A Brief Introduction to the Consumption Based Asset Pricing Model (CCAPM)

We have seen that CAPM identifies the risk of any security as the covariance between the security's rate of return and the rate of return on the market portfolio. According to the CAPM, the uncertainty associated with the return on the market portfolio is the sole source of risk in the economy but CAPM has no theoretical structure that allows us to readily identify what it is that causes the market portfolio to be risky. Macroeconomics does have such a theoretical structure. It tells us, for example, how the profits of firms are related to such things as overall economic activity (GDP) and the government's conduct of monetary and fiscal policies. Macroeconomics provides us with models that enable us to not only identify various sources of aggregate uncertainty but to also understand the mechanisms by which these affect security returns and prices.

The asset pricing model that is embedded in stochastic models of macroeconomics is called the Consumption Based Asset Pricing Model (CCAPM). The name derives from the fact that the equations that describe the behaviour of asset prices and returns in the CCAPM devolve from the consumption/saving and asset choice decisions of households.

In the CCAPM the economy is assumed to be populated by a large number of households that are identical in all respects, including preferences and endowments. This assumption permits decision making to be analyzed by examining the behaviour of a single, representative household. No matter what the macroeconomic setting, one consequence of the CCAPM assumption that all households are identical is that households will never exchange assets with one another. For instance it will never be the case that one household will borrow from another. Why? All households are identical; if one wishes to borrow, all will wish to borrow and there will be no household that wishes to lend. If there are any assets that exist in positive net supply, these must come form outside the household sector (e.g. from governments, businesses, or the rest of world).

Example 1, Perfect Certainty

The simplest example of CCAPM is a two period endowment economy with perfect certainty. Let us suppose that the representative household has lifetime utility given by

\[ U = u(c_0) + \beta u(c_1). \]

Let us further suppose that the representative agent has endowment \( Y_0 \) units of consumption in the current time period and \( Y_1 \) units of consumption in the future time period. Consider a discount bond that pays 1 unit of consumption in time period 1 and sells for the price \( p \) in time period 0. Let \( q_0 \) denote the quantity of discount bonds purchased by the representative household during time period 0. (\( q_0 \) may be positive or negative). The representative will choose the value of \( q_0 \) so as to maximize lifetime utility subject to the budget constraints:

\[ c_0 = Y_0 - pq_0 \quad \text{and} \quad c_1 = Y_1 + q_0. \]

Formally, the optimization problem is
$Max_{q_0} \ u(Y_0 - pq_0) + \beta u(Y_1 + q_0)$.

The first order condition (FOC) is

$$-u'(Y_0 - pq_0)p + \beta u'(Y_1 + q_0) = 0.$$  (1)

We should regard the value of $q_0$ in FOC (1) as the quantity of discount bonds demanded by the representative household. Now let us suppose that the discount bond under consideration is issued by the government in this economy. Suppose that for each household in this economy the government issues a quantity $q^g$ of the discount bond. Then for the bond market to be in equilibrium, quantities demanded and supplied must be equal; hence the equilibrium price of the bond $p$ must be such that the quantity $q_0$ demanded by the representative household is exactly equal to the quantity $q^g$ supplied by the government. The equilibrium price is the value of $p$ that solves FOC (1) with $q_0 = q^g$. In other words, FOC (1) is an expression that describes the equilibrium price of the discount bond.

A numerical example will help to clarify this. Suppose the utility function in our example is the logarithmic form $u(c) = \ln(c)$ and the time discount factor is $\beta = 0.95$. Also suppose the endowments are $Y_0 = 10$ and $Y_1 = 7$; and the quantity of bonds supplied is $q^g = 2$ per household. Then FOC (1) becomes

$$(1)' \quad -\frac{p}{10 - 2p} + 0.95(\frac{1}{7 + 2}) = 0.$$  

The equilibrium value for the bond price is $p = 0.8716$. And, since the price of a discount bond is always equal to $\frac{1}{1 + r}$, where $r$ is the rate of interest, the equilibrium interest rate in the economy is $r = 0.1473$ (or 14.73%).

As a second numerical example, suppose that preferences and endowments are as given above but there exists no government so that the equilibrium supply of bonds is $q^g = 0$. In this case FOC (1) becomes

$$(1)'' \quad -\frac{p}{10} + 0.95(\frac{1}{7}) = 0.$$  

The equilibrium bond price is $p = 1.3571$ and the equilibrium interest rate is $r = -0.2632$ (or negative 26.32%). The intuition for a negative interest rate here is that with endowments of $Y_0 = 10$ and $Y_1 = 7$ the representative household would have natural incentive to smooth consumption by buying bonds (lending) in time period 0 if the rate of
interest were positive. But with bonds in zero net supply lending is impossible and the interest rate must be negative to make the representative household choose net demand for lending that is exactly equal to zero.

End Example 1.

The general case is a multi-period model in which it is assumed that the representative household is infinitely-lived; i.e. the household lives forever. To the uninitiated it will seem odd that a household is assumed to live forever, but it is not so odd if we view a household as consisting of related individuals of differing ages. As the older members of a household die off, they are replaced by younger members of the next generation. In this representation the household is, in effect, a never-ending "dynasty". The representative household is assumed to have expected lifetime utility function given by

\[
E_t[U_t] = u(c_t) + E_t\sum_{i=1}^{\infty} \beta^i u(c_{t+i}).
\]

Here \(E_t\) denotes the expected value operator taken during time period \(t\) – and, therefore, conditional upon the information that is available to the representative household during that time period. \(u(c_t)\) is the utility derived from consumption undertaken during time period \(t\) and \(E_tu(c_{t+i})\) is the expected utility of consumption in time period \(t+i\). \(\beta\) denotes a time discount factor. It is assumed that \(0<\beta<1\) so that the weight in \(U_t\) of consumption during any future time period \(t+i\) is smaller, the larger is \(i\). That is, consumption in the near future is valued more highly than consumption in the more distant future. Empirically, it is found that the value of \(\beta\) is in fact only slightly less than 1, which implies that consumption in consecutive time periods are very close substitutes and that households desire to smooth consumption so that period-to-period variations in the value of \(c\) are small.

In each time period \(t\) the representative household chooses current period consumption \(c_t\) and formulates a plan for future consumption \(c_{t+1}, c_{t+2}, \ldots\), subject to a budget constraint. The nature of the budget constraint can vary, depending upon the complexity of the economic environment in which the representative household is assumed to dwell. However, for determining equilibrium prices of assets the form of the budget constraint really does not matter. We simply ask the question: If the representative household were able to purchase some security for a price of \(p_{x,t}\) in time period \(t\) and redeem (e.g. sell) this asset for the quantity of consumption \(x_{t+1}\) in time period \(t+1\), what would be the optimal quantity of this security that the representative household would choose to purchase? The answer is given by the following First Order Condition:

\[
-u'(c_t)p_{x,t} + \beta E_t[x_{t+1}u'(c_{t+1})] = 0.
\]

Observe that the first term in FOC (3) is the utility lost in time period \(t\) from the purchase of a marginal unit of the security and the second term is the expected utility to be gained
in time period \( t+1 \) from the redemption of this marginal unit. FOC (3) applies to any security.

By definition \( \frac{x_{t+1}}{p_x} = 1 + r_{x,t+1} \), where \( r_{x,t+1} \) is the one-period rate of return to be received on the security in time period \( t+1 \). Using this definition, FOC (3) may be rewritten as

\[
(4) \quad 1 = \beta E_t(1 + r_{x,t+1}) \frac{u'(c_{t+1})}{u'(c_t)}.
\]

Equation (4) is the standard expression of CCAPM. It tells us that in economy-wide equilibrium the expected value of \((1 + \text{the one-period-ahead rate of return on any asset}) \times \text{the marginal rate of substitution of consumption between time periods } t \text{ and } t+1 \) must equal 1.0. This equation must be satisfied for all assets. For a risk-free asset, the corresponding FOC is

\[
(5) \quad 1 = \beta(1 + r_j)E_t \frac{u'(c_{t+1})}{u'(c_t)}.
\]

We can subtract Equation (5) from Equation (4) and express the CCAPM relationship in terms of excess returns -- eliminating in the process the parameter \( \beta \):

\[
(6) \quad E_t[(r_{x,t+1} - r_j) \frac{u'(c_{t+1})}{u'(c_t)}] = 0.
\]

To gain some intuition into this result, let us make use of the statistical property that says that the expected value of the product of two random variables is the sum of their covariance and the product of their respective expected values and rewrite Equation (6) as

\[
(7) \quad \bar{r}_{x,t+1} - r_j = -\text{Cov}[r_{x,t+1}, \frac{u'(c_{t+1})}{u'(c_t)}] / \left[ E_t \left( \frac{u'(c_{t+1})}{u'(c_t)} \right) \right].
\]

Equation (7) tells us that the risk of any asset is proportional to the negative of the covariance of its rate of return with the marginal rate of substitution. This makes some sense in so far as a main motivation for acquiring assets is to facilitate the smoothing of consumption (and utility) across time periods. Suppose the asset \( x \) appearing in Equation (7) has a rate of return that is high whenever \( c_{t+1} \) is high. Then the marginal utility \( u'(c_{t+1}) \) will be low when \( r_{x,t+1} \) is high, and the covariance in Equation (7) will be negative in sign. In this case asset \( x \) does not help to smooth consumption over time and will command a positive risk premium; i.e. \( \bar{r}_{x,t+1} > r_j \).
Now suppose, instead, that $r_{x,t+1}$ is high whenever future consumption $c_{t+1}$ is low. In this case the asset does assist in smoothing consumption over time and the covariance in Equation (7) will be positive in sign, which implies $\bar{r}_{x,t+1} < r_f$.

In CAPM a security's systematic risk depends on the covariance between its return and the return on the market portfolio. In CCAPM a security's systematic risk depends on the covariance of its return with future consumption.

While Equations (4) – (7) deal only with returns one period into the future, CCAPM can accommodate assets with payoffs in any future time period. In this sense CCAPM is truly a multi-period asset pricing theory. Consider a security that will pay $y_{t+j}$ units of consumption in future time period $t+j$ and nothing prior or after that time period. Let $p_{y,t}$ denote the price of this security in time period $t$. Then the FOC that describes the equilibrium value of this price is

\[ (8) \quad -u'(c_t)p_{y,t} + \beta^j E_t[y_{t+j}u'(c_{t+j})] = 0. \]

We will not further pursue CCAPM in this general setting. Instead, we will apply it in a very simple example.

**Example 2**

Let us suppose that the current time period is period 0 and that the representative household has a current endowment of $Y_0 = 6$. For each future time period the endowment is assumed to be random with two possible outcomes:

with probability 0.6, $Y_t = 10$; with probability 0.4, $Y_t = 5$ for all $t > 0$.

The realizations of $Y_t$ are assumed to be independent from one time period to the next.

The utility function is $u(c) = \ln(c)$ and the time discount factor is $\beta = 0.98$.

Let us suppose that all assets are in zero net supply so that $c_0 = Y_0$ and $c_t = Y_t$ for all values of $t$. We can use Equation (4) to determine the value of the one-period risk-free rate of interest during time period 0 that will cause the representative household to demand zero units of the risk-free asset. We know $u'(c_0) = \frac{1}{6} = 0.1667$ and can easily compute $E[u'(c_t)] = 0.6(\frac{1}{10}) + 0.4(\frac{1}{5}) = 0.14$. Then from Equation (4), $r_f = 0.2148$.

Now let us determine the period 0 price of a discount bond that will pay 1 unit of consumption in time period $t = 2$ no matter which state occurs. Let $P_0^{(2)}$ denote this price, which from Equation (8) must satisfy the following:
\[ P_0^{(2)} = \beta^2 \left( E \frac{u'(c_2)}{u'(c_0)} \right). \]

\( E[u'(c_2)] = 0.14 \) and \( u'(c_0) = 0.1667 \); consequently \( P_0^{(2)} = 0.8066 \). Note that the 2-year rate of interest is the value \( r_0^{(2)} \) that solves \( P_0^{(2)} = \frac{1}{(1 + r_0^{(2)})^2} \). So here the 2-year rate is \( r_0^{(2)} = 0.1135 \).

Let \( q_0^H \) denote the price of the pure security that pay will pay 1 unit of consumption in time period 1 if the high consumption state occurs and let and let \( q_0^L \) denote the price of the pure security that pay will pay 1 unit of consumption in time period 1 if the low consumption state occurs. In determining the values of these two prices we must take some care in evaluating \( E[u'(c_i)] \). The pure security that pays 1 only in the high consumption state has \( E[u'(c_i)] = 0.6(\frac{1}{10}) = 0.06 \). The pure security that pays 1 only in the low consumption state has \( E[u'(c_i)] = 0.4(\frac{1}{5}) = 0.08 \). Then using Equation (3), we can infer that \( q_0^H = 0.3528 \) and \( q_0^L = 0.4703 \). We can confirm that these values are correct because we know that \( (q_0^H + q_0^L) \) is always equal to \( \frac{1}{1 + r_f} \), and from our earlier computations we found \( \frac{1}{1 + r_f} = \frac{1}{1.2148} = 0.8231 \), which does indeed equal \( (q_0^H + q_0^L) \).

Observe that there are two possible states in time period 1 and we have recovered the prices of the two pure securities; consequently the security market is complete here. (Securities markets are always complete under CCAPM). Now let us determine the price of a European put option written on \( c_1 \) with an exercise price of 8.50. (The maturity date is in time period \( t = 1 \)). [Note: In the classroom we have not yet discussed put and call options. That is still to come. Suffice it to say the put option under consideration here is a security that will pay 0 if \( Y_1 = 10 \) and will pay 3.5 if \( Y_1 = 5 \). With this information regarding the payoffs, the student should have no difficulty determining the equilibrium price of this security.]

There are two different ways of determining the price of this put option. First, we know that the payoff of the option will be non-zero only if \( c_1 = Y_1 = 5 \), in which case the payoff will be 3.5 (units of period 1 consumption). This is equivalent to owning 3.5 units of the pure security with price \( q_0^L = 0.4704 \) consequently the time period 0 price of the put must be \( p_{put,0} = 3.5(0.4703) = 1.6464 \).

Alternatively, we can use Equation (3) to compute the equilibrium put price directly. In this case Equation (3) becomes
\[ (3') \quad -\left(\frac{1}{6}\right) p_{put,0} + 0.98(3.5)(0.4)\left(\frac{1}{5}\right) = 0, \]

which yields \( p_{put,0} = 1.6464 \).

I leave for the student to show that the price of a European call option written on \( c_2 \) with an exercise price of 7.50 is \( p_{call,0} = 0.08644 \). (The maturity is in period \( t = 2 \)).

[In this case the call option is a security that will pay 2.5 at date 2 if \( Y_2 = 10 \) and will pay 0 at date 2 if \( Y_2 = 5 \).]

Now let us interpret the future endowments to be received by the representative household as being the result of both labour effort and investment on the part of the household. We will assume that the representative household is to receive real wages of 5 units of consumption with certainty in every future time period, starting with time period \( t = 1 \). The household is also assumed to own one share of stock in a corporation and will receive real dividends in each future time period in amount 5 with probability 0.6 or in amount 0 with probability 0.4, also starting in time period \( t = 1 \). In other words, in the "high" state (which occurs with probability 0.6) the endowment will be \( \text{wages} = 5 \) plus dividends = 5; in the "low" state (which occurs with probability 0.4) the endowment will be \( \text{wages} = 5 \) plus dividends = 0. (Note that we have not changed the total endowments here; we have just identified where the endowments come from).

Let \( S_0 \) be the time period \( t = 0 \) price of the share of stock owned by the representative household and let \( Div_t \) be the value of the dividend to be received in time period \( t > 0 \). It can be deduced from Equation (8) that the "present value" of this risky dividend is given by the expression \( \beta^t E_0[Div_t \frac{u'(c_t)}{u(c_0)}] \), which prompts the following generalization of that equation:

\[ (9) \quad S_0 = E_0 \sum_{t=1}^{\infty} \beta^t Div_t \frac{u'(c_t)}{u(c_0)}. \]

Equation (9) says that the share price is the sum of the present discounted values of all future dividends.

Here \( E_0[Div_t \frac{u'(c_t)}{u(c_0)}] = (0.6)(5)(\frac{0.1}{0.1667}) + (0.4)(0)(\frac{0.2}{0.1667}) = 1.8 \) for all \( t > 0 \); hence

\[ S_0 = 1.8 \sum_{t=1}^{\infty} 0.98^t = 1.8(0.98)\frac{1}{1-0.98} = 88.2. \]

That is, the equilibrium price of one share of stock is 88.2 units of consumption in time period 0.
End Example 2.