1. (20 marks)

A coupon bond with a maturity of 2 years has a face value of $100 and an annual coupon rate of 6%. (The coupon is paid once a year at the end of the year). The yield to maturity on this bond is \( r_{ym} = 0.075 \) (or 7.5%). The current 1-year rate of interest is \( r_{10} = 0.08 \).

(a) Determine the current price of the coupon bond.

(b) Determine the value of the current 2-year rate of interest \( r_{20} \).

(c) Determine the value of the 1-year forward rate of interest \( f_{1} \).

(d) An investment dealer currently owns the coupon bond described above. She plans on "stripping" the coupons, which means she will sell the right to receive all future coupon payments to some investor for price \( A_0 \) and she will sell the right to receive the payment of the face value to some other investor for price \( B_0 \). Determine the values of \( A_0 \) and \( B_0 \).

2. (10 marks)

In a world of perfect certainty the one-year rate of interest is constant over time at a value \( r = 0.05 \). ABC Inc. will pay a cash dividend of $0.50 per share at the end of the current year. This dividend will rise to $0.60 next year and grow thereafter at a rate of 3% per year into the indefinite future. Determine the current price of a share of equity in ABC Inc.
3. (12 marks)
An individual derives utility from wealth via the utility function $U(W) = W^{1/2}$. The individual faces the following gamble: $W = 16$ with probability $p$, or $W = 4$ with probability $1-p$.

(a) For $p = \frac{1}{2}$, what is the certainty equivalent of this gamble?

(b) For $p = \frac{1}{2}$, what is the maximum amount the individual would pay to exchange the gamble for its expected value?

(c) At what value for $p$ would the individual be indifferent between taking the gamble or receiving $W = 10$ with certainty?

4. (18 marks)

The following table describes the rate of return possibilities for two risky securities. The table states that there is a probability of 0.8 that future economic conditions will be good and a probability of 0.2 that future economic conditions will be poor. If economic conditions are good, Security A will yield a return of +0.10 (or +10 %) and Security B will yield a return of +0.06. If economic conditions are poor, Security A will yield a return of -0.10 (or -10 %) and Security B will yield a return of 0.00.

<table>
<thead>
<tr>
<th>Economic Conditions</th>
<th>probability</th>
<th>Return on Security A</th>
<th>Return on Security B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Good</td>
<td>0.80</td>
<td>+0.10</td>
<td>+0.06</td>
</tr>
<tr>
<td>Poor</td>
<td>0.20</td>
<td>-0.10</td>
<td>0.00</td>
</tr>
</tbody>
</table>

(a) Determine the expected values and standard deviations of the returns on each security.

(b) Determine the coefficient of correlation between the two returns.

(c) An investor with an expected utility function of the form $E[U] = \bar{r} - 1/2 A \sigma^2$ must choose either Security A or Security B. What value for the risk aversion parameter $A$ will make the investor indifferent between the two securities?

[Note: The answer to Q. 4, Part (c) is sensitive to the units in which $\bar{r}$ and $\sigma$ are expressed. Please use decimals instead of percentages.]
The following information is used in questions 5. and 6.

Two risky securities, denoted as securities 1 and 2, respectively, have the following properties.

\[
\begin{align*}
\bar{r}_1 &= 0.16 & \bar{r}_2 &= 0.10 \\
\sigma_1 &= 0.20 & \sigma_2 &= 0.10 \\
\end{align*}
\]

The coefficient of correlation between \( r_1 \) and \( r_2 \) is \( \rho_{1,2} = +0.25 \)

5. (16 marks)

(a) What are the investment proportions of the minimum variance portfolio consisting of the two securities? What are the expected value and standard deviation of the return on this minimum variance portfolio?

(b) What is the optimal portfolio for an investor with an expected utility function given by \( E(U) = \bar{r} - 5\sigma^2 \)? [Short sales of either security are permitted].

6. (24 marks)

Suppose that in a CAPM setting the market portfolio consists of equal investments in securities 1 and 2; i.e.

\[
r_M = 0.5r_1 + 0.5r_2 .
\]

(a) What are the values for \( \bar{r}_M \) and \( \sigma_M \)?

(b) Find the values of \( \beta_1 \) and \( \beta_2 \).

(c) Given your answers for Part (b), if there is a risk-free security, what must be the value of its return \( r_f \)?

(d) Suppose there is a risk-free security with a return \( r_f \) equal to your answer to Part(c). Then what is the optimal portfolio for an investor with an expected utility function given by \( E(U) = \bar{r} - 5\sigma^2 \)? [Short sales of all securities are permitted].
**SOLUTIONS**

1. Normally we know the price of a bond and its future payments, and from these we are able to calculate its yield to maturity. For the 2-period coupon bond described in Question 1., its yield to maturity is the value of $r_{ytm}$ that satisfies the following equation:

$$\text{Price of the Bond} = \frac{\$6}{(1 + r_{ytm})} + \frac{\$6 + \$100}{(1 + r_{ytm})^2}.$$

In this Question, we are given the value $r_{ytm} = 0.075$ and asked to compute the price of the bond. The simple computation yields price of $\$97.3067$.

(b) We know that the current price of the bond is the Present Discounted Value (PDV) of its future payouts, using interest rates as discount factors. Therefore the following equation applies:

$$\$97.3067 = \frac{\$6}{(1 + 0.08)} + \frac{\$6 + \$100}{(1 + r_{20})^2}.$$

Solving this equation for the value of the 2-year interest rate yields $r_{20} = 0.07485$.

(c) By definition, the one-year forward rate of interest is the value for $f_1$ that satisfies

$$1.08(1 + f_1) = (1.07485)^2.$$

The solution is $f_1 = 0.06972$.

(d) Here $A_0$ is the PDV of the future stream of coupon payments and $B_0$ is the PDV of the future principal (or face value) payment. Consequently,

$$A_0 = \frac{\$6}{(1 + 0.08)} + \frac{\$6}{(1 + 0.07485)^2} = \$10.7490.$$

$$B_0 = \frac{\$100}{(1 + 0.07485)^2} = \$86.5574.$$

2. After date 1 (1 year from now) the cash dividend will grow indefinitely at a rate of 3% per year. We have learned that under these circumstances the share price can be expressed as the sum of an infinite geometrically declining series. That is, the share price at date 1 will be

$$e_1 = \frac{D_1}{r - g} = \frac{\$0.60}{0.05 - 0.03} = \$30.$$

The current share price $e_0$ must be such that the 1-year holding period return between dates 0 and 1 is equal to the 1-year rate of interest of 5%. Therefore
\[ 1.05 = \frac{D_0 + e_1}{e_0} = \frac{$0.50 + $30.00}{e_0}, \]

which implies \( e_0 = $20.0476. \)

3.  
(a) The expected utility of wealth is \( E[U] = 1/2 \sqrt{16} + 1/2 \sqrt{4} = 3 \). Certainty equivalent wealth is then the value of \( CE \) that solves \( \sqrt{CE} = 3 \); consequently \( CE = 9 \).

(b) If the individual accepts the gamble he/she has certainty equivalent wealth of 9. If, instead, the individual pays some amount \( \pi \) to exchange the gamble for its expected payoff, he/she will have wealth equal to \( E[W] = \frac{1}{2} (16) + \frac{1}{2} (4) = 10 \). The maximum amount the individual would pay to make this exchange is \( \pi = E[W] - CE = 10 - 9 = 1 \).

(c) If the individual takes the gamble with probability \( p \) of receiving the outcome 16, he/she will have expected utility of \( (p \sqrt{16} + (1 - p) \sqrt{4}). \) The individual will be indifferent between this outcome and \( W = 10 \) with certainty whenever the 2 alternatives yield the same expected utility; i.e., whenever \( p \sqrt{16} + (1 - p) \sqrt{4} = \sqrt{10} \). Indifference, then requires that \( p = 0.581 \).

4.  
(a) Applying the definition of expected value, we can readily compute that \( E[r_A] = 0.80(10.10) + 0.20(-0.10) = 0.06 \). Similarly, \( E[r_B] = 0.048 \).

Applying the definition of variance, we can readily compute that \( \sigma_A^2 = 0.80(0.10 - 0.06)^2 + 0.20(-0.10 - 0.06)^2 = 0.0064; \quad \sigma_A = 0.08 \). Similarly, \( \sigma_B^2 = 0.00056; \quad \sigma_B = 0.024 \).

(b) By definition the coefficient of correlation between the two returns is \( \rho_{1,2} = \frac{Cov(r_A,r_B)}{\sigma_A \sigma_B} \). Applying the definition of covariance, we can readily compute that
\[
Cov(r_A, r_B) = 0.80[(0.10 - 0.06)(0.06 - 0.048)] + 0.20[(-0.10 - 0.06)(0.00 - 0.048)] \\
= +0.00192.
\]

Then, \( \rho_{1,2} = \frac{0.00192}{(0.08)(0.024)} = +1 \).

(c) For Security A \( E[U^A] = 0.06 - 1/2(0.0064). \)  
For Security B \( E[U^B] = 0.048 - 1/2(0.000576). \)
An investor will be indifferent between the two securities whenever they yield the same expected utility. This occurs when the risk aversion parameter $A = 4.1209$.

5. (a) Let $w_1$ denote the share of the portfolio in Security 1 and $(1-w_1)$ the share in Security 2. The variance of the return on the portfolio is

\[ \sigma_p^2 = w_1^2 (0.2)^2 + (1-w_1)^2 (0.1)^2 + 2w_1(1-w_1)(0.25)(0.2)(0.1). \]

To find the minimum variance portfolio, determine the value of $w_1$ that makes $\frac{\partial \sigma_p^2}{\partial w_1} = 0$:

\[ 0.08w_1 - 0.02(1 - w_1) + 0.01(1 - 2w_1) = 0, \]

which solves for $w_1 = \frac{1}{8}$ (or 0.125).

The minimum variance portfolio has a 0.125 proportion invested in Security 1 and a 0.875 proportion invested in Security 2.

(b) For any portfolio $p$ with a proportion $w_1$ invested in Security 1

\[ E[U] = [0.16w_1 + 0.10(1-w_1)] - 5 \left[ w_1^2 (0.2)^2 + (1 - w_1)^2 (0.1)^2 + 2w_1(1-w_1)(0.005) \right]. \]

To find the optimal portfolio, determine the value of $w_1$ that makes $\frac{\partial E[U]}{\partial w_1} = 0$:

\[ [0.16 - 0.10] - 5 [0.08w_1 - 0.02(1 - w_1) + 0.01(1 - 2w_1)] = 0, \]

which solves for $w_1 = 0.275$.

The optimal portfolio has a 0.275 proportion invested in Security 1 and a 0.725 proportion invested in Security 2.

6. (a) \[ \bar{r}_M = 0.5(0.16) + 0.5(0.10) = 0.13 \]

\[ \sigma_M^2 = (0.5)^2(0.2)^2 + (0.5)^2(0.1)^2 + 2(0.5)^2(0.25)(0.2)(0.1) = 0.015 \]

\[ \sigma_M = \sqrt{0.015} = 0.1225 \]

(b) Here $\beta_i = \frac{Cov(r_i, r_M)}{\sigma_M^2} = \frac{Cov[r_i, (0.5r_1 + 0.5r_2)]}{\sigma_M^2}$.

We know from Part (a) that $\sigma_M^2 = 0.015$ and we must evaluate the Covariance term appearing in the nominator of the equation above.

Applying the definition of Covariance:
\[ \text{Cov}\{r_1, (0.5r_1 + 0.5r_2)\} = E\{(r_1 - \bar{r}_1)(0.5(r_1 - \bar{r}_1) + 0.5(r_2 - \bar{r}_2))\} \]
\[ = 0.5E\{(r_1 - \bar{r}_1)^2 + 0.5E[(r_1 - \bar{r}_1)(r_2 - \bar{r}_2)]\} \]
\[ = 0.5\sigma_1^2 + 0.5\text{Cov}(r_1, r_2). \]

Here \( \sigma_1^2 = 0.04 \) and \( \text{Cov}(r_1, r_2) = 0.25(0.20)(0.10) = 0.005 \); therefore
\[ \beta_1 = \frac{\text{Cov}(r_1, r_M)}{\sigma_M^2} = \frac{0.02 + 0.0025}{0.015} = 1.5. \]

Similar reasoning yields
\[ \beta_2 = \frac{\text{Cov}(r_2, r_M)}{\sigma_M^2} = \frac{0.5[\sigma_2^2 + \text{Cov}(r_2, r_M)]}{\sigma_M^2} = 0.5. \]

(c) This is CAPM, so we know that \( \bar{r}_i = r_f + \beta_i(r_M - r_f) \). Then using values from Parts (a) and (b), this implies \( 0.16 = r_f + 1.5(0.13 - r_f) \), which yields \( r_f = 0.07. \)

[We get the same result for \( r_f \) if we use the CAPM equation \( \bar{r}_2 = r_f + \beta_2(r_M - r_f) \).]

(d) We know the optimal portfolio lies on the \textit{Capital Market Line} (CML) and that any portfolio \( p \) on the CML has some proportion \( \alpha \) invested in the market portfolio and a proportion \((1 - \alpha)\) invested in the risk-free security. This means that \( \bar{r}_p = 0.13\alpha + 0.07(1 - \alpha) \) and \( \sigma_p^2 = 0.015\alpha^2. \)

Then, \( E[U] = 0.07 + 0.06\alpha - 5(0.015)\alpha^2 \).

To find the optimal portfolio, determine the value of \( \alpha \) that makes \( \frac{\partial E[U]}{\partial \alpha} = 0 \):
\[ 0.06 - 0.15\alpha = 0, \]
which solves for \( \alpha = 0.40. \)

The optimal portfolio has a 0.40 proportion invested in the market portfolio of risky securities and a 0.60 proportion invested in the risk-free security.