1. (20 marks)
Consider a world of perfect certainty that will last 3 years. The current date is date 0, to be followed in succession by dates 1, 2 and 3. The world ends at date 3.

The current price of a 1-year zero coupon bond that will pay $100 just prior to date 1 and nothing thereafter is $96.15.

The current price of a 2-year zero coupon bond that will pay $100 just prior to date 2 and nothing before or after is $90.70.

(a) (12 marks)
Use the preceding information to determine the values of each of the following:

(i) the current 1-year rate of interest.
(ii) the current 2-year rate of interest.
(iii) the current price of a share of stock that will pay dividends of $D_0 = 12, D_1 = 15 just prior to dates 1, 2, respectively.

(b) (8 marks)
Let $r_{10}$ denote the current 1-year rate of interest and let $r_{20}$ denote the current 2-year rate of interest. Explain why the 1-year interest rate that will prevail at date 1 -- denoted by $r_{11}$ -- must satisfy the following relationship:

$$1 + r_{11} = \frac{(1 + r_{20})^2}{(1 + r_{10})}.$$
2. (20 marks)
An individual has wealth that is risky: \( W \) will either be 16 or 4, each with probability \( \frac{1}{2} \). Determine the certainty equivalent wealth (CE) and the maximum risk premium (\( \pi \)) that the individual would pay to exchange risky wealth for its expected value for each of the following utility functions:

(i) \( u(W) = \ln(W) \).
(ii) \( u(W) = -\frac{1}{2W^2} \).
(iii) \( u(W) = 10E[W] - 0.5\text{Var}[W] \).

[Note that in (iii) \( \text{Var}[W] \) denotes the variance of total wealth.]

3. (20 marks)
Two risky securities 1 and 2 have returns with the following expected values and standard deviations:
\[
\bar{r}_1 = 0.09, \quad \sigma_1 = 0.08; \quad \bar{r}_2 = 0.05, \quad \sigma_2 = 0.03
\]
The coefficient of correlation between the two returns is \( \rho_{12} = -0.3 \).

Consider portfolios made up of only these two securities in which \( \alpha \) is the share of the portfolio invested in Security 1. (Short sales of either security are permitted so there are no restrictions on the value of \( \alpha \)).

(a) (12 marks)
Determine the range of values for \( \alpha \) that defines the efficient set of portfolios (i.e. the efficient frontier).

(b) (8 marks)
Determine the optimal portfolio for an investor with the following expected utility function:
\[
E[U] = \bar{r}_p - 8\sigma_p^2.
\]
(Measure means and variances as decimals).

4. (20 marks)
In a CAPM world the rate of return on the market portfolio has expected value \( \bar{r}_M = 0.12 \) with standard deviation \( \sigma_M = 0.20 \). The betas and relevant properties of the returns on each of two risky securities are given below.
\[
\beta_1 = 0.8 \quad \beta_2 = 1.4 \\
\bar{r}_1 = 0.10 \quad \bar{r}_2 = 0.16 \\
\sigma_1 = 0.24 \quad \sigma_2 = 0.36
\]
The coefficient of correlation between the two returns is \( \rho_{12} = 0 \).

(a) (6 marks)
Determine the value of the risk-free rate \( r_f \).

(b) (14 marks)
Let \( r_p \) denote the rate of return on a portfolio consisting solely of risky securities 1 and 2. Find the portfolio that has a beta \( \beta_p = 0 \), then compute the values of \( \bar{r}_p \) and \( \sigma_p \) for that portfolio.

5. (20 marks)
An economic agent can choose among all portfolios consisting of a risky security and the risk-free security. The risky security has a return with expected value \( \bar{r}_1 = 0.08 \) and standard deviation \( \sigma_1 = 0.15 \). The risk-free rate is \( r_f = 0.05 \). The agent has the following utility function defined over the mean and variance of the return on her portfolio:

\[
\text{Utility} = \bar{r}_p - 0.4\sigma_p^2.
\]

(a) Find the portfolio that maximizes the agent's utility. (Measure means and variances as decimals).

(b) Now suppose that the risky security is really the market portfolio so that \( \bar{r}_M = \bar{r}_1 \) and \( \sigma_M = \sigma_1 \). What is the equation for the Capital Market Line (CML)? What is the value of beta (\( \beta_p \)) for the portfolio that maximizes the agent's utility?
SOLUTIONS

Q1. Part (a) (12 marks)

Using the notation defined in Part (b) of the question, the solution values are:

(i) \(1 + r_{10} = \frac{\$100}{\$96.15} = 1.04\); the current 1-year interest rate is 4%.

(ii) \((1 + r_{20})^2 = \frac{\$100}{\$90.70} = 1.025 = (1.05)^2\); the current 2-year interest rate is 5%.

(iii) current share price = \(\frac{\$12}{(1.04)} + \frac{\$15}{(1.05)^2} = \$25.14\).

Part (b) (8 marks)

The relationship must be satisfied; otherwise there will exist a profitable arbitrage opportunity. To see this, rewrite the relationship as follows:

\((1 + r_{10})(1 + r_{11}) = (1 + r_{20})^2\).

The right-hand-side of this equation is the payout just prior to date 2 from investing $1 in a 2-year bond at date 0.
The left-hand-side is the payout just prior to date 2 from investing $1 in a 1-year bond at date 0, then re-investing the payout from that bond in another 1-year bond at date 1.
If these two payouts are not equal, there is clearly a profitable arbitrage opportunity, and the actions of persons undertaking this opportunity will cause interest rates to change until the relationship above is satisfied and bond markets are in equilibrium.

Q. 2. (20 marks)

\(E[W] = \frac{1}{2}(16+4) = 10\).

(i) \(E[u(W)] = \frac{1}{2}[\ln(16) + \ln(4)] = 2.07944\); \(CE = \exp(2.07944)\), which implies \(CE = 8.0\).
The maximum risk premium is \(\pi = E[W] – CE = 2\).

(ii) \(E[u(W)] = \frac{1}{2}[\frac{1}{2(16)^2} - \frac{1}{2(4)^2}] = -0.0166; \quad -\frac{1}{2(CE)^2} = -0.0166\), which implies \(CE = 5.488\). Then \(\pi = E[W] – CE = 4.512\).

(iii) \(Var[W] = \frac{1}{2}[ (16-10)^2 + (4-10)^2 ] = 36\).
The utility derived from the risky wealth is \(10(10) - 0.5(36) = 82\). The certainty equivalent wealth is the wealth that will yield the same utility with zero Variance. \(10(CE) + 0 = 82\), which implies \(CE = 8.2\). Then \(\pi = E[W] – CE = 1.8\).
Q3. (20 marks)

(12 marks) (a) We want to first find the value of $\alpha$ for the minimum variance portfolio.

$$\sigma_p^2 = \alpha^2 \sigma_1^2 + (1-\alpha)^2 \sigma_2^2 + 2\alpha(1-\alpha)\text{cov},$$

where $\text{cov} = \rho_{1,2}\sigma_1\sigma_2 = -0.00072$.

Set $\frac{\partial \sigma_p^2}{\partial \alpha} = 2[\alpha(\sigma_1^2 + \sigma_2^2 - 2\text{cov}) - (\sigma_2^2 - \text{cov})] = 0$

And solve for $\alpha_{\text{min}} = 0.1854$.

Then the efficient frontier consists of all portfolios with $\alpha \geq 0.1854$.

(8 marks) (b) $E[U] = \alpha \bar{r}_1 + (1-\alpha)\bar{r}_2 - 8(\alpha^2 \sigma_1^2 + (1-\alpha)^2 \sigma_2^2 + 2\alpha(1-\alpha)\text{cov})$

Set $\frac{\partial E[U]}{\partial \alpha} = (\bar{r}_1 - \bar{r}_2) - 16[\alpha(\sigma_1^2 + \sigma_2^2 - 2\text{cov}) - (\sigma_2^2 - \text{cov})] = 0$

And solve for $\alpha^* = 0.4714$.

The optimal portfolio for this investor has a share 0.4714 invested in Security 1 and a share 0.5286 invested in Security 2.

Q4. (20 marks)

(6 marks) (a) The CAPM equation for any security or portfolio $j$ is

$$[1] \quad \bar{r}_j = r_f + \beta_j (\bar{r}_m - r_f).$$

We can use this equation with values for either Security 1 or Security 2 to find the value of the risk-free rate. Here is Equation [1] with the given values for Security 1:

$$0.10 = r_f + 0.8(0.12-r_f),$$

which implies $r_f = 0.02$.

(14 marks) (b) Next we must find the value of $\alpha$ that yields a value $\beta_p = 0$ for a portfolio with a share $\alpha$ invested in Security 1 and a share $(1-\alpha)$ invested in Security 2.

There are at least two ways to do this.

First, with $\beta_p = 0$ the CAPM Equation [1] implies $E[r_p] = r_f = 0.02$. But the expected return on the portfolio must also equal $[\alpha \bar{r}_1 + (1-\alpha)\bar{r}_2] = [\alpha(0.10) + (1-\alpha)(0.16)]$.

From this we can readily deduce that the value $\alpha = 2.333$ will produce a portfolio that has $\beta_p = 0$ and $\bar{r}_p = 0.02$. The standard deviation of the return on this portfolio is

$$\sigma_p = \sqrt{[(2.333)^2(0.24)^2 + (-1.333)^2(0.36)^2} = \sqrt{0.544} = 0.7376.$$
An alternative method utilizes the property that the beta of a portfolio consisting of a weighted sum of individual securities is equal to that weighted sum of the betas of the individual securities. That means $\beta_p = 0.8\alpha + 1.4(1 - \alpha)$, which implies $\alpha = 2.333$.

Q5. (20 marks)

(a) $E[U] = \alpha \bar{r}_1 + (1 - \alpha)r_f - 0.4(\alpha^2 \sigma_1^2)\)  
Set $\frac{\partial E[U]}{\partial \alpha} = (\bar{r}_1 - \bar{r}_2) - 0.8[\alpha\sigma_1^2] = 0$  
And solve for $\alpha^* = 1.6667; (1-\alpha^*) = -0.6667$  
The investor should borrow (sell short) the risk-free security in amount 0.6667 times wealth, add the proceeds to wealth and invest 1.6667 times wealth in the risky security.

(b) The CML (CAL in Bodie's terminology) is simply the straight line that defines the efficient frontier in this case. The equation for this line is  
$$r_p = r_f + \left(\frac{\bar{r}_M - r_f}{\sigma_M}\right)\sigma_p,$$  
where $p$ denotes any portfolio on the efficient frontier.  
Using values given in Question 5, the equation becomes  
$$\bar{r}_p = 0.05 + 0.2\sigma_p.$$  
The optimal portfolio from Part (a) has $r_p = 1.6667r_M - 0.6667r_f.$  
The $\beta$ for this portfolio is (by definition) $\beta_p = \frac{\text{cov}(r_p, r_M)}{\sigma_M^2} = \frac{1.6667\sigma_M^2}{\sigma_M^2} = 1.6667.$