Optimal Delegation with Limited Commitment

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Abstract. In this paper we introduce the model of delegation mechanisms under limited commitment. A privately informed agent is delegated by a principal to choose from a pre-specified set of actions, with the principal committed to not changing the agent's action to any action outside the set but free to switch to some action within the set. For the uniform-quadratic example first introduced by Crawford and Sobel (1982), we characterize the optimal delegation mechanism under limited commitment. In an optimal mechanism, the principal specifies fewer actions in the delegation set in order to relax the incentive compatibility constraints due to limited commitment.

1. Introduction

In the framework of Holmstrom (1984), Melumad and Shibano (1991), and most recently Alonso and Matouschek (2008), a delegation mechanism is a set of actions that the principal allows the agent to choose from. In one trivial case, the principal chooses a singleton action set, which is the same as dictating the decision (making an uninformed decision); in another special case, the principal can choose the set of discrete actions induced in the corresponding cheap talk game. In either case, there is no incentive for the principal to overrule the decision—commitment is not an issue. However, optimal delegation mechanisms in general are not robust to lack of commitment.

Comparison between delegation and communication, for example by Dessein (2002) and by Gilligan and Kriehbel (1987), can be seen as studies on the issue of delegation without full commitment. However, when all delegation mechanisms are allowed, communication is trivially dominated, which makes the problem uninteresting (which is also the reason that Dessein (2002) looked at the comparison between communication and unrestricted delegation, ie letting the agent decide). Similarly, Gilligan and Kriehbel's (1987) paper looks at the issue of how to provide incentives for information acquisition by committing not to overrule the decision, but again only unrestricted delegation is considered.

The issue of design of delegation mechanisms becomes more interesting when we have an environment that is between full commitment and no commitment. When there is no commitment, the principal keeps the flexibility of reacting to the agent's action according to the information revealed by the agent's choice, but since this is anticipated by the agent, the quality of information revealed by the agent's is poor and so is the quality of the principal's decision in the end. Under full commitment, the principal completely gives up the flexibility of reacting to the agent's choice, and the optimal delegation mechanism under full commitment optimally balances the loss of making ex post suboptimal decisions with the gain from making better use of the agent's information. Under limited commitment, the principal cannot credibly commit to completely giving up the flexibility, and this affects the optimal balance between the loss of making ex post suboptimal decisions with the gain from making better use of the agent's information.

This note examines one way of thinking about limited commitment. To begin with, it is useful to separate two ingredients in the assumption of full commitment: the ability to make certain actions infeasible, both for the agent and for the principal himself; and the ability to commit not to change the action chosen by the agent to another action that has not been excluded. It's possible to argue that the first ingredient is a more realistic feature of commitment, while the second one is harder or even impossible to accomplish. If so, it is interesting to look at a model of limited delegation in which a delegation mechanism is the set of actions that the agent can choose from and the principal can change to. In this model, there is no parameter that represents the degree of commitment, but it may still be interesting to look at how much discretion that the agent is optimally given.

In the same setup as in the present paper, a few recent papers have tried to address the issue of commitment (Dessein, 2002; Marino, 2007; Mylovanov, 2008). These papers have a different definition of limited commitment: they assume that the principal may not be able to commit to not vetoing the agent's choice when there is a default action that the principal could take ex post. Mylovanov (2008) reconciles contradictory findings in Dessein (2002) and Marino (2007) by showing that if the principal can choose a default action ex ante, then the optimal delegation mechanism under full commitment (for the case of uniformly distributed signal of the agent and quadratic preferences for the agent and the principal) can be implemented veto-free, so that the inability to commit to not taking the default action ex post is inconsequential.

2. Limited Commitment in Delegation Mechanisms

We will introduce the model using the uniform-quadratic example of the setup of Crawford and Sobel (1982). As will be clear from the description, both the concept of delegation mechanisms under limited commitment and the approach to characterizing the optimal mechanism are immediately applicable to more general structures of payoff and information.

2.1. A uniform-quadratic model

Consider the uniform-quadratic example of the setup of Crawford and Sobel (1982). An agent is privately, and fully informed of the state of the world s, which is uniformly distributed on $[0, 1]$. A principal needs to choose an action $a \in \mathbb{R}$. For any s and a, the payoff to the agent is $-(a - (s + \beta))^2$, and the payoff to the principal is $-(a - s)^2$, where $\beta > 0$ is a parameter that represents an upward bias of the agent relative to the principal.

A delegation mechanism is a subset $A \subset \mathbb{R}$. Under limited commitment, any A induces the following delegation game. First, the state s is realized and revealed to the agent privately; the agent then chooses an action $a \in A$; next the principal decides either to approve the agent's choice, or to overrule the agent and choose another action $a' \in A$; either decision by the principal ends the game and yields the corresponding payoffs to the agent and the principal according to the quadratic loss functions described above.

2.2. Optimal delegation mechanism

The main goal of this note is to characterize the optimal delegation mechanism under limited commitment. It is also interesting to understand how the optimal delegation mechanism changes with the bias of the agent.

A "delegation outcome" of a delegation mechanism A is a mapping from the state space $[0, 1]$ to A induced by a perfect Bayesian equilibrium of the delegation game associated with A. A delegation outcome d of some delegation mechanism A is minimal and veto-free if for each $a \in A$, there is $s \in [0, 1]$ such that $d(s) = a$, and if there is a perfect Bayesian equilibrium of the delegated game associated with A where the equilibrium strategy of the agent coincides with d and the equilibrium strategy of the principal is an identity mapping from A to itself.

A straightforward revelation-principle type of argument establishes that any delegation outcome of some delegation mechanism can be replicated by a minimal and veto-free delegation outcome. In searching for the optimal delegation mechanism under limited commitment, it is without loss of generation to restrict to minimal and veto-free delegation outcomes.

Any minimal and veto-free delegation outcome associated with some delegation mechanism A is characterized by the following two conditions. First, for each $a \in A$, there is a subset $I(a)$ of the state space $[0,1]$ such that $I(a) \cap I(a') = \emptyset$ for all $a, a' \in A$ and $a \neq a'$, and $\cup_{a \in A} I(a) = [0,1]$. Second, for any $a \in A$ and any $s \in I(a)$, the action a is weakly optimal in A for the agent conditional on the state being s and for the principal conditional on the state being in $I(a)$.

In the present uniform-quadratic model, any minimal and veto-free delegation outcome is characterized by a finite set $A = \{a_1, \ldots, a_n\}$ and the corresponding intervals $I(a_i) = [t_{i-1}, t_i)$ for each $i = 1, \ldots, n-1$, with $t_0 = 0$, and $I(a_n) = [t_{n-1}, t_n]$ with $t_n = 1$, such that: (i) (agent-IC) for each $i = 1, \ldots, n-1$, at $s = t_i$, the agent is indifferent between a_i and a_{i+1}, or

$$(t_i + \beta) - a_i = a_{i+1} - (t_i + \beta); \tag{2.1}$$

and (ii) (principal-IC) for each $i = 2, \ldots, n$, the principal weakly prefers a_i to a_{i-1} conditional on $s \in [t_{i-1}, t_i)$. The program of optimal delegation mechanism design under limited commitment is to find a finite set A such that conditions (i) and (ii) above are satisfied.

In setting up the above program, we have made the following claims: (i) any minimal and veto-free delegation mechanism is finite (has a finite number of actions); (ii) given any such mechanism the agent's incentive compatibility constraints are all satisfied, if the delegation outcome is a partition of the state space into the same number of intervals with the thresholds of the partition determined by the agent's indifference conditions; and (iii) the principal's incentive constraints are all satisfied if, given the agent's thresholds determined according to (2.1), the principal has no incentive to replace any action chosen by the agent by the adjacent lower action. These claims can be established in a straightforward manner.

3. Two Benchmarks

To characterize the optimal delegation mechanism under limited commitment, it is useful to first consider two benchmarks. These two benchmarks correspond respectively to a lower bound and an upper bound on what can be achieved for the principal under limited commitment.

3.1. No commitment

A lower bound on the optimal delegation mechanism under limited commitment is no commitment. This is achieved by setting $A = \mathbb{R}$ and replicating the most informative equilibrium in the cheap talk game of Crawford and Sobel (1982).

As is well-known, the bias β determines a non-negative integer $N^{NC}(\beta)$, such that for each $n = 1, \ldots, N(\beta)$, there is a cheap talk equilibrium corresponding to a partition of the state space $[0, 1]$ into n intervals with a single equilibrium action taken for all states in each interval, and there are no other equilibrium outcomes. Since the principal's payoff is increasing (loss is decreasing) in the number of intervals in an equilibrium, the lower bound on the optimal delegation mechanism under limited commitment corresponds to the cheap talk equilibrium with $N^{NC}(\beta)$ intervals, and we refer to the equilibrium as the "optimal" delegation mechanism under no commitment. For the sake of comparison, the cheap talk equilibrium for each $n = 1, \ldots, N^{NC}(\beta)$ is characterized in terms of the corresponding delegation mechanism.

Fix any $n \leq N^{NC}(\beta)$ (to be characterized below). Denote the $n - 1$ thresholds of the equilibrium with n partition elements as $t_1, \ldots, t_{n-1}$, and let $t_0 = 0$ and $t_n = 1$. Then, the corresponding delegation mechanism is given by $A^{NC}(n) = \{a_1, \ldots, a_n\}$ where, for each $i = 1, \ldots, n - 1$,

$$a_i = \frac{1}{2}(t_{i-1} + t_i). \tag{3.1}$$

In the corresponding equilibrium of the game induced by $A^{NC}(n)$, there is a perfect Bayesian equilibrium such that for each $i = 1, \ldots, n - 1$, the agent chooses a_i for any state $s \in [t_{i-1}, t_i]$, and the principal approves all choices by the agent. Thus, the cheap talk equilibrium with n intervals corresponds to a minimal and veto-free delegation outcome with n actions.

For each $i = 1, \ldots, n - 1$, at $s = t_i$, the agent is indifferent between a_{i-1} and a_i, implying

$$(t_i + \beta) - \frac{1}{2}(t_{i-1} + t_i) = \frac{1}{2}(t_i + t_{i+1}) - (t_i + \beta),$$

or

$$t_{i+1} = 2t_i - t_{i-1} + 4\beta. \tag{3.2}$$

By successive iterations, we have

$$t_i = it_1 + 2i(i-1)\beta \tag{3.3}$$

for each $i = 2, \ldots, n$. Using $t_n = 1$, we can solve for t_1 as

$$t_1 = \frac{1}{n} - 2(n-1)\beta. \tag{3.4}$$

We have found an equilibrium if $t_1 > 0$, which is equivalent to

$$\beta < \beta^{NC}(n) \equiv \frac{1}{2n(n-1)}. \tag{3.5}$$

The corresponding delegation mechanism can be then found by using equations (3.3) and (3.1). By definition, $N^{NC}(\beta)$ is the largest non-negative integer n that satisfies the above inequality.

Since the delegation mechanism $A^{NC}(n)$ represented by the cheap talk equilibrium with n intervals is incentive compatible for the principal, the number $N^{NC}(\beta)$ provides a lower bound on the number of actions in the optimal delegation mechanism under limited commitment. This is true so long as the principal's expected loss is decreasing in the number of actions in the delegation mechanism, as can easily verified. If $\beta \geq \frac{1}{4}$, there is no informative cheap talk equilibrium, and correspondingly the optimal delegation mechanism under no commitment is equivalent to uninformed dictation of the decision by the principal. We can write $\beta^{NC}(2) = \frac{1}{4}$. We summarize the above discussion as the following lemma.

LEMMA 3.1. *For any $\beta < \frac{1}{4}$, the optimal delegation mechanism under no commitment is given by $A^{NC}(n)$, where $n = N^{NC}(\beta)$.*

For any fixed β, the relation between each t_i and t_1, equation (3.3), is the same for all n. To have a cheap talk equilibrium with a greater n, the first threshold t_1 must decrease to squeeze in more thresholds according to the same relation (3.3) in order to satisfy the necessary condition that $t_n = 1$. This eventually becomes impossible when n is greater than $N^{NC}(\beta)$.

Fix any n and β such that $\beta < \beta^{NC}(n)$, or equivalently $n < N^{NC}(\beta)$, and consider changes in the delegation mechanism $A^{NC}(n)$ as β decreases. Note that as β decreases,

$N^{NC}(\beta)$ increases so it remains true that $n < N^{NC}(\beta)$. By equation (3.4), the first threshold t_1 increases as β decreases. By equation (3.3), all thresholds move to the right, and so do the actions in the delegation mechanism $A^{NC}(n)$ according to (3.1). When $\beta = 0$, the thresholds become evenly spaced on $[0, 1]$, and so are the actions in $A^{NC}(n)$. In contrast, from equation (3.2), for any positive β, each interval is larger in size than the adjacent interval to the left by a constant 4β. By equation (3.1), the difference between two adjacent actions in the corresponding delegation mechanism also increases by the constant 4β in the direction of the agent's bias (higher actions); that is,

$$a_{i+1} - a_i = a_i - a_{i-1} + 4\beta$$

for each $i = 2, \ldots, n-1$.

The above comparative analysis suggests that the principal would like to somehow make the thresholds and correspondingly the actions more evenly spaced, starting from the above delegation mechanism induced by the cheap talk equilibrium with n actions. Indeed, in the delegation mechanism derived from a cheap talk equilibrium, the principal's incentive constraints, condition (ii) in the program of optimal delegation mechanism, are not binding. That is, for each $i = 2, \ldots, n$, the action a_i is in fact optimal conditional on $s \in [t_{i-1}, t_i)$, while all that is required is that the principal weakly prefer a_i to the adjacent lower action a_{i-1}. From the principal's point of view, the problem with a delegation mechanism with higher actions spaced at increasingly large distances is that the information quality of these higher actions is poor. With any commitment power, the principal could credibly adjust the distances of the actions and make them more evenly spaced. This would introduce ex post inefficient actions, but the cost of inefficiency is of second order starting from the ex post efficiency of the cheap talk equilibrium, while the gain from making more informative action choices would be of first order.

3.2. Full commitment

An upper bound on the optimal delegation mechanism under limited commitment is full commitment. In this section we consider the problem of minimizing the expected loss to the principal by choosing a delegation mechanism with a fixed finite number n of actions, subject only to the incentive constraints of the agent (condition (i) in the program of optimal delegation mechanism). Solution to this hypothetical problem has the interpretation of the case of full commitment (with the restriction of a finite number n of actions), and is informative about the case of limited commitment.

The choice variables are the n actions $a_1, \ldots, a_n$. The objective function is

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (a_i - s)^2 \, \mathrm{d}s, \tag{3.6}$$

where $t_1, \ldots, t_{n-1}$ are the thresholds satisfying condition (i) in the program of optimal delegation mechanism, or equivalently

$$t_i = \frac{1}{2}(a_i + a_{i+1}) - \beta, \tag{3.7}$$

for each $i = 1, \ldots, n-1$, with $t_0 = 0$ and $t_n = 1$.

The first order condition with respect to a_1 is:

$$a_1^2 - (t_1 - a_1)^2 + \frac{1}{2}(t_1 - a_1)^2 - \frac{1}{2}(a_2 - t_1)^2 = 0.$$

The first and the second terms alone would imply that $a_1 = \frac{1}{2}t_1$, which is the optimal action given that the agent has revealed that $s \in [0, t_1]$. The third and the fourth terms capture the effect on the principal's objective of raising a_1 through the indifference condition, equation (3.7) for $i = 1$: raising a_1 increases t_1, and since $t_1 < \frac{1}{2}(a_1 + a_2)$ by the indifference condition, this yields a benefit to the principal. Thus, from this first order condition we already know that a_1 in the optimal delegation mechanism under full commitment is higher than what is ex post optimal. We can rewrite the first order condition with respect to a_1 as

$$a_1^2 = \frac{1}{2}(t_1 - a_1)^2 + \frac{1}{2}(a_2 - t_1)^2. \tag{3.8}$$

For each $i = 2, \ldots, n-1$, the first order condition with respect to a_i can be written as

$$\frac{1}{2}(t_{i-1} - a_{i-1})^2 + \frac{1}{2}(a_i - t_{i-1})^2 = \frac{1}{2}(t_i - a_i)^2 + \frac{1}{2}(a_{i+1} - t_i)^2. \tag{3.9}$$

Finally, the first order condition with respect to a_n is

$$\frac{1}{2}(t_{n-1} - a_{n-1})^2 + \frac{1}{2}(a_n - t_{n-1})^2 = (1 - a_n)^2. \tag{3.10}$$

As is true for a_1, these first order conditions imply that in the optimal delegation mechanism under full commitment, all actions $a_2, \ldots, a_n$ are higher than their corresponding ex post optimal level.

For each $i = 2, \ldots, n-1$, using the constraints (3.7) for i and $i-1$, from the first order condition (3.9) for i we obtain

$$a_i = \frac{1}{2}(a_{i-1} + a_{i+1}). \tag{3.11}$$

By successive iteration, we have

$$a_i = (i-1)a_2 - (i-2)a_1 \tag{3.12}$$

for each $i = 2, \ldots, n$. Next, using all the first order conditions (3.8) through (3.10), we have

$$a_1 = 1 - a_n. \tag{3.13}$$

Thus, in contrast with the delegation mechanism under no commitment, the actions $a_2, \ldots, a_{n-1}$ are evenly spaced in the interval $[a_1, a_n]$, and further more, a_1 and a_n are positioned at equal distance from the end points 0 and 1 respectively.

Combining equation (3.12) for $i = n$ and (3.13), we obtain

$$(n-1)a_2 - (n-3)a_1 = 1. \tag{3.14}$$

Using equation (3.7) for $i = 1$ and (3.14), we can solve for a_1 and a_2 in terms of t_1:

$$\begin{aligned} a_1 &= \frac{2(n-1)(t_1+\beta) - 1}{2(n-2)}; \\ a_2 &= \frac{2(n-3)(t_1+\beta) + 1}{2(n-2)}. \end{aligned} \tag{3.15}$$

Substituting the above two equations into the first order condition (3.8), we obtain the following quadratic equation in t_1:

$$nt_1^2 + (2n\beta - 1)t_1 + 2\beta^2 - \beta = 0.$$

The solution to the above equation is valid if and only if it is positive. It is straightforward to see that if $\beta \geq \frac{1}{2}$, there is no valid solution to the above quadratic equation for any n; while if $\beta < \frac{1}{2}$, for any n only one of the two roots is valid and is given by

$$t_1 = \frac{1}{2n}\left(\sqrt{1 + 4n(n-2)\beta^2} + 1 - 2n\beta\right). \tag{3.16}$$

The optimal delegation mechanism $A^{FC}(n)$ with n actions under full commitment is then determined by equations (3.15) and (3.12).

It is interesting to compare the delegation mechanisms $A^{NC}(n)$ under no commitment and under full commitment $A^{FC}(n)$ for the same n, assuming that $n < N^{NC}(\beta)$. In both cases, the agent's thresholds satisfy the indifference conditions given the actions. Under no commitment, the principal's actions are ex post optimal given the information inferred from the agent's choices, resulting in cheap talk equilibrium actions that are increasingly farther apart in the direction of the agent's bias (higher actions). Under full commitment, the principal's actions are instead chosen to take into account the effects on the thresholds through the agent's indifference conditions (3.7), resulting in evenly spaced actions and thresholds. As the bias parameter β decreases, the two delegation mechanisms become closer to each other.

As in the case of no commitment, if there is a delegation mechanism under full commitment with n distinct actions that satisfies all the first order conditions, then there is also a delegation mechanism under full commitment with fewer actions that satisfies all the corresponding first order conditions. However, unlike the case of no commitment, under full commitment for any finite n there is a delegation mechanism with n actions that satisfies the first order conditions when $\beta < \frac{1}{2}$. Thus, the optimal delegation mechanism under full commitment is to make n arbitrarily large. It is clear from the solution given by (3.16) and equations (3.15) that as n goes to infinity, the smallest threshold t_1 converges to 0, and the smallest two actions a_1 and a_2 converge to β. By equations (3.11) and (3.13), all actions are evenly spaced on the interval $[\beta, 1-\beta]$. In the limit of n going to infinity, the optimal delegation mechanism with n actions, $A^{FC}(n)$, converges to the interval $[\beta, 1-\beta]$. This coincides with the solution given by Melumad and Shibano (1991). The above analysis is summarized in the following lemma.

Lemma 3.2. *For any $\beta < \frac{1}{2}$ and for any $n \geq 2$, the optimal delegation mechanism under full commitment is given by $A^{FC}(n)$.*

It is of some interest to find the optimal delegation mechanism with 2 actions under full commitment. The above solution formulas do not completely apply, as can be seen from equations (3.14) and (3.15). Instead, we can find t_1, a_1 and a_2 directly from (3.7) for $i = 1$, the first order condition (3.8) for a_1, and the first order condition (3.10) for a_n with $n = 2$. The solution is given by $A^{FC}(2) = \{a_1, a_2\}$ such that $t_1 = \frac{1}{2} - \beta$, $a_1 = \beta^2 + \frac{1}{4}$ and $a_2 = 1 - a_1$. It turns out that at the solution, the principal's incentive compatibility constraint is satisfied: the principal never has any incentive to replace the agent's choice of a_2 with a_1, as

$$\frac{1}{2}(1 + t_1) - a_1 > a_2 - \frac{1}{2}(1 + t_1)$$

regardless of β.

In general, $A^{FC}(n)$ does not satisfy the principal's incentive compatibility constraints (condition (ii) in the program of optimal delegation mechanism under limited commitment). To begin, observe that we have already established from the first order conditions (3.8) through (3.10) that $a_i > \frac{1}{2}(t_i + t_{i-1})$ for each $i = 1, \ldots, n$. It follows that necessary and sufficient conditions for the principal's incentive compatibility constraints to be satisfied are, for each $i = 1, \ldots, n-1$,

$$\frac{1}{2}(t_i + t_{i+1}) - a_i \geq a_{i+1} - \frac{1}{2}(t_i + t_{i+1}), \tag{3.17}$$

or equivalently

$$t_{i+1} - t_i \geq 2\beta.$$

For all $i = 1, \ldots, n-2$, using the indifference conditions (3.7) and the relations (3.12), we can verify that the above inequalities reduce a single condition

$$a_2 - a_1 \geq 2\beta. \tag{3.18}$$

For the principal's incentive compatibility constraint $t_n - t_{n-1} \geq 2\beta$, we need a separate argument because $t_n = 1$ does not satisfy any incentive compatibility constraint for the agent. Using the indifference condition (3.7) for t_{n-1}, equation (3.13), and then the relation (3.12) for a_{n-1}, we can rewrite the requirement $t_n - t_{n-1} \geq 2\beta$ as $a_2 + a_1 \geq 2\beta$, which is implied by (3.18). Thus, condition (3.18) is both necessary and sufficient for the optimal delegation mechanism under full commitment for fixed n to satisfy the principal's incentive compatibility constraints under limited commitment.

Using equations (3.15) and the solution t_1 in equation (3.16), we can write condition (3.18) as

$$\sqrt{1 + 4n(n-2)\beta^2} < (n-1) - 2n(n-2)\beta. \tag{3.19}$$

A necessary condition for the above to hold is

$$\beta < \frac{n-1}{2n(n-2)}. \tag{3.20}$$

Provided that (3.20) is satisfied, after some calculations we can show that (3.19) is equivalent to

$$\beta < \beta^{FC}(n) \equiv \frac{1}{2(n-1+\sqrt{2})}. \tag{3.21}$$

It can be verified that (3.21) implies (3.20). Let $N^{FC}(\beta)$ be the largest positive integer n such that (3.21) holds. Then, for any $n < N^{FC}(\beta)$ the optimal delegation mechanism with n actions under full commitment satisfies the principal's incentive compatibility constraints.

Finally, we may ask whether it is true that $N^{FC}(\beta) \geq N^{NC}(\beta)$, that is, for any β, whether the maximum number of actions in the optimal delegation mechanism under full commitment that is incentive compatible for the principal is at least as great as the maximum number of actions in the optimal delegation mechanism under no commitment. Compare (3.21) with (3.5). Then $N^{FC}(\beta) \geq N^{NC}(\beta)$ if

$$\left(\frac{1}{2\beta} - \sqrt{2}\right)\left(\frac{1}{2\beta} - \sqrt{2} + 1\right) \geq \frac{1}{2\beta},$$

which is equivalent to

$$\left(\frac{1}{2\beta} - \sqrt{2}\right)^2 \geq \sqrt{2}.$$

The above holds strictly when $\beta \leq \frac{1}{4}(2-\sqrt{2})$, or equivalently, if $N^{FC}(\beta) \geq 3$. Further, $N^{FC}(\beta) = 2$ for any $\beta \in \left(\frac{1}{2}, \frac{1}{4}(2-\sqrt{2})\right)$, while $N^{NC}(\beta) = 1$ for any $\beta > \frac{1}{4}$ and by (3.5) we have $N^{NC}(\beta) = 2$ for $\beta \in \left(\frac{1}{4}, \frac{1}{12}\right)$. Since $\frac{1}{12} < \frac{1}{4}(2-\sqrt{2})$, we have $N^{FC}(\beta) \geq N^{NC}(\beta)$ for all β. It follows that $N^{FC}(\beta)$ is generally a tighter lower bound than $N^{NC}(\beta)$ on the number of actions in the optimal delegation mechanism under limited commitment.

4. Limited Commitment

If the optimal delegation mechanism with n actions under full commitment derived in the previous section satisfies the principal's incentive compatibility constraints (condition (ii) in the program of optimal delegation mechanism under limited commitment), or equivalently, if $n < N^{FC}(\beta)$, then it is the optimal delegation mechanism with n actions under limited commitment. However, to find the optimal delegation mechanism under limited commitment, we need to identify the number of actions in the mechanism.

We will try to find the optimal delegation mechanism under limited commitment in two steps. First, we solve for the optimal delegation mechanism $A^{LC}(n)$ for each fixed n. The solution is clearly related to the optimal delegation mechanism under full commitment characterized in the previous section. Second, we try to establish the optimal delegation mechanism under limited commitment by allowing n to vary.

Fix some n. The problem of optimal delegation mechanism under limited commitment shares the same objective function, the choice variables, and the agent's incentive compatibility constraints (3.7) with the problem of optimal delegation mechanism under full commitment stated in the previous section, with the addition of the incentive compatibility constraints of the principal, given by conditions (3.17), or equivalently through the agent's incentive compatibility constraints (3.7), by

$$a_{i+1} - a_{i-1} \geq 4\beta \tag{4.1}$$

for each $i = 2, \ldots, n-1$, together with

$$a_n + a_{n-1} \leq 2(1-\beta). \tag{4.2}$$

The last constraint takes a different form because $t_n = 1$ by definition which is not given by any of the agent's incentive constraints.

Let λ_i, $i = 2, \dots, n-1$, be the Lagrangian multiplier associated with the constraint (4.1), and let λ_n be the multiplier associated with (4.2). The first-order necessary conditions are, for a_1, given by

$$a_1^2 = \frac{1}{2}(t_1 - a_1)^2 + \frac{1}{2}(a_2 - t_1)^2 - \lambda_2; \tag{4.3}$$

for a_2, given by

$$\frac{1}{2}(t_1 - a_1)^2 + \frac{1}{2}(a_2 - t_1)^2 = \frac{1}{2}(t_2 - a_2)^2 + \frac{1}{2}(a_3 - t_2)^2 - \lambda_3; \tag{4.4}$$

for a_i, $i = 3, \dots, n-1$, given by

$$\frac{1}{2}(t_{i-1} - a_{i-1})^2 + \frac{1}{2}(a_i - t_{i-1})^2 - \lambda_{i-1} = \frac{1}{2}(t_i - a_i)^2 + \frac{1}{2}(a_{i+1} - t_i)^2 - \lambda_{i+1}; \tag{4.5}$$

and for a_n, given by

$$\frac{1}{2}(t_{n-1} - a_{n-1})^2 + \frac{1}{2}(a_n - t_{n-1})^2 - \lambda_{n-1} = (1 - a_n)^2 - \lambda_n. \tag{4.6}$$

The above first order conditions are also sufficient for optimality, as the objective function (3.6) is convex in the choice variables $a_1, \dots, a_n$, the agent's incentive compatibility constraints (3.7) are linear equality constraints, and the principal's incentive compatibility constraints (4.1) are linear and therefore convex.

We conjecture that (4.2) is not binding and $\lambda_n = 0$. Given this conjecture, adding up the first order conditions (4.3) through (4.6) yields $a_1 = 1 - a_n$, which is identical to the relation (3.13) under full commitment.

4.1. Odd mechanisms

Suppose for now that n is odd. There is a natural conjecture in this case about the values of the multipliers, such that there is a delegation mechanism $A^{LC}(n) = \{a_1, \dots, a_n\}$ that satisfies all first order necessary conditions and is tightly connected to the optimal delegation mechanism $A^{FC}(n)$ under full commitment. The conjecture is

$$\lambda_i = \lambda > 0$$

for each $i = 2, 4, \ldots, n-1$ even, and

$$\lambda_i = 0$$

for each $i = 3, 5, \ldots, n-2$ odd.

Given the above conjecture, from the first order necessary conditions (4.4) and (4.5) for $a_2, \ldots, a_{n-1}$, we immediately have (3.11). This means that as is the case under full commitment, all the actions are evenly placed in the interval $[a_1, a_n]$, with the two extreme actions satisfying (3.13). Under the conjecture, the principal's incentive compatibility constraints (4.1) are binding for each $i = 2, 4, \ldots, n-1$ even. Combing these constraints with (3.11) and using induction, we have

$$a_i = a_1 + 2(i-1)\beta \tag{4.7}$$

for each $i = 2, \ldots, n$. Note that this implies that all incentive compatibility constraints (4.1) for the principal are satisfied with equality. Together with the agent's incentive compatibility constraints (3.7), the above then implies

$$t_i = a_i \tag{4.8}$$

for each $i = 1, \ldots, n-1$. Since (3.13) holds, from (4.7) we immediately have the solution for a_1, given by

$$a_1 = \frac{1}{2} - (n-1)\beta, \tag{4.9}$$

and consequently for every a_i from (4.7).

For the above solution to be valid, first we need the value of λ that satisfies the first order conditions to be positive. All the first order conditions (4.3) through (4.6) reduce to a single one

$$\lambda = 2\beta^2 - a_1^2.$$

Substituting in the solution for a_1 in equation (4.9), we can write the above condition as

$$\beta > \frac{1}{2(n-1+\sqrt{2})},$$

which is the negation of (3.21). This makes sense, because if (3.21) is not satisfied, the optimal delegation mechanism $A^{FC}(n)$ under full commitment is incentive compatible for the principal and therefore coincides with the optimal delegation mechanism under limited commitment. We also need to verify that the ignored constraints are satisfied. The odd-numbered incentive compatibility constraints for the principal, conditions (4.1) for $i = 3, 5, \ldots, n-2$, are clearly satisfied as they hold with equalities by equations (4.7). The constraint (4.2) is equivalent to $a_n \leq 1$, because $a_{n-1} = a_n - 2\beta$ by (4.7), and is satisfied. Finally, the solution given by (4.9) and (4.7) is valid only if the implied threshold t_1 is positive. Since $t_1 = a_1$ by (4.8), we need

$$\beta < \frac{1}{2(n-1)}.$$

To summarize, when n is odd, the delegation mechanism $A^{LC}(n)$ defined by (4.9) and (4.7) satisfies all necessary conditions if

$$\frac{1}{2(n-1+\sqrt{2})} = \beta^{FC}(n) < \beta < \beta^{LC}(n) \equiv \frac{1}{2(n-1)}. \tag{4.10}$$

We refer to $A^{LC}(n)$ as an "odd mechanism." The following lemma summarizes the above derivation.

LEMMA 4.1. *For any $n \geq 3$ and odd, and $\beta \in \left(\beta^{FC}(n), \beta^{LC}(n)\right)$, the odd mechanism $A^{LC}(n)$ solves the principal's optimal delegation problem under limited commitment with n actions.*

The interpretation of an odd mechanism is clear. When (3.21) is violated so that the optimal delegation mechanism $A^{FC}(n)$ under full commitment is not incentive compatible for the principal, $A^{FC}(n)$ has too many actions $a_2, \ldots, a_{n-1}$ in the interval $[a_1, a_n]$, which is the same as $[a_1, 1-a_1]$. To satisfy the incentive compatibility constraints for the principal, the mechanism is adjusted such that a_1 is reduced and a_n raised, while keeping a_1 and a_n equally distanced from their respective end point and the other actions $a_2, \ldots, a_{n-1}$ even spaced in $[a_1, a_n]$. This adjustment becomes infeasible when the right inequality of (4.10) becomes binding, as $t_1 = a_1$ has to stay positive for the solution to be valid. It is now impossible to effectively use all n actions to balance the need for well-informed decision by the agent and the credibility for not overruling the agent.

4.2. Even mechanisms

For n even, starting from the delegation mechanism $A^{FC}(n)$ at β just greater than $\beta^{FC}(n)$, it is also possible to adjust the mechanism in the same way as when n is odd to create a delegation mechanism with n actions that is incentive compatible for the principal, in particular, a mechanism satisfying (3.11) and (3.13). However, the first order necessary conditions (4.3) through (4.6) cannot be satisfied by this mechanism, implying that it is not optimal. To see this, compare the first order condition (3.9) for a_2 under full commitment to the corresponding condition (4.4) under limited commitment. In order for (3.11) to hold for $i = 2$, we need $\lambda_3 = 0$. Next, for each $i = 3, \ldots, n-1$, compare the first order condition (3.9) for a_i under full commitment to the corresponding condition (4.5) under limited commitment. Since n is even, for (3.11) to hold for each $i = 3, \ldots, n-1$, we need $\lambda_3 = \lambda_5 = \ldots = \lambda_{n-1} = 0$, and $\lambda_2 = \lambda_4 = \ldots = \lambda_{n-2} = \lambda_n$. But then the first order conditions (4.3) and (4.6) imply that, for (3.13) to hold, all multipliers must be zero, which is impossible if $\beta > \beta^{FC}(n)$.

Instead, when n is even, an "even mechanism" can be constructed with the following set of values for the multipliers. There exists $\lambda > 0$, to be determined later, such that

$$\lambda_{2k} = \Big(\frac{1}{2}n - k\Big)\lambda$$

for each $k = 1, 2, \ldots, \frac{1}{2}(n-2)$, and

$$\lambda_{2k-1} = (k-1)\lambda$$

for each $k = 2, 3, \ldots, \frac{1}{2}n$. For any given λ, the configuration of the values $\lambda_2, \ldots, \lambda_{n-1}$ is chosen such that for each $i = 3, 4, \ldots, n-2$, summing up the first order conditions for a_i and a_{i+1} in (4.4) through (4.5) yields

$$\frac{1}{2}(t_{i-1} - a_{i-1})^2 + \frac{1}{2}(a_i - t_{i-1})^2 = \frac{1}{2}(t_{i+1} - a_{i+1})^2 + \frac{1}{2}(a_{i+2} - t_{i+1})^2,$$

with all the multipliers canceled out. Using the agent's indifference condition (3.7) for $i-1$ and $i+1$, the above condition immediately implies that

$$a_i - a_{i-1} = a_{i+2} - a_{i+1}.$$

Since all multipliers are positive (except that $\lambda_n = 0$), the principal's incentive compatibility constraint (4.1) binds for each $i = 2, \ldots, n-1$. Thus, to satisfy all the first order conditions (4.3) through (4.6), we just need the values of three independent variables, a_1, $a_2 - a_1$ and λ such that: (4.3) holds, or by using (3.7) for $i = 1$,

$$a_1^2 = \beta^2 + \frac{1}{4}(a_2 - a_1)^2 - \Big(\frac{1}{2}n - 1\Big)\lambda; \tag{4.11}$$

(4.4) holds, or by using (3.7) for $i = 2$ and the binding constraint (4.1) for $i = 2$

$$\frac{1}{4}(a_2 - a_1)^2 = \frac{1}{4}(4\beta - (a_2 - a_1))^2 - \lambda; \tag{4.12}$$

and (3.13) holds, which together with binding incentive compatibility constraints (4.1) for the principal implies

$$a_n - a_1 = 1 - 2a_1 = (a_2 - a_1) + 2(n-2)\beta. \tag{4.13}$$

Solving the three equations for a_1, we have

$$a_1 = \frac{1}{4} - (n^2 - 2n - 1)\beta^2. \tag{4.14}$$

The even mechanism with n actions $A^{LC}(n)$ is then given by a_1 from (4.14), a_2 from (4.13), and $a_3, \ldots, a_n$ from binding incentive compatibility constraints (4.1) for the principal.

The even mechanism defined above is valid, if the following two conditions are satisfied. First, the implied value of λ is positive. Using (4.12), this condition is equivalent to

$$a_3 + a_1 > 2a_2, \tag{4.15}$$

or, using the binding constraint (4.1) for $i = 2$ and (4.13), equivalent to

$$a_1 > \frac{1}{2} - (n-1)\beta.$$

Using the solution (4.14), the above is the same as the negation of (3.21). Thus, just as in the case of odd mechanisms, a necessary condition for the even mechanism with n actions to be valid is that $\beta > \beta^{FC}(n)$, so that the optimal full commitment delegation mechanism $A^{FC}(n)$ is not incentive compatible for the principal. Second, the implied smallest threshold t_1 is positive. Using the agent's incentive constraint (3.7) for $i = 1$, this condition is equivalent to

$$t_1 = \frac{1}{2}(a_1 + a_2) - \beta = \frac{1}{2} - (n-1)\beta > 0,$$

where the second equality follows from (4.13), independent of the solution for a_1 given in (4.14). The above inequality imposes an upper bound on the parameter value β, which takes the same form as in (4.10). Thus, just as in the case of odd mechanisms, a necessary condition for the even mechanism with n actions to be valid is that $\beta < \beta^{LC}(n)$, so that the agent uses the smallest action a_1 with a positive probability.

Lemma 4.2. *For any $n \geq 2$ and even, and $\beta \in \left(\beta^{FC}(n), \beta^{LC}(n)\right)$, the even mechanism $A^{LC}(n)$ solves the principal's optimal delegation problem under limited commitment with n actions.*

To compare odd mechanisms and even mechanisms, let n be some even number. In the odd mechanism with n actions, actions $a_2, \ldots, a_{n-1}$ are evenly spaced on the interval $[a_1, a_n]$, with the distance between any a_{i+1} and a_i for $i = 1, 2, \ldots, n-1$ equal to 2β. In contrast, in the even mechanism, from (4.15) and the binding incentive constraints for the principal, we have

$$a_2 - a_1 = a_4 - a_3 = \ldots = a_n - a_{n-1} < 2\beta < a_3 - a_2 = a_5 - a_4 = \ldots = a_{n-1} - a_{n-2}, \quad (4.16)$$

so that the actions are not all evenly spaced but have equal distances between every other pair of two adjacent ones. Nonetheless, in both odd and even mechanisms, from the agent's indifference condition (3.7) for $i+1$ and i and the binding incentive constraint (4.1) for i we have

$$t_{i+1} - t_i = 2\beta$$

for each $i = 2, \ldots, n-2$, so that the agent's thresholds $t_2, \ldots, t_{n-2}$ are even spaced on $[t_1, t_{n-1}]$ at the distance of 2β from each other.

Finally, the smallest action a_1 in an even mechanism is greater than the smallest action in an odd mechanism in the relevant range of parameter values for β. That is,

$$\frac{1}{4} - (n^2 - 2n - 1)\beta^2 > \frac{1}{2} - (n-1)\beta,$$

which can be shown to equivalent to $\beta > \beta^{FC}(n)$. This result has an intuitive interpretation. Starting at β just above $\beta^{FC}(n)$, both the odd mechanism and the even mechanism with n actions are feasible. As argued above, the odd mechanism is obtained from the full commitment mechanism $A^{FC}(n)$ by lowering a_1 and raising a_n simultaneously, keeping both actions at an equal distance from the respective end point of 0 and 1 and all other actions $a_2, \ldots, a_{n-1}$ evenly spaced on the interval $[a_1, a_n]$. Since deviations from $A^{FC}(n)$ are costly from the principal's point of view, the adjustment is stopped, and the odd mechanism is obtained, when a_1 is lowered just enough such that the resulting distance $a_2 - a_1$ is precisely 2β, satisfying (3.18) and hence all incentive constraints for the principal (3.18). When n is even, however, the odd mechanism obtained in this way is not optimal. This is because the incentive constraints for the principal are not (3.18) in the original optimization program, but rather (4.1). By keeping $a_2 - a_1$ smaller than 2β and simultaneously $a_3 - a_2$ greater than 2β so that the sum of the distances remains at 4β, and making the same adjustments to all other actions according to (4.16), the principal obtains the even mechanism $A^{LC}(n)$. This even mechanism is strictly better than the odd mechanism for the principal, because the former has a greater smallest action a_1, and thus smaller deviations or adjustments from the full commitment mechanism $A^{FC}(n)$.

4.3. Optimal delegation mechanisms

We have completed the first step in characterizing the optimal delegation mechanism A^{LC} under limited commitment, by characterizing $A^{LC}(n)$, the optimal delegation mechanism $A^{LC}(n)$ under limited commitment with n actions for each n. We now try to identify A^{LC} for all β, by searching among the delegation mechanisms $A^{FC}(n)$ and $A^{LC}(n')$ that are feasible.

To begin, for $\beta \geq \frac{1}{2}$, we have $A^{LC} = \left\{\frac{1}{2}\right\}$. This simply means that the best the principal can do if $\beta \geq \frac{1}{2}$ is to dictate $\frac{1}{2}$ without any information from the agent. Define $\beta^{FC}(2) = \beta^{LC}(2) = \frac{1}{2}$. For any $\beta < \frac{1}{2}$, the worst that the principal can do under limited commitment is to choose $A^{FC}(2)$.

From the definitions (3.21) and (4.10) we have, for each $n = 3, 4, \ldots$,

$$\beta^{FC}(n) < \beta^{LC}(n+1) < \beta^{FC}(n-1). \tag{4.17}$$

We search for A^{LC} for each β on the interval $\Big(\beta^{FC}(n), \beta^{FC}(n-1)\Big)$. Note that $\beta^{LC}(n) > \beta^{FC}(n-1)$ and $\beta^{FC}(n+1) < \beta^{FC}(n)$. Also, the principal's expected loss is decreasing with the number of actions n under the full mechanisms $A^{FC}(n)$. There are thus two cases in searching for A^{LC} for each β in $\Big(\beta^{FC}(n), \beta^{FC}(n-1)\Big)$. For any $\beta \in \Big(\beta^{FC}(n), \beta^{LC}(n+1)\Big)$, we need to consider delegation mechanisms $A^{FC}(n-1)$, $A^{LC}(n+1)$ and $A^{LC}(n)$; for any $\beta \in \Big(\beta^{LC}(n+1), \beta^{FC}(n-1)\Big)$, the mechanism $A^{LC}(n+1)$ becomes unavailable, and we need to consider $A^{FC}(n-1)$ and $A^{LC}(n)$.

We start by making a number of observations. First, the principal's expected loss jumps discontinuously at $\beta^{FC}(n)$ when he switches from $A^{FC}(n)$ to $A^{FC}(n-1)$. This follows because we can express the principal's loss under any $A^{FC}(n)$ as a function of the difference of the two smallest actions $a_2 - a_1$, or equivalently as a function of the smallest action a_1, which varies discretely as n decreases. A similar observation obtains when the principal switches from $A^{FC}(n-1)$ to $A^{FC}(n-2)$ at $\beta^{FC}(n-1)$. To derive the expression for the principal's expected loss, fix any $n \geq 3$. For each $i = 1, 2, \ldots, n-1$, using the agent's indifference condition (3.7) for i, and the relations (3.12) for i and $i+1$, we have

$$t_i = a_i + \frac{1}{2}(a_2 - a_1) - \beta = a_{i+1} - \frac{1}{2}(a_2 - a_1) - \beta. \tag{4.18}$$

Using (4.18), together with (3.13), we can rewrite the objective function (3.6) as (ignoring the constant $\frac{1}{3}$)

$$2(a_1)^3 + (n-1)\Big(\frac{1}{2}(a_2 - a_1) - \beta\Big)^3 + (n-1)\Big(\frac{1}{2}(a_2 - a_1) + \beta\Big)^3. \tag{4.19}$$

From (3.14) we have

$$a_2 - a_1 = \frac{1 - 2a_1}{n-1}, \tag{4.20}$$

or equivalently

$$a_1 = \frac{1}{2} - \frac{1}{2}(n-1)(a_2 - a_1).$$

Thus, the program of optimal delegation mechanism with n actions under full commitment is to choose a_1, or equivalently, choose $a_2 - a_1$, to minimize (4.19), without any constraint. The first order necessary condition with respect to a_1 is given by

$$a_1^2 = \frac{1}{4}\Big(\frac{1 - 2a_1}{n-1}\Big)^2 + \beta^2. \tag{4.21}$$

The above is a single-variable equation in a_1. Note that from (4.18) and (4.20), the above is identical to the condition (3.8) derived earlier.

Second, there is no payoff discontinuity for the principal when he switches from $A^{FC}(n)$ to $A^{LC}(n)$ at β just above $\beta^{FC}(n)$; and a similar observation obtains when the principal switches from $A^{FC}(n-1)$ to $A^{LC}(n-1)$ at $\beta^{FC}(n-1)$. In fact, the two mechanisms $A^{FC}(n)$ to $A^{LC}(n)$ are identical at $\beta^{FC}(n)$. Take the odd mechanism for example. The delegation mechanism $A^{LC}(n)$, for $n = 3, 5, \ldots$ odd, satisfies equations (3.7) and (3.12) used to derive (4.18), and equation (3.14). Thus, for $n = 3, 5, \ldots$ odd, if the solution to (4.19) satisfies the principal's incentive compatibility constraint (3.18), or equivalently through (4.20), if the solution in a_1 satisfies

$$a_1 \leq \frac{1}{2} - (n-1)\beta, \tag{4.22}$$

then a_1 also solves the problem of optimal delegation mechanism with n actions under limited commitment. Of course, (4.22) is equivalent to $\beta \leq \beta^{FC}(n)$. Compare (4.22) with (4.9). When $\beta > \beta^{FC}(n)$, the solution to (4.19) violates the principal's incentive compatibility constraint, in which case the optimal a_1 under limited commitment is given by binding (4.22), or equivalently by (4.9).

Third, the cutoff value $\beta^{LC}(n+1)$ is not relevant for the principal in searching for the optimal delegation mechanism A^{LC} under limited commitment. That is, at β just below $\beta^{LC}(n+1)$, when the optimal delegation mechanism $A^{LC}(n+1)$ under limited commitment with $n+1$ actions is just valid, A^{LC} cannot be equal to $A^{LC}(n+1)$. To see this, note that regardless of whether $n+1$ is odd or even, at $\beta^{FC}(n+1)$, we have $t_1 = 0$, so that the probability that the agent uses the smallest action a_1 is 0. It follows that $A^{LC}(n+1)$ has just n actions, as in the delegation mechanism $A^{LC}(n)$, which is feasible at $\beta^{LC}(n+1)$ because $\beta^{LC}(n+1) \in (\beta^{FC}(n), \beta^{LC}(n))$. For the principal, $A^{LC}(n+1)$ is strictly dominated by $A^{LC}(n)$ at $\beta^{LC}(n+1)$ and thus can not be optimal.

Putting the above observations together, we have the following conjecture, stated as a proposition.

Proposition 4.3. *For any $\beta \geq \beta^{FC}(2) = \frac{1}{2}$, the optimal delegation mechanism A^{LC} under limited commitment is $A^{LC}(2) = \left\{\frac{1}{2}\right\}$. For each $n \geq 3$, there exists $\beta(n, n-1) \in$*

$\left(\beta^{FC}(n), \beta^{FC}(n-1)\right)$, *such that the optimal delegation mechanism* A^{LC} *under limited commitment is given by* $A^{LC}(n)$ *for all* $\beta \in \left(\beta^{FC}(n), \beta(n, n-1)\right)$, *and by* $A^{FC}(n-1)$ *for all* $\beta \in \left(\beta(n, n-1), \beta^{FC}(n-1)\right)$.

The above proposition illustrates the point that the optimal mechanism under limited commitment does not maximize the number of actions in the delegation set. This is reflected in following two ways.

First, for each interval $\left(\beta^{FC}(n), \beta^{FC}(n-1)\right)$, when β falls between $\beta^{FC}(n)$ and $\beta^{LC}(n+1)$, the delegation mechanism $A^{LC}(n+1)$ is available, but according to the above proposition it is never optimal. Thus, unlike the case of optimal full commitment delegation mechanisms, the principal's payoff does not necessarily increase with the number of actions in the optimal limited commitment delegation mechanisms. This may be surprising, but recall that we have applied a version of the revelation principle and restricted the search for the optimal delegation mechanism under limited commitment to delegation mechanisms that are minimal and veto free. Thus, in characterizing the mechanisms $A^{LC}(n)$ for each fixed n, we have imposed the condition that all n actions are used with positive probabilities. As a result, we have precluded the standard reasoning that adding an action cannot make the principal worse off because the principal could always choose a mechanism that effectively does not involve the added action.

Second, for each interval $\left(\beta^{FC}(n), \beta^{FC}(n-1)\right)$, when β falls between $\beta^{LC}(n+1)$ and $\beta^{FC}(n-1)$, the delegation mechanism $A^{LC}(n)$ is available, but according to the proposition the principal strictly prefers $A^{FC}(n-1)$ for β between $\beta(n, n-1)$ and $\beta^{FC}(n-1)$. Thus, in an optimal mechanism, the principal specifies fewer actions in the delegation set in order to relax the incentive compatibility constraints due to limited commitment. The number of actions in a delegation set represents how much private information that the principal is able to elicit from the agent in an incentive compatible manner. But under limited commitment, eliciting more private information can create credibility problem for the principal and therefore the additional information may not be used effectively by the principal. In this case, it is better for the principal to elicit less information by reducing the number of actions in the delegation set so as to use the elicited information more effectively.

To prove the proposition, it is useful to derive an alternative expression to (4.19) that applies to both the odd and even mechanisms. Integrating out the objective function (3.6) (and ignoring the constant $\frac{1}{3}$), we have

$$(a_1)^3 - (a_1 - t_1)^3 + \sum_{i=2}^{n-1} \Big((a_i - t_{i-1})^3 - (a_i - t_i)^3 \Big) + (a_n - t_{n-1})^3 + (1 - a_n)^3. \quad (4.23)$$

Note that both odd and even mechanisms satisfy (3.13). Further, under both mechanisms, for each $i = 1, \ldots, n-2$, using the agent's incentive constraints (3.7) for $i+1$ and i, and the principal's binding incentive constraint (4.1) for $i+1$, we have

$$a_{i+1} - t_{i+1} = -(a_i - t_i),$$

with

$$a_1 - t_1 = -\frac{1}{2}(a_2 - a_1) + \beta \quad (4.24)$$

by the agent's incentive constraint for t_1. Finally, using the binding incentive constraints (4.1) for the principal and the agent's incentive constraints (3.7) for t_1 and t_2, for each $i = 4, 6, \ldots, n$ even, we have

$$a_i - t_{i-1} = a_2 - t_1 = \frac{1}{2}(a_2 - a_1) + \beta; \quad (4.25)$$

and for each $i = 5, 7, \ldots, n-1$ odd, we have

$$a_i - t_{i-1} = a_3 - t_2 = -\frac{1}{2}(a_2 - a_1) + 3\beta. \quad (4.26)$$

When n is odd, the objective function (4.23) becomes

$$2(a_1)^3 + \frac{1}{2}(n-1)(a_2 - t_1)^3 + \frac{1}{2}(n-1)(a_3 - t_2)^3. \quad (4.27)$$

Again, in the above expression both $a_2 - t_1$ and $a_3 - t_2$ depend on $a_2 - a_1$ through (4.25) and (4.26). Further, since n is odd, we have

$$a_n - a_1 = 1 - 2a_1 = 2(n-1)\beta,$$

which is independent of $a_2 - a_1$, and which implies a_1 is given by (4.9) as in an odd mechanism. In (4.27) the variable $a_2 - a_1$ then becomes a free variable, and minimizing (4.27) requires that $a_3 - t_2 = a_2 - t_1$, or $a_2 - a_1 = 2\beta$, as in an odd mechanism.

When n is even, using the above relations, we can write the objective function (4.23) as

$$2(a_1)^3 - (a_1 - t_1)^3 + \frac{1}{2}n(a_2 - t_1)^3 + \left(\frac{1}{2}n - 1\right)(a_3 - t_2)^3. \quad (4.28)$$

The above can be thought of as a function of a single variable a_1, as $a_1 - t_1$, $a_2 - t_1$ and $a_3 - t_2$ all depend on $a_2 - a_1$ through (4.24), (4.25) and (4.26), and $a_2 - a_1$ is a function of a_1 through (4.13). Equivalently, we can view the objective function (4.28) as a single-variable function of $a_2 - a_1$. The first order condition with respect to $a_2 - a_1$ is given by

$$a_2 - a_1 = \frac{1}{2} - 2(n-2)\beta + 2(n^2 - 2n - 1)\beta^2. \quad (4.29)$$

Note that from (4.13) that the above is equivalent to the solution given in (4.14) in even mechanisms.

Using the expression (4.19) for the principal's expected loss under the optimal full commitment mechanism $A^{FC}(n)$, and the corresponding expressions (4.27) and (4.28) derived for odd and even mechanisms, we can establish the following two claims: (i) for each $n \geq 3$, there exists $\beta(n, n-1) \in \left(\beta^{FC}(n), \beta^{FC}(n-1)\right)$, such that the principal prefers $A^{LC}(n)$ to $A^{FC}(n-1)$ for all $\beta \in \left(\beta^{FC}(n), \beta(n, n-1)\right)$, and $A^{FC}(n-1)$ to $A^{LC}(n)$ for all $\beta \in \left(\beta(n, n-1), \beta^{FC}(n-1)\right)$; and (ii) for each $n \geq 2$, the principal prefers $A^{LC}(n)$ to $A^{LC}(n+1)$ for all $\beta \in \left(\beta^{FC}(n), \beta^{LC}(n+1)\right)$. The proposition then follows immediately. Since the proofs of the two claims involve straightforward but tedious calculations, they are relegated to the appendix.

5. Discussion

Under limited commitment, it might be interesting to model the degree of commitment. A straightforward way of accomplishing this is to imagine that there is some probability that after the decision has been made by the agent, the principal can choose to change the decision based on what has been learned from the agent's decision. In this model, the probability of overruling the decision by the agent can be thought of as the degree of commitment. In terms of real-life representation, we might think of the degree of commitment by the principal as the size of uncertainty faced by the agent as to whether the principal can over-rule the agent or not.

It's also possible to endogenize this degree of commitment, by for example, assuming that the principal is privately informed, and thus depending on the action taken by the agent the principal may or may not find it optimal to over-rule the agent. We may then ask whether there is an optimal degree of limited commitment. Such exercise would further our understanding of the principal's trade-off between maintaining the flexibility of responding to unanticipated events and establishing the credibility of letting the agent best use his private information.

Appendix

LEMMA A.1. *For each $n \geq 3$, there exists $\beta(n, n-1) \in \Big(\beta^{FC}(n), \beta^{FC}(n-1)\Big)$, such that the principal prefers $A^{LC}(n)$ to $A^{FC}(n-1)$ for all $\beta \in \Big(\beta^{FC}(n), \beta(n, n-1)\Big)$, and $A^{FC}(n-1)$ to $A^{LC}(n)$ for all $\beta \in \Big(\beta(n, n-1), \beta^{FC}(n-1)\Big)$.*

PROOF. First, note that regardless of whether n is odd or even, $A^{LC}(n)$ is identical to $A^{FC}(n)$ at $\beta = \beta^{FC}(n)$. Since the principal strictly prefers $A^{FC}(n)$ to $A^{FC}(n-1)$ at $\beta^{FC}(n)$, he strictly prefers $A^{LC}(n)$ to $A^{FC}(n-1)$ at the same β.

Second, we claim the principal strictly prefers $A^{FC}(n-1)$ to $A^{LC}(n)$ at $\beta^{FC}(n-1)$. Note that at $\beta = \beta^{FC}(n-1)$, regardless of whether n is odd or even, we have that $a_2 - a_1 = 2\beta$, and from (4.19) the principal's expected loss under $A^{FC}(n-1)$ is given by

$$2\left(\frac{1}{2} - (n-2)\beta\right)^3 + (n-2)(2\beta)^3.$$

We distinguish two cases. First, suppose that n is odd. Then, from (4.27) the principal's expected loss under $A^{LC}(n)$ is given by

$$2\left(\frac{1}{2} - (n-1)\beta\right)^3 + (n-2)(2\beta)^3. \tag{A.1}$$

Using the value of $\beta = \beta^{FC}(n-1)$ given by (3.21), we can directly verify that the principal's loss is lower under $A^{FC}(n-1)$ than under $A^{LC}(n)$ at $\beta = \beta^{FC}(n-1)$. Second, suppose that n is even. Then, using (4.28) and (4.29), we can explicitly calculate the principal's expected loss under $A^{LC}(n)$ and rewrite it as

$$2\left(\frac{1}{2}-(n-2)\beta\right)^3+14(n-2)\beta^3-3\left(\left(\frac{1}{2}-(n-2)\beta\right)^2+(2n-5)^2\beta^2\right)^2. \qquad \text{(A.2)}$$

Using the value of $\beta=\beta^{FC}(n-1)$ given by (3.21), we can verify that

$$\left(\frac{1}{2}-(n-2)\beta\right)^2=2\beta^2.$$

It is then straightforward to verify that the principal's loss is lower under $A^{FC}(n-1)$ than under $A^{LC}(n)$ at $\beta=\beta^{FC}(n-1)$.

Third, we claim that at any $\beta\in\Big(\beta^{FC}(n),\beta^{FC}(n-1)\Big)$, the derivative with respect to β of the principal's expected loss under $A^{LC}(n)$ is strictly greater than the expected loss under $A^{FC}(n-1)$. Using (4.19) and the Envelope Theorem, we can write the derivative of the principal's expected loss under $A^{FC}(n-1)$ with respect to β as

$$6\Big(1-2a_1^{FC}(n-1;\beta)\Big)\beta,$$

where $a_1^{FC}(n-1;\beta)$ satisfies the first order condition (4.21) for $n-1$. This first order condition implies that $a_1^{FC}(n;\beta)$ increases with β for any n in the relevant range, and decreases in n for any β in the relevant range. Thus,

$$a_1^{FC}(n-1;\beta)>a_1^{FC}(n-1;\beta^{FC}(n))>a_1^{FC}(n;\beta^{FC}(n))=\frac{1}{2}-(n-1)\beta^{FC}(n),$$

where the last equality follows because $A^{FC}(n)$ is identical to $A^{LC}(n)$ at $\beta=\beta^{FC}(n)$. As a result, the derivative of the principal's expected loss under $A^{FC}(n-1)$ with respect to β is bounded from above by

$$12(n-1)\beta^2. \qquad \text{(A.3)}$$

Now we distinguish two cases. First, suppose that n is odd. Using (A.1), the derivative of the principal's expected loss with respect to β is

$$6(n-1)\left(4\beta^2-\left(\frac{1}{2}-(n-1)\beta\right)^2\right), \qquad \text{(A.4)}$$

which is strictly greater than (A.3) because $\beta > \beta^{FC}(n)$. Second, suppose that n is even. Using the (A.2), the derivative of the principal's expected loss with respect to β is

$$12\beta\left(n(n-1)(n-2)\beta - (n^2-2n-1)\left(\frac{1}{4} + (n^2-2n-1)\beta^2\right)\right), \tag{A.5}$$

which is strictly greater than (A.3) because $\beta > \beta^{FC}(n)$.

The lemma then follows immediately. *Q.E.D.*

LEMMA A.2. *For each $n \geq 2$, the principal prefers $A^{LC}(n)$ to $A^{LC}(n+1)$ for all $\beta \in \left(\beta^{FC}(n), \beta^{LC}(n+1)\right)$.*

PROOF. By Lemma A.1, at $\beta^{FC}(n)$ the principal's expected loss is lower under $A^{LC}(n)$, which is identical to $A^{FC}(n)$, than under $A^{LC}(n+1)$. The present lemma follows immediately after we demonstrate that the derivative of the expected loss with respect to β for any $\beta \in \left(\beta^{FC}(n), \beta^{LC}(n+1)\right)$ is greater under $A^{LC}(n+1)$ than under $A^{LC}(n)$. We distinguish two cases.

For n odd, using (A.4) for n and (A.5) for $n+1$, we can rewrite the comparison result of the derivatives we need as

$$2\left(n-2-2(n^2-2)\beta\right)\beta^2 + 2(n^2-2)\left(\frac{1}{2}-n\beta\right)^2\beta - (n-1)\left(\frac{1}{2}-(n-1)\beta\right)^2 < 0.$$

Using the expression (3.21) for $\beta^{FC}(n)$, we can verify that since $\beta > \beta^{FC}(n)$, the first difference in the left-hand-side of the above inequality is negative, and thus it suffices to show that the second difference is also negative, or

$$2(n^2-2)\left(\frac{1}{2}-n\beta\right)^2\beta < (n-1)\left(\frac{1}{2}-(n-1)\beta\right)^2.$$

By taking derivatives with respect to β and using $\beta > \beta^{FC}(n)$, we can verify that the left-hand-side of the above inequality decreases with β. Further, we can easily check that the left-hand-side evaluated $\beta = \beta^{FC}(n)$ is less than the right-hand-side evaluated at $\beta = \beta^{LC}(n+1)$ for any $n \geq 3$.

For n even, using (A.4) for $n+1$ and (A.5) for n, we can rewrite the desired comparison result of the derivatives as

$$2\left(n+1-2(n^2-2n-1)\beta\right)\beta^2 + 2(n^2-2n-1)\left(\frac{1}{2}-(n-1)\beta\right)^2\beta - n\left(\frac{1}{2}-n\beta\right)^2 > 0.$$

Using the expression (3.21) for $\beta^{FC}(n)$, we can verify that since $\beta > \beta^{FC}(n)$, the first difference in the left-hand-side of the above inequality is positive, and thus it suffices to show that the second difference is also positive, or

$$2((n-1)^2 - 2)\left(\frac{1}{2} - (n-1)\beta\right)^2 \beta > n\left(\frac{1}{2} - n\beta\right)^2.$$

By taking derivatives with respect to β and using $\beta > \beta^{FC}(n)$, we can verify that the left-hand-side of the above inequality decreases with β. Further, we can easily check that the left-hand-side evaluated $\beta = \beta^{LC}(n+1)$ is less than the right-hand-side evaluated at $\beta = \beta^{FC}(n)$ for any $n \geq 4$. *Q.E.D.*

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