1. (20 marks)

In a world of perfect certainty, the Smith Corporation pays all of its net earnings out as dividends to shareholders. Net earnings this year will be $48 per share and they will grow year-by-year at the rate of 2% into the indefinite future. The current \((date 0)\) one-year rate of interest is 6% and it will remain constant at this value over the indefinite future.

(a) (5 marks)
What is the current price of a share in the Smith Corporation? What will be the price of a share 1 year from now?

(b) (15 marks)
The Smith Corporation is contemplating undertaking an investment with a one-time cost of $24 per share at \(date 0\). If the investment is undertaken, it will cause net earnings per share to increase by $10 this year and by $10 for each of the following two years. (These earnings will be paid out as dividends at the ends of each of the three years). The investment will be financed by borrowing with the principal and all accrued interest repaid at the end of the third year. If the investment is undertaken, what will be the current price of a share? What will be the price of a share 1 year from now? What will be the price of a share 2 years from now (at \(date 2\))? 

2. (12 marks)
Fred has invested his entire wealth in a new luxury car that is valued at \(W_0 = 50\). In the next little while there is a probability = 0.20 that Fred will be involved in an automobile accident. If this occurs, the car will suffer damages of 10 and Fred's wealth will fall to \(W = 40\). With probability = 0.80 there will be no accident and Fred's wealth will remain at \(W = 50\). Fred is considering buying automobile insurance and has a choice of two different policies.
Policy A: For a price of 2.5 this policy will pay Fred 10 in the event of an accident (and 0 otherwise).

Policy B: For a price of 1.8 this policy will pay Fred 8 in the event of an accident (and 0 otherwise).

Fred's utility function is $U(W) = -\frac{1}{W}$. Will he choose to buy insurance and, if so, which of the two policies will he purchase?

3. (18 marks)
An individual has initial wealth $W_0 = 50$ with certainty and faces a gamble with the following payoffs:

- with probability 0.4 the gamble pays +10
- with probability 0.6 the gamble pays +20.

(a) (8 marks)
If the individual's utility function is $U(W) = \ln(W)$, what is the maximum amount she will pay to exchange the gamble for its expected value?

(b) (10 marks)
Suppose the individual has the following mean/variance utility function:

$$E[U(W)] = 5E[W] - Var[W].$$

Note here that $Var[W]$ is the variance of wealth.

What is the maximum amount she will pay to take the gamble?

4. (16 marks)
Investors form portfolios by combining one risky security with a risk-free security. The securities' returns are described by the following.

$$r_f = 0.03 \quad \bar{r}_1 = 0.15 \quad \sigma_1 = 0.25$$

Let $w_1$ denote the share of wealth invested in the risky security and determine the optimal portfolio for each of the following investors.

Investor A has expected utility: $E[U] = \bar{r} - 4\sigma^2$

Investor B has expected utility: $E[U] = \frac{1}{2} \bar{r}^2 - \frac{1}{2} \sigma^2$.

[Note: Please measure $r$'s and $\sigma$ as decimals.]
5. (20 marks)
Two risky securities have rates of return described by the following.

\[
E[r_1] = 0.18 \quad E[r_2] = 0.10 \\
\sigma_1 = 0.20 \quad \sigma_2 = 0.15 \\
\text{cov}[r_1, r_2] = +0.012
\]

Consider portfolios consisting of only these 2 securities with a share \( w_1 \) invested in security 1.

(a) (8 marks)
Find \( \bar{r}_p \) and \( \sigma_p \) for the portfolio that has the minimum variance.

(b) (4 marks)
Use your solution to Part (a) to describe the "efficient frontier". [Even if your answer to Part (a) is incorrect, you can still earn full marks on Part (b)].

(c) (8 marks)
If we now add a risk-free security with \( r_f = 0.04 \), the efficient frontier becomes a straight line in \( (E[r], \sigma) \) space that starts at \( r_f \) and is just tangent to the efficient frontier of the two risky securities. Suppose the tangency occurs at the value \( w_1 = 0.75 \). What is the equation for the straight line efficient frontier?

6. (14 marks)
In a world in which CAPM is valid, the following table describes the expected returns on two stocks for two different values for the market return.

<table>
<thead>
<tr>
<th>Market Return</th>
<th>Return on Stock A</th>
<th>Return on Stock B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>-0.06</td>
<td>0.036</td>
</tr>
<tr>
<td>0.18</td>
<td>0.30</td>
<td>0.108</td>
</tr>
</tbody>
</table>

(a) What are the betas of the two stocks?

(b) If the market rate of return is equally likely to be 0.00 or 0.18, what are the expected returns on the two stocks? What must be the value of the risk-free rate of return?
Question 1.
(a) With constant interest rate \( r \) and constant dividend growth rate \( g \), the share price at any date \( t \) is 
\[
e_t = \frac{Div_t}{r - g}.
\]
At date 0 (the present time) 
\[
e_0 = \frac{\$48}{0.06 - 0.02} = \$1,200.00.
\]
One year from now (at date 1) the share price will be 
\[
e_1 = \frac{\$48(1.02)}{0.04} = 1.02e_0 = \$1,224.00.
\]
(b) Undertaking the investment will cause dividends to change for each of the first three years as follows:

\[
\Delta Div_0 = \$10 \quad \Delta Div_1 = \$10 \quad \Delta Div_2 = \$10 - \$24(1.06)^3.
\]
This will cause the share price at date 0 to change by the present discounted value of the changes in dividends.

\[
\Delta e_0 = \frac{\$10}{(1.06)^0} + \frac{\$10}{(1.06)^2} + \frac{\$10}{(1.06)^3} - \$24 = \$2.73.
\]
The new level of the share price will be 
\[
e_0 = \$1,200.00 + \$2.73 = \$1,202.73.
\]

Similarly, 
\[
\Delta e_1 = \frac{\$10}{(1.06)^1} + \frac{\$10}{(1.06)^2} - \$24(1.06) = -\$7.11.
\]
Then the new level of this share price will be 
\[
e_1 = \$1,224.00 - \$7.11 = \$1,216.89.
\]
Finally, the new date 2 share price will be
\[
e_2 = \frac{\$10 - \$24(1.06)^3}{(1.06)} + (1.02)\$1,224.00 = \$1,230.95.
\]

Question 2.
(This question can be answered either by comparing expected utilities across the alternatives or by comparing certainty equivalents of the alternatives. It is not necessary to do both.)

With no insurance: 
\[
E[U] = -0.2 \frac{1}{40} - 0.8 \frac{1}{50} = -0.02100 \quad \text{and} \quad CE = 47.619.
\]

With Policy A: 
\[
E[U] = -\frac{1}{50 - 2.5} = -0.021053 \quad \text{and} \quad CE = 47.5.
\]

With Policy B: 
\[
E[U] = -0.2 \frac{1}{48 - 1.8} - 0.8 \frac{1}{50 - 1.8} = -0.0202926 \quad \text{and} \quad CE = 47.49.
\]
Fred will choose Policy B because it yields the greatest expected utility (certainty equivalent).
Question 3.
(a) For this utility function \( E[U] = 0.4 \ln(50 + 10) + 0.6 \ln(50 + 20) = 4.1868349 \). The certainty equivalent of this is \( CE = \exp(4.1868349) = 65.814 \). If she takes the gamble she will have certainty equivalent wealth of 65.814. If she exchanges the gamble for its expected value of 16, she will have certain wealth of 50 + 16 = 66. The maximum she will pay to make this exchange is 66 – 65.814 = 0.186.

(b) In order to use this utility function we must first compute \( E[W] = 66 \) and \( \text{Var}[W] = 24 \). Then the expected utility from taking the gamble is \( E[U] = 5(66) - 24 = 306 \). The certainty equivalent of this satisfies \( E[U] = [5(CE) + 0] = 306 \); therefore, \( CE = 61.2 \). If she does not take the gamble her wealth is 50 with certainty; if she does take it, her certainty equivalent wealth is 61.2. The maximum she will pay to take the gamble is the difference, 61.2 – 50 = 11.2.

Question 4.
There are 2 different ways of solving this problem, each of which leads to the same solution. First, we can find the slope of an indifference curve as a function of \( w_1 \) for each utility function, set this equal to the slope of the straight-line efficient frontier, then solve for optimal \( w_1 \). Alternatively, we can replace \( \bar{r} \) and \( \sigma \) in the utility function with \( \left[ w_i \bar{r}_i + (1 - w_i) r_f \right] \) and \( w_i \sigma_i \), respectively, and then find the value of \( w_1 \) that makes \( \frac{\partial E[U]}{\partial w_1} = 0 \).

The slope of the straight-line efficient frontier is \( S = \frac{\bar{r}_i - r_f}{\sigma_i} = 0.48 \).

For Investor A write the utility function as \( \bar{r}_p = E[U] + 4 \sigma_p^2 \). Along an indifference curve \( E[U] \) is constant, so the slope is \( \frac{\partial \bar{r}_p}{\partial \sigma_p} = 8 \sigma_p = 8w_1 \sigma_1 = 2w_1 \).

The optimal portfolio has \( 2w_1 = 0.48 \), or \( w_1 = 0.24 \). (Investor A should invest 24 % of wealth in the risky security and the remaining 76 % of wealth in the risk-free security).

Alternatively, write the utility function as \( E[U] = w_i \bar{r}_i + (1 - w_i) r_f - 4w_i^2 \sigma_i^2 \). Maximizing directly yields \( \frac{\partial E[U]}{\partial w_1} = 0 = (\bar{r}_i - r_f) - 8w_1 \sigma_i^2 = 0.12 - 0.5w_1 \),

which implies that optimal \( w_1 = 0.24 \).

For Investor B write the slope of an indifference curve can be obtained by taking the total differential of the expected utility function:
Along an indifference curve, \( dE[U] = 0 \), so the slope of an indifference curve is

\[
\frac{d\bar{r}_p}{d\sigma_p} = \frac{\sigma_p}{\bar{r}_p} = \frac{w_1\sigma_i}{w_1\bar{r}_i + (1 - w_1)r_f} = \frac{0.25w_i}{0.03 + 0.12w_i}.
\]

Set this equal to the slope \( S \) and determine that optimal \( w_1 = 0.075 \) for Investor B.

Alternatively, write the utility function as

\[
E[U] = \frac{1}{2}[w_1\bar{r}_i + (1 - w_1)r_f]^2 - \frac{1}{2}w_1^2\sigma_i^2.
\]

Maximizing directly yields

\[
\frac{\partial E[U]}{\partial w_1} = 0 = (\bar{r}_i - r_f)[w_i\bar{r}_i + (1 - w_i)r_f] - w_1\sigma_i^2.
\]

Inserting the given values:

\[
0.12[0.03 + 0.12w_i] - 0.0625w_i = 0,
\]

which implies that optimal \( w_1 = 0.075 \).

**Question 5.**

(a) As per a classroom handout, the value of \( w_1 \) for the minimum variance portfolio satisfies

\[
w_i^{\text{min var}} = \frac{\sigma_2^2 - \text{Cov}(r_1, r_2)}{\sigma_1^2 + \sigma_2^2 - 2\text{Cov}(r_1, r_2)}.
\]

[Alternatively, this expression may be derived from the first-order condition \( \frac{\partial \sigma_p^2}{\partial w_1} = 0 \).]

In any event, here \( w_i^{\text{min var}} = 0.2727 \), which yields \( \bar{r}_p^{\text{min var}} = 0.1218 \) and \( \sigma_p^{\text{min var}} = 0.1401 \).

(b) The efficient frontier is the set of all portfolios that have \( \bar{r}_p \geq \bar{r}_p^{\text{min var}} = 0.1218 \). Alternatively, the efficient frontier is the set of all portfolios with \( w_1 \geq w_i^{\text{min var}} = 0.2727 \).

(c) The tangency portfolio (with \( w_T = 0.75 \)) has \( \hat{r}_T = 0.016 \) and \( \sigma_T = 0.1685 \).

The equation for the efficient frontier is

\[
\bar{r}_p = 0.04 + 0.7122\sigma_p.
\]
Question 6.
(a) For CAPM the expected return on any security \( j \) satisfies

\[
\bar{r}_j = r_j + \beta_j (\bar{r}_m - r_j).
\]

Then, for changes in expected returns \( \Delta \bar{r}_j = \beta_j \Delta \bar{r}_m \), which may be rewritten as

\[
\beta_j = \frac{\Delta \bar{r}_j}{\Delta \bar{r}_m}.
\]

Applying this equation to data given in the table:

\[
\beta_A = \frac{0.30 - (-0.06)}{0.18} = 2.0 \quad \text{and} \quad \beta_B = \frac{0.108 - 0.036}{0.18} = 0.4.
\]

(b) \( \bar{r}_A = 0.12 \) and \( \bar{r}_B = 0.072 \).

Using Equation (1) from Part (a), we can compute the risk-free rate from

\[
r_f = \frac{\bar{r}_j - \beta_j \bar{r}_m}{1 - \beta_j}.
\]

It does not matter whether we use data for Stock A or Stock B. Each yields the solution \( r_f = 0.06 \).