners of a new private venture and want to know what would be the market value of this venture, should we decide to go public with an IPO). Let's consider a hypothetical
Security $X$ that has payoffs across the $K$ states of the world denoted by $X_1, X_2, \ldots, X_K$. Let the unknown price of this security be denoted $P_X$. Under what conditions will we be able to determine the value of $P_X$ from the observed prices of the $N+1$ traded securities?

A necessary condition is that the payoffs of hypothetical Security $X$ be linearly dependent on the payoffs of some subset of the traded securities. This will always be the case when the security market is complete. It may or may not be the case when the security market is incomplete.

Valuation Methods when the Security Market is Complete

Let's first investigate the issue with a complete security market. In this case there exists a subset of $K$ of the traded securities whose payoffs are linearly independent. Let's suppose that this subset consists of Securities $A$ through $K$. Because the market is complete, we can always use these securities to recover the prices of the pure securities associated with each of the $K$ states of the world. Let $p_1, p_2, \ldots, p_K$ denote the prices of these pure securities. Then to determine their values we must simply solve the following set of $K$ linear equations:

\[
\begin{align*}
P_A &= p_1A_1 + p_2A_2 + \ldots + p_KA_K \\
P_B &= p_1B_1 + p_2B_2 + \ldots + p_KB_K \\
&\quad \vdots \\
P_k &= p_1k_1 + p_2k_2 + \ldots + p_Kk_K
\end{align*}
\]

Once we have determined the values of $p_1, p_2, \ldots, p_K$, we can apply the no arbitrage condition and readily compute the price of the hypothetical Security $X$ as

\[
(1) \quad P_X = \sum_{i=1}^{K} p_i X_i.
\]

An additional feature of a complete security market is that the following relationship always holds between the risk-free rate of interest and the prices of the pure securities:

\[
(2) \quad \frac{1}{1+r_f} = \sum_{i=1}^{K} p_i.
\]

This follows directly from the fact that the risk-free asset pays $1 in each state and its price is $1/(1+r_f)$.

Recovering the values for $p_1, p_2, \ldots, p_K$ and using them is not the only way to determine the price of a security when the security market is complete. An alternative, but equivalent, valuation method utilizes what is known as a hedge portfolio. A hedge portfolio is a portfolio of risky assets (and, possibly, also the risk-free security) that replicates the state-by-state payouts of the risk-free security. In other words, a hedge portfolio is risk free. Here is how we can construct a hedge portfolio and use it to determine the value of the hypothetical Security $X$: 
Construct a portfolio consisting of 1 unit of security $X$, some number $a$ units of traded Security $A$, some number $b$ units of traded Security $B$, and so on through $k$ units of traded Security $K$. (Remember that Securities $A$ through $K$ are linearly independent and comprise a complete security market here). Choose the values for $a$, $b$, ……,$k$ to make the portfolio a hedge portfolio by solving the following set of $K$ simultaneous, linear equations.

\[
\begin{align*}
X_1 + aA_1 + bB_1 + cC_1 + \ldots + kK_1 &= 1 \\
X_2 + aA_2 + bB_2 + cC_2 + \ldots + kK_2 &= 1 \\
X_3 + aA_3 + bB_3 + cC_3 + \ldots + kK_3 &= 1 \\
\vdots \hspace{1cm} \vdots \hspace{1cm} \ldots \hspace{1cm} \vdots \hspace{1cm} \vdots \hspace{1cm} \ldots \hspace{1cm} \vdots \hspace{1cm} \vdots \\
X_K + aA_K + bB_K + cC_K + \ldots + kK_K &= 1.
\end{align*}
\]

(payoff in State 1)

(payoff in State 2)

(payoff in State 3)

Once values have been obtained for the parameters $a$, $b$, ……,$k$, we can apply the no arbitrage condition to infer that

\[
(3) \quad P_X + aP_A + bP_B + cP_C + \ldots + kP_X = \frac{1}{1 + r_f}.
\]

The left-hand-side is the cost of the hedge portfolio; the right-hand-side is the price of the risk-free asset whose payouts the hedge portfolio replicates. There is a unique value for $P_X$ that will satisfy this equation and this must be the price of Security $X$.

The value of $P_X$ that satisfies Equation (3) will always be exactly the same value that satisfies Equation (1). The use of a hedge portfolio and the use of pure security prices are equivalent methods of determining the prices of any security when the market is complete. Both methods require solving a set of $K$ linear equations, but the use of pure security prices requires us to only do this once and we can use the results to easily value any number of non-traded or hypothetical securities. The hedge portfolio method requires that we solve a different set of $K$ linear equations for every security we wish to value. The following is a numerical example that illustrates the two methods in a 2-state world with a complete security market in which there are two (linearly independent) traded securities – a risky Security $A$ and a risk-free discount bond. The payoffs and current prices of these two securities in each of the 2 states are given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>Payoffs</td>
<td>Security $A$</td>
<td>$2$</td>
</tr>
<tr>
<td></td>
<td>Risk Free Security</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Let us determine the price of non-traded Security $X$ that pays $1.50$ if state 1 occurs and
$0.50$ if state 2 occurs.

Using the method of *pure security* prices we must solve the following set of equations to
determine the values for $p_1$ and $p_2$:

\[
\begin{align*}
2.40 &= 2p_1 + 3p_2 \\
0.94 &= p_1 + p_2
\end{align*}
\]

The solution values are $p_1 = 0.42$ and $p_2 = 0.52$.
Then $P_X = 0.42(1.5) + 0.52(0.5) = 0.89$.

Using the method of a *hedge portfolio*, we must solve the following set of equations for
the parameters $a$ and $b$.

\[
\begin{align*}
1.50 + 2a + 1b &= 1 \\
0.50 + 3a + 1b &= 1
\end{align*}
\]

The solutions are $a = 1$ and $b = -2.5$, which imply that a portfolio consisting of 1 unit of
Security $X$, 1 unit of Security $A$ and -2.5 units of the risk-free security is a *hedge portfolio*.
Then the value of $P_X$ solves $P_X + 2.40 - 2.5(0.94) = 0.94$, which yields $P_X = 0.89$.

[The student should be aware that there is more than one way to form a *hedge portfolio*. I
have chosen a particular normalization rule for use in this document – one which
combines 1 unit of the non-traded security with all of the traded securities. But if one
looks at equations (4) above, it is obvious that these equations could be re-normalized as

\[
\begin{align*}
1.50 \frac{1}{1-b} + 2 \frac{a}{1-b} &= 1 \\
0.50 \frac{1}{1-b} + 3 \frac{a}{1-b} &= 1
\end{align*}
\]

Equations (5) imply that a portfolio consisting of $\frac{1}{1-b}$ units of non-traded Security $X$ and
$\frac{a}{1-b}$ units of traded Security $A$ is also *hedge portfolio*. Using the previous solution
values for $a$ and $b$, this *hedge portfolio* combines 0.2857 of a unit of Security $X$ with
0.2857 units of Security $A$.]

Now let's tackle a little tougher 3-state problem.
Problem 1

States

<table>
<thead>
<tr>
<th></th>
<th>State 1</th>
<th>State 2</th>
<th>State 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.2</td>
<td>0.35</td>
<td>0.45</td>
</tr>
<tr>
<td>Payoffs</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Security A</td>
<td>$2</td>
<td>$3</td>
<td>$4</td>
</tr>
<tr>
<td>Security B</td>
<td>$3</td>
<td>$0</td>
<td>$2</td>
</tr>
<tr>
<td>Risk Free Security</td>
<td>$1</td>
<td>$1</td>
<td>$1</td>
</tr>
</tbody>
</table>

\[ P_A = 2.84, \quad P_B = 1.42 \quad \text{and} \quad r_f = 0.0417 \quad (\text{so} \quad P_{rf} = 1/1.0417 = 0.96). \]

Here there are 3 states and 3 linearly independent, traded securities, so markets are complete and we are able to determine the prices of all the pure securities.

\[
\begin{align*}
2p_1 + 3p_2 + 4p_3 &= 2.84 \\
3p_1 + 2p_3 &= 1.42 \\
p_1 + p_2 + p_3 &= 0.96
\end{align*}
\]

The solution values are \( p_1 = 0.30; \quad p_2 = 0.40; \quad p_3 = 0.26. \)

Now we can easily price any hypothetical security via Equation (1). For example, the price of a security with payoffs of $6, $10, $13 in states 1, 2, and 3, respectively would be \( 9.18 \quad (=6(0.30) + 10(0.40) + 13(0.26)). \) A security that pays $100, -$50, $15 in states 1, 2, and 3, respectively, would have a price of \( 13 \quad (=100(0.3) -50(0.4) +15(0.26)). \) It is left as an exercise for the student to find hedge portfolios for these two securities and to verify that their prices are as given above using the method of hedge portfolios to value the securities.

Valuation of Non-trade and Hypothetical Securities when the Market is Incomplete

When the security market is incomplete, it will be possible to determine the price of a non-traded or hypothetical Security \( X \) only when the payouts of that security are a linear combination of the payouts of traded securities. This will not always be the case and, even when it is, we cannot use the method of pure securities to value the non-traded security because we can recover the values of \( p_1, p_2, \ldots, p_K \) only when the security market is complete. In what I will call "special cases" it will sometimes be quite easy to determine the value of \( P_X \). For example, suppose that the payoff of Security \( X \) in every state is equal to $3 plus 2 times the payoff of Security \( A \); i.e. \( X_j = 3 + 2A_j \) for every state \( j = 1, 2, 3, \ldots, K \). Then holding 1 unit of Security \( X \) is equivalent to holding 3 units of the risk free security plus 2 units of Security \( A \). The price of Security \( X \) must in this
case be \( P_X = 3 \left[ 1/(1+r_f) \right] + 2P_A \); otherwise there exists a profitable arbitrage opportunity.

In general, we can determine the value of \( P_X \) in an incomplete market if only if we can combine Security \( X \) with existing securities to form a hedge portfolio.

**Risk-Neutral Valuation in a Complete Security Market**

If you go back and look at our two solution methods in a complete security market, you will observe that neither makes any use of the probabilities associated with the various states of the world. In other words, the price of any non-trade security that has different payoffs in different states of the world appears to be completely independent of the probabilities associated with those states. This seems an unlikely result and the explanation for it is that the probabilities associated with various different states have already been incorporated into the prices of the traded securities (along with investors' preferences). When we price a hypothetical security using either pure security prices or a hedge portfolio we are utilizing the observed prices of existing securities and these implicitly contain all relevant information about future states and their associated probabilities.

The lack of direct dependence of non-traded security prices on state probabilities gives rise to a very interesting variation of the use of pure securities to determine security prices. Use the general notation of Table 1 and recall that when markets are complete, the price of any Security \( J \) -- existing or hypothetical -- may be computed via Equation (1) which is reproduced below.

\[
P_J = \sum_{i=1}^{K} p_i J_i,
\]

where \( J_i \) denotes the payout of Security \( J \) in state \( i \).

Now define for each state \( i \) the variable \( \pi_i^{RN} = (1+r_f) p_i \). We know from earlier Equation (2) that \( \sum \pi_i^{RN} = 1 \). We also know that because the \( p_i \) are prices, none of them can be negative. This, together with Equation (2) implies \( 0 \leq \pi_i^{RN} \leq 1 \) for all states \( i \). The \( \pi_i^{RN} \)'s look a lot like probabilities.

Now rewrite Equation (1) as

\[
P_J = \sum_{i=1}^{K} \pi_i^{RN} J_i.
\]

(6)

If the probability of state \( i \) occurring was actually equal to \( \pi_i^{RN} \), then the numerator of Equation (6) is the expected value of the payoff of Security \( J \), and the Equation says that the price of any security is just the present value of its expected payoff, discounted at the risk free rate of interest.
Equation (6) is exactly what we would see in a world in which all investors were risk neutral; consequently we refer to the $\pi_i^{RN}$ as risk-neutral probabilities. This provides us with yet another method of determining the prices of non-traded securities. When securities markets are complete, we can replace the actual probabilities associated with the various states of the world with their risk-neutral counterparts, then compute the price of any security via Equation (6). It should be clear from the way in which we define risk-neutral probabilities that this method is absolutely equivalent to valuing securities by the use of pure security prices.

Students will at this point logically pose the following question. If the use of risk-neutral probabilities to value securities is absolutely equivalent to the use of pure security prices, why would we bother to use this new method? The answer is that risk-neutral valuation is widely used to determine the prices of derivative securities and students should have at least a rudimentary knowledge of its basis. The method is not described in our textbook but the use of risk-neutral probabilities does occur -- without much explanation -- at a critical point in Chapter 7. If you have not been familiarized with risk-neutral probabilities prior to Chapter 7, you will be very confused by their brief appearance there.

As an exercise, solve the following 2-state problems using each one of the three different valuation methods.

**Problem 3**

<table>
<thead>
<tr>
<th>States</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>State 1</td>
<td>State 2</td>
</tr>
<tr>
<td>Probability</td>
<td>0.33</td>
</tr>
</tbody>
</table>

**Payoffs**

<table>
<thead>
<tr>
<th>Payoffs</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Hypothetical Security A</td>
<td>$3</td>
</tr>
<tr>
<td>Traded Security B</td>
<td>$2.35</td>
</tr>
<tr>
<td>Risk Free Security</td>
<td>$1</td>
</tr>
</tbody>
</table>

$P_A = ?, \quad P_B = $2.84 \quad r_f = 0.05$. Determine the values of $p_1, p_2, \pi_1^{RN}, \pi_2^{RN}, P_A$, and find a hedge portfolio that contains Security A.

**Problem 4.** Repeat Problem 3 with a value $r_f = 0.02$.

**Problem 5.** An equity security that pays no dividends currently has price $S_0$. Over the next time period the price of the security will either
rise to the level $uS_0$ with probability $q$ (State 1)
or fall to the level $dS_0$ with probability $1-q$. (State 2)

The risk free rate is $r_f$.

(a) Show that the no arbitrage condition is satisfied here only if $d < 1+r_f < u$.

(b) Under the assumption that the no arbitrage condition is satisfied, show that the values of the risk neutral probabilities are

$$\frac{(1+r_f) - d}{u-d} \text{ for State 1 and } \frac{u - (1+r_f)}{u-d} \text{ for State 2.}$$

Solutions:

Problem 3.
$p_1 = $0.20 $\quad p_2 = $0.7524 $\quad \pi_1^{RN} = 0.21 \quad \pi_2^{RN} = 0.79 \quad P_A = $1.3524

There are more than one hedge portfolios but the one that corresponds to the normalization of this document consists of 1 unit of Security $A$, 2.5 units of Security $B$, and -7.875 units of the risk-free security.

Problem 4.
$p_1 = $0.3103 $\quad p_2 = $0.6701 $\quad \pi_1^{RN} = 0.3165 \quad \pi_2^{RN} = 0.6835 \quad P_A = $1.601

There are more than one hedge portfolios but the one that corresponds to the normalization of this document consists of 1 unit of Security $A$, 2.5 units of Security $B$, and -7.875 units of the risk-free security – the same as for Problem 3.

Problem 5.
Part (a) Suppose $(1+r_f) < d < u$. Then if one borrows $S_0$ dollars at the risk-free rate and uses this to purchase 1 share of the security, the future payoffs will be

if State 1 occurs: $uS_0 - (1+r_f)S_0 > 0$
if State 2 occurs: $dS_0 - (1+r_f)S_0 > 0$.

With a 0 initial investment one will earn a profit no matter which state occurs. This is a profitable arbitrage opportunity, which we rule out; consequently, it cannot be the case that $(1+r_f) < d < u$. A similar argument can be made to rule out the possibility that $(1+r_f) > u > d$. The only case consistent with the "no arbitrage" condition is $d < (1+r_f) < u$.

Part (b) Let $\pi^{RN}$ denote the risk neutral probability that the outcome $uS_0$ occurs.
Then we know that the following must hold:

\[
S_0 = \frac{\pi^{RN} u S_0 + (1 - \pi^{RN}) d S_0}{1 + r_f}.
\]

Solve this equation to find the expressions for \(\pi^{RN}\) and \((1 - \pi^{RN})\) given in Problem 5.