Conditional Jump Dynamics in Stock Market Returns

Wing H. Chan
Department of Economics, University of Alberta, Edmonton Alberta, T6G 2H4, Canada
whchan@gpu.srv.uaalberta.ca

John M. Maheu
Department of Economics, University of Toronto, Toronto, Ontario, M5S 3G7, Canada
jmaheu@chass.utoronto.ca

This article develops a new conditional jump model to study jump dynamics in stock market returns. We propose a simple filter to infer ex post the distribution of jumps. This permits construction of the shock affecting the time $t$ conditional jump intensity and is the main input into an autoregressive conditional jump intensity model. The model allows the conditional jump intensity to be time-varying and follows an approximate autoregressive moving average (ARMA) form. The time series characteristics of 72 years of daily stock returns are analyzed using the jump model coupled with a generalized autoregressive conditional heteroskedasticity (GARCH) specification of volatility. We find significant time variation in the conditional jump intensity and evidence of time variation in the jump size distribution. The conditional jump dynamics contribute to good in-sample and out-of-sample fits to stock market volatility and capture the rallies often observed in equity markets following a significant downturn.

KEY WORDS: Conditional intensity; Filter; Jump size.

1. INTRODUCTION

Over the past several decades, some stylized facts have emerged about the statistical behavior of speculative market returns. The most important of these empirical findings are that asset returns are approximately a martingale difference sequence, the conditional variance is time-varying, and the unconditional distribution displays leptokurtosis. Conventional wisdom on volatility dynamics is that generalized autoregressive conditional heteroskedasticity (GARCH) and stochastic volatility (SV) models provide a good first approximation to these stylized facts by modeling the autoregressive structure in the conditional variance. Both the GARCH and SV models are designed to capture smooth persistent changes in volatility. But these models are not suited to explaining the large discrete changes found in asset returns. In most speculative markets, discrete jumps in returns are necessary to better match statistical features observed in the data (Andersen, Benzoni, and Lund 1999; Gallant, Hsieh, and Tauchen 1997), as well as reconcile mispricing in options markets (Bakshi, Cao, and Chen 1997; Bates 1996; Das and Sundaram 1999; Jorion 1988). A large literature has investigated the importance of jumps from statistical and asset pricing perspectives.

The basic Poisson jump model of stock returns used in finance was introduced by Press (1967), who called his approach a compound events model, because it can be motivated as the aggregation of a random number of price changes within a fixed time interval. The Poisson distribution is assumed to govern the number of events that result in price movements, and the average number of events in a time interval is called the intensity. The model is capable of producing skewness and excess kurtosis in returns. All volatility dynamics are assumed to be the result of discrete jumps in stock returns, and the size of a jump is stochastic and normally distributed. Several early empirical applications have demonstrated the usefulness of the Press model. Akigiray and Booth (1988), Tucker and Pond (1988), and Hsieh (1989) found that a normal-Poisson jump model provides a good statistical characterization of daily exchange rates. Similar results were found by Ball and Torous (1983) using stock returns.

The basic jump model has been extended in a number of directions. Estimation of continuous-time SV jump diffusion models requires simulation methods and only recently has been investigated by Anderson et al. (1999), Craine, Loehstoer, and Syyrveit (2000), Eraker, Johannes, and Polson (1999) and Chernov, Gallant, Ghysels, and Tauchen (1999). A tractable alternative is to combine jumps with an ARCH/GARCH model in discrete time. In this case the GARCH parameterization explains the smooth changes in volatility, whereas the jumps explain infrequent large discrete movements in asset returns. Applications of a GARCH-jump mixture model have been given by Jorion (1988), Vlaar and Palm (1993), and Nieuwland, Verschoor, and Wolff (1994).

A common thread in these GARCH-jump mixture models is the assumption that a constant Poisson distribution directs the jump probability through time. However, it seems likely that the jump probability will change over time. Would we expect the probability of a jump in stock market returns before the 1987 stock market crash to be the same as other periods? The results of Bates (1991) would suggest the answer to this is no. Using Standard & Poor 500 futures options and assuming an underlying jump diffusion, Bates (1991) found systematic behavior in expected jumps before the 1987 crash.

Recent research has extended the theoretical framework to permit a time-varying jump distribution. For exam-
ple, Das (1998) and Fortune (1999) used dummy variables to allow the jump intensity to change over the week. Chernov et al. (1999) estimated specifications that allow the jump intensity to depend on the size of previous jumps, and a stochastic volatility factor. Eraker et al. (1999) modeled jumps in both returns and volatility.

To explore the importance of time variation in the jump intensity, we propose a new discrete time model in which the conditional jump intensity follows an endogenous autoregressive process. To make estimation straightforward, we assume that the conditional jump intensity can be projected onto observables contained in the most recent information set. In our model, jumps are unobserved and thus are difficult to analyze directly. The first step in our approach is to propose a filter to infer ex post the distribution of jumps at time $t$. Using the filter, we next construct the shock to the expected number of jumps as observed by the econometrician. This shock at time $t$ provides the basic input into next period’s conditional jump intensity. Our model of autoregressive conditional jump intensity (ARJI) specifies the conditional jump intensity as an approximate autoregressive moving average (ARMA) process.

The model studied here bears some resemblance to Markov switching models, such as that of Hamilton and Susmel (1994), which assume that an unobserved Markov chain directs the dynamics of returns. Like the Markov switching model, jumps are discrete and unobserved; however, they are governed by a serially correlated Poisson process that is closely related to recently proposed models for the arrival rate of events. For example, Hamilton and Jorda (2000) noted that Engle and Russell’s (1998) autoregressive conditional duration model of the time between events is equivalent to a Poisson process where the intensity follows an ARMA process. Similarly, Davis, Rydberg, Shephard, and Streett (2001) used an ARMA model for the intensity of a Poisson process that describes count data over a fixed time interval.

Our approach to exploring time variation in the jump intensity has several advantages. First, because the jump intensity has an ARMA functional form, it is capable of parsimoniously capturing many forms of autocorrelation. Second, the model is easy to estimate, and maximum likelihood estimation and asymptotic inference are available. A byproduct of estimation is the filter that provides ex post inference regarding the latent jump dynamics. Although we project the conditional jump intensity onto past observables, we expect our model to provide a good approximation to a process where the jump intensity follows a latent stochastic process. Such a model would require simulation methods for estimation, whereas our model does not. Finally, multiperiod forecasts of future expected jumps can be directly calculated in our model.

Time variation in jump intensity may only be one aspect of the conditional jump dynamics in stock returns. In particular, the distribution governing the jump size may be time varying. To investigate this, we make the standard assumption that the jump size distribution is normally distributed, but allow the conditional mean and variance to be a function of observables. For example, we estimate models that permit the jump size variance to be related to lagged squared returns and a GARCH variance factor. In addition, we explore whether the mean of the conditional jump size distribution is asymmetrically related to the most recent positive or negative return. The motivation for this asymmetry is to consider whether jump dynamics can capture the stock market rally frequently observed after a crash.

We study jump dynamics in stock market returns using the ARJI model coupled with a GARCH specification, and apply them to more than 72 years of daily returns on the Dow Jones Industrial Average (DJIA) price index. Using the filter, we find that a constant jump intensity for the DJIA is violated, and that a low-order ARJI model adequately captures the time variation in the conditional jump intensity.

The empirical results indicate that the autocorrelation in the conditional jump intensity in stock returns is positive and very persistent. Similar to the GARCH parameterization of volatility, a high probability of many (few) jumps today tends to be followed by a high probability of many (few) jumps tomorrow. Unconditionally, jumps are infrequent; however, conditionally, jumps show significant time variation over our data sample. For example, during the 1940s, daily jump intensity ranges from only .03 to 2.02. This indicates that in the 1940s there were periods where almost no jumps were expected (.03) and periods where several jumps (2.02) were expected. We find evidence of an increase in the conditional expected jump intensity before both the 1929 (in-sample) and the 1987 (out-of-sample) stock market crashes.

Allowing the conditional variance of the jump size distribution to be linearly related to a measure of market volatility, such as past squared returns, improves the models’ in-sample fit and the out-of-sample forecasts of volatility. Our model identifies the rally after a significant stock market downturn through a change in the conditional jump size mean. For instance, any decrease in the market of 2.5% or more, implies a positive conditional mean in next period’s jump size distribution. Therefore, the day after a stock market crash, the likelihood of a jump in the next period does not necessarily decrease, but the likelihood of a negative jump decreases and the likelihood of a positive jump increases.

Finally, previous research (Bates 2000; Chernov et al. 1999) has investigated whether there is a relationship between the jump intensity and a SV specification of volatility. In our model, the analog relationship is between the conditional jump intensity parameterization and the GARCH specification. In general, we found mixed results. After permitting the variance of the jump size distribution to be a function of the GARCH variance, we found no evidence that the GARCH process affects the conditional intensity specification for our dataset. However, in a similar specification in which lagged squared returns affect the jump size distribution rather than the GARCH variance, we found that the GARCH variance is positive and significantly affects the conditional intensity.

The time variation in the conditional jump intensity implies time variation in the volatility and also in the conditional skewness and conditional kurtosis of returns. Thus conditional jump dynamics may be important in explaining higher-order moments in speculative returns. The systematic persistence in the likelihood of jumps uncovered in this study may have important effects in forecasting volatility, risk management, and derivative pricing.
The article is organized as follows. Section 2 presents a model of conditional jump dynamics coupled with a GARCH parameterization for financial market returns. A filter and the statistical features of the model are emphasized. Section 3 reviews some specification tests used to evaluate the performance of the models. Section 4 details the data used in the empirical application, and Section 5 reports the estimates and features of the conditional jump models applied to DJIA returns. Section 6 presents a discussion of the results, and Section 7 gives conclusions.

2. A DYNAMIC CONDITIONAL JUMP MODEL FOR STOCK RETURNS

In this section we present a discrete time jump model with a time-varying conditional jump intensity and jump size distribution. Because of the vast literature showing GARCH models to be a good first approximation to the conditional variance, we combine the jump specification with a GARCH parameterization of volatility. Define the information set at time t to be the history of returns, \( \Phi_t = \{R_{i, t} \} \). Consider the following jump model for stock returns:

\[
R_t = \mu + \sum_{i=1}^{t} \phi_i R_{i-1} + \sqrt{\eta_i} z_t + \sum_{i=1}^{t} \gamma_{i, t},
\]

\[
z_t \sim NID(0, 1), \quad Y_{i, t} \sim N(\theta_i, \delta_i^2).
\]

The conditional jump size \( Y_{i, t} \), given \( \Phi_{i-1} \), is presumed to be independent and normally distributed with mean \( \theta_i \) and variance \( \delta_i^2 \). To simplify construction of the likelihood, we specify both \( z_t \) and the jump size \( Y_{i, t} \) as independent normal random variables; however, our model of the conditional jump dynamics does not depend on this assumption.

Let \( n_t \) denote the discrete counting process governing the number of jumps that arrive between \( t-1 \) and \( t \), which is distributed as a Poisson random variable with the parameter \( \lambda_i > 0 \) and density

\[
P(n_t = j | \Phi_{i-1}) = \frac{\exp(-\lambda_i) \lambda_i^j}{j!}, \quad j = 0, 1, 2, \ldots.
\]

The mean and variance for the Poisson random variable are both \( \lambda_i \), which is often called the (jump) intensity. We permit the jump intensity to be time-varying and discuss the process that \( \lambda_i \) follows later, but for now we assume that knowledge of the information set at time \( t-1 \) implies knowledge of \( \lambda_i \). To complete the conditional volatility dynamics for returns, let \( h_t \) be measurable with respect to the information set \( \Phi_{i-1} \) and follow a GARCH(\( p, q \)) (Bollerslev 1986) process, that is,

\[
h_t = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{i-1}^2 + \sum_{i=1}^{p} \beta_i h_{i-1},
\]

where \( \epsilon_t = R_t - \mu - \sum_{i=1}^{\infty} \phi_i R_{i-1} \). This specification of \( \epsilon_t \) contains the expected jump component and thus allows it to propagate and affect future volatility through the GARCH variance factor. An alternative definition for \( \epsilon_t \), that includes the conditional expectation from the jump component is \( R_t - \mu - \sum_{i=1}^{\infty} \phi_i R_{i-1} - \lambda_i \theta_i \). In our empirical applications, we found that the former specification gives a substantially better log-likelihood value.

Imposing the restrictions of a constant jump intensity (\( \lambda_i = \lambda \)) and constant jump size distribution (\( \theta_i = \theta, \delta_i^2 = \delta^2 \)) in our model nests several mixed jump models that have been investigated in the literature. For example, Jorion (1988) estimated a constant jump intensity-ARCH model for foreign exchange and stock market returns, and Vlaar and Palm (1993) and Nieuwland et al. (1994) used a constant jump intensity-GARCH model to capture exchange rate dynamics.

Several extensions to this model have been proposed that permit the intensity to be time-varying, typically driven by some exogenous vector \( X_t \), thought to identify the likelihood of jumps. For instance, \( X_t \) might include dummy variables (Das 1998; Fortune 1999) or macro variables such as interest rates (Bekaert and Gray 1998; Neely 1999). A problem with this approach is the choice of news events or information to include in \( X_t \). Rather than follow this approach, we allow \( \lambda_i \) to endogenously evolve according to a parsimonious ARMA structure.

Consider the following ARJI model, denoted ARJI(\( r, s \)). Let \( \lambda_i = E[n_t | \Phi_{i-1}] \) be the conditional expectation of the counting process that is assumed to follow

\[
\lambda_i = \lambda_0 + \sum_{i=1}^{r} \rho_i \lambda_{i-1} + \sum_{j=1}^{s} \gamma_j \xi_{i-1}.
\]

The conditional jump intensity at time \( t \) is related to \( r \) past lags of the conditional jump intensity plus lags of \( \xi_i \). Here \( \xi_{i-1} \) represents the innovation to \( \lambda_{i-1} \) as measured ex post by the econometrician. This shock, or jump intensity residual, is calculated as

\[
\xi_{i-1} = E[n_{i-1} | \Phi_{i-1}] - \lambda_{i-1} = \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} n_{i-1} P(n_{i-1} = j | \Phi_{i-1}) - \lambda_{i-1}.
\]

The first term on the right side of (5) is our inference on the average number of jumps at time \( t - i \) based on time \( t - i \) information, while the second term in (5) is our expectation of the number of jumps using information at time \( t - i - 1 \). Therefore, \( \xi_{i-1} \) represents the unpredictable component affecting our inference about the conditional mean of the counting process \( n_{i-1} \).

Let \( f(R_t | n_t = j, \Phi_{i-1}) \) denote the conditional density of returns given that \( j \) jumps occur and the information set \( \Phi_{i-1} \). HAVING OBSERVED \( R_t \) AND USING THE BAYES RULE, WE CAN INFERENCE EX POST THE PROBABILITY OF THE OCCURRENCE OF \( J \) JUMPS AT TIME \( T \), WITH THE FILTER DEFINED AS

\[
P(n_t = j | \Phi_{i-1}) = \frac{f(R_t | n_t = j, \Phi_{i-1}) P(n_t = j | \Phi_{i-1})}{P(R_t | \Phi_{i-1})}, \quad j = 0, 1, 2, \ldots.
\]

where \( P(n_t = j | \Phi_{i-1}) \) is from (2). The filter in (6) is an important component of our model of time-varying jump dynamics, because it enters (5) but also can be constructed and used for inference purposes. For example, the probability that at least one jump occurred could be assessed using \( 1 - P(n_t = 0 | \Phi_{i-1}) \). The filter may be particularly useful in revealing misspecification in the simpler constant intensity specification.
The conditional density of returns is completed by integrating out the discrete-valued variable \( n_i \) governing the number of jumps. The conditional density of returns is

\[
P(R_j|\Phi_{t-1}) = \sum_{j=0}^{\infty} f(R_j|n_i = j, \Phi_{t-1})P(n_i = j|\Phi_{t-1}) .
\]  

(7)

Equation (7) shows that this model is nothing more than a discrete mixture of distributions where the mixing is driven by a time-varying Poisson distribution. The assumptions in (1) imply that the distribution of returns conditional on the most recent information set and \( j \) jumps is normally distributed as

\[
f(R_j|n_i = j, \Phi_{t-1}) = \frac{1}{\sqrt{2\pi(h_j + j\delta_j^2)}} \exp\left(-\frac{(R_j - \mu - \sum_{i=1}^{j} \Phi_{t-1} - \delta_j)^2}{2(h_j + j\delta_j^2)}\right).
\]

Construction of the likelihood and maximum likelihood estimation follows by iterating on (4), (6), and (7). Equation (7) involves an infinite sum over the possible number of jumps \( n_i \). In practice, we truncate the maximum number of jumps to a large value \( \tau \), so the probability of \( \tau \) or more jumps is 0. We empirically check for a particular dataset and estimate that, to machine precision, Equation (2) is 0 for \( j > \tau \). A second check on the choice of \( \tau \) is to investigate \( \tau > \tau \) to ensure that the likelihood and the parameter estimate do not change.

In this model the conditional jump intensity is time varying and under certain circumstances has an unconditional value. To derive the unconditional value of \( \lambda_i \), first note that \( \xi_i \) is a martingale difference sequence with respect to \( \Phi_{t-1} \), because

\[E[\xi_i|\Phi_{t-1}] = E[E[n_i|\Phi_{t-1}]|\Phi_{t-1}] - \lambda_i = \lambda_i - \lambda_i = 0, \]

and thus \( E[\xi_i] = 0 \) and \( \text{cov}(\xi_i, \xi_{i+j}) = 0, i > 0 \). Another way to see this result is to note that \( \xi_i \) is by definition nothing more than the rational forecast error associated with updating the information set. This is \( \xi_i = E[n_i|\Phi_{t}] - E[n_i|\Phi_{t-1}] \).

Using the approximate ARMA form for the evolution of \( \lambda_i \) in (4), many of the results for ARMA models are directly applicable to this model. For example, assuming that the roots of the polynomial \((1 - \sum_{i=1}^{m} \rho_i L^i)\), where \( L \) is the lag operator associated with (4), lie outside the unit circle, the unconditional value of \( \lambda_i \) exists and is

\[
E[\lambda_i] = \frac{\lambda_0}{1 - \sum_{i=1}^{m} \rho_i}.
\]

Furthermore, conditional forecasts of the future jump intensity can be formed using (4). To illustrate, consider the case where \( r = s = 1 \); then

\[E[\lambda_{r+s}|\Phi_{t-1}] = \begin{cases} \lambda_i, & i = 0 \\ \rho^i \lambda_{i} + \lambda_0 \sum_{j=0}^{i-1} \rho^j, & i \geq 1 \end{cases}, \]

and recall that \( \lambda_i \) is measurable with respect to the information set \( \Phi_{t-1} \). If \(|\rho| < 1\), then as \( i \) becomes large, the forecast approaches the unconditional value in (9).

For the Poisson distribution to be well defined, \( \lambda_i \) must be positive. Note that in the case of \( r = s \), (4) can be rewritten as

\[
\lambda_i = \lambda_0 + \sum_{i=1}^{r} (\rho^i - \gamma_i)\lambda_{i-1} + \sum_{i=1}^{r} \gamma_i E[n_{i-1}|\Phi_{i-1}].
\]

(11)

Subject to reasonable startup conditions that ensure \( \lambda_i > 0, i = 0, \ldots, r \), a sufficient condition for \( \lambda_i > 0 \) for all \( t \) is \( \lambda_0 > 0, \rho_i \geq \gamma_i \), and \( \gamma_i \geq 0 \). To estimate the ARJI model, startup values of \( \lambda_i \), and \( \xi_i, i \leq 0 \), must be set. In our empirical application discussed in the next section, we set startup values of the jump intensity to the unconditional value in (9) and \( \xi_i = 0 \). Alternatively, these values could be estimated or arbitrarily set to some value, because asymptotically the first observation is negligible to the likelihood function.

Consider the intuition behind the evolution of the conditional jump intensity in the ARJI model. Suppose that we observe \( \xi_i > 0 \) for several periods. This suggests that the jump intensity is temporarily trending away from its unconditional mean. This model effectively captures systematic changes in jump risk in the market. The likelihood of large discrete changes in foreign exchange markets or crashes in stock markets may change considerably over time. The ARJI model can capture systematic changes and also forecast increases (decreases) in jump risk into the future.

The ARJI model of \( \lambda_i \) is convenient from both an estimation and forecasting perspective and should be useful in capturing time series dynamics of the conditional jump intensity. As we show in Section 5, the linear specification for \( \lambda_i \) appears to work well for daily stock returns. Nonetheless, other functional forms that include nonlinearity also may be very useful. In this case lags of \( \lambda_i, \xi_i \), and other variables in the information set may enter a nonlinear function driving the conditional intensity parameter at time \( t \).

The time series model of \( \lambda_i \) is not a true ARMA model in that it is not driven by an unforecastable innovation, but rather a measurable one with respect to \( \Phi_{t-1} \). Nonetheless, we would expect this model to provide a good approximation if \( \lambda_i \) did follow a true latent ARMA structure. Such a model would require simulation methods to compute the likelihood, whereas our model does not.

To this point, we have focused on the conditional dynamics governing the number of jumps; however, the distribution of the jump size, which is postulated to be Gaussian, may also change and display conditional dynamics. To explore this further, we consider two extensions of the model. The first extension allows the conditional mean and conditional variance of the jump size distribution to be conditionally normal and a function of past returns,

\[\theta_i = \eta_0 + \eta_1 R_{t-1}D(R_{t-1}) + \eta_2 R_{t-1}(1 - D(R_{t-1}))\]

(12)

and

\[\delta_i^2 = \xi_0^2 + \xi_1 R_{t-1}^2,\]

(13)

where \( D(x) = 1 \) if \( x > 0 \) and 0 otherwise and \( \eta_0, \eta_1, \eta_2, \xi_0, \) and \( \xi_1 \) are parameters to be estimated. This specification of the conditional mean of the jump size allows some flexibility regarding where jumps are centered. For example, if in
the last period the market experienced a gain (decline), then today’s conditional mean of the jump size is \( \eta_0 + \eta_1 R_{t-1} \) (\( \eta_0 + \eta_1 R_{t-1} \)). Thus the first moment of the jump size distribution can respond to whether the last period’s market return was positive or negative and to its magnitude. This formulation may capture the rally after a stock market crash through a change in jump direction. For this model to capture this effect, we would expect \( \eta_j < 0 \). To investigate whether the jump size variance is sensitive to the overall level of market volatility, we allow \( R_{t-2}^2 \) to affect \( \delta_i^2 \). We label the extension in (12)-(13) as ARJI-R\(_{t-1}^2\).

A second extension of interest is whether the variance of the jump size is a function of the GARCH volatility. The formulation for \( \theta_i \) is the same as in (12), but now

\[
\delta_i^2 = \xi \delta_i^2 \frac{\xi_i}{h_i},
\]

which we refer to as ARJI-\( h_i \). The difference between these two specifications [(13) and (14)] of the jump size variance is that whereas the lagged squared return is a proxy for the last period’s market volatility, \( h_i \) is a prediction of the time \( t \) GARCH volatility component of our model. If the variance of the jump size is sensitive to contemporaneous market volatility, then this new specification of the conditional variance of jumps may capture this effect better than in (13).

A comparison of the distribution of jumps identified by the ARJI-R\(_{t-1}^2\) and ARJI-\( h_i \) models with the ARJI specification should be interpreted with care. The introduction of time variation in the jump size distribution may result in considerably different inference regarding jumps. In addition to the changing arrival rate of jumps, both the ARJI-R\(_{t-1}^2\) and ARJI-\( h_i \) specifications allow the characterization of jumps to change over time. For example, what may be identified by the ARJI model as two or more jumps could be seen as one jump using a larger jump size variance in the ARJI-\( h_i \) framework.

To derive the conditional mean and variance of our model, first redefine (1) so that \( R_t = B_t + C_t \) where \( C_t = \sum_{i=0}^{t} Y_{i-k} \) is the jump component and \( B_t = R_t - C_t \) is the remainder. Because our GARCH-jump model is a discrete mixture of distributions, the \( \theta \) th uncentered moment of \( C_t \) is

\[
E[C_t|\Phi_{t-1}] = \sum_{j=0}^{\infty} E[C_t|n_t = j, \Phi_{t-1}] \times P(n_t = j|\Phi_{t-1}), \quad i > 0.
\]

Standard calculations show the first two conditional moments of \( C_t \) to be

\[
E[C_t|\Phi_{t-1}] = \theta_i \lambda_i
\]

and

\[
\text{var}(C_t|\Phi_{t-1}) = (\delta_i^2 + \theta_i^2)\lambda_i.
\]

Using these results, the conditional mean and variance for returns are

\[
E[R_t|\Phi_{t-1}] = \mu + \sum_{i=1}^{\theta} \phi_i R_{t-1} + \theta_i \lambda_i
\]

and

\[
\text{var}(R_t|\Phi_{t-1}) = h_t + (\delta_i^2 + \theta_i^2)\lambda_i.
\]

Note that time variation in \( \lambda \), and the conditional jump size distribution affect both the conditional mean and conditional variance of returns. The conditional variance of returns is an increasing function in the jump intensity, whereas the conditional mean of returns can be increasing or decreasing depending on the sign of \( \theta_i \). The conditional jump dynamics also imply conditional skewness and kurtosis. Similar calculations or the results of Das and Sundaram (1997) show the conditional skewness (skew) and conditional kurtosis (kurt) to be

\[
\text{skew}(R_t|\Phi_{t-1}) = \frac{\lambda_i(\theta_i^3 + 3\theta_i \delta_i^2)}{(h_t + \lambda_i \delta_i^2 + \lambda_i \theta_i^3)^{3/2}}
\]

and

\[
\text{kurt}(R_t|\Phi_{t-1}) = 3 + \frac{\lambda_i(\theta_i^4 + 6\theta_i^2 \delta_i^2 + 3\delta_i^4)}{(h_t + \lambda_i \delta_i^2 + \lambda_i \theta_i^3)^2}.
\]

### 3. Specification Tests

To evaluate the in-sample and out-of-sample performance of the proposed models, we consider several specification tests based on the integral transformation of Rosenblatt (1952).

Consider a candidate model with conditional density \( f(R_t|\Phi_{t-1}, \Theta) \), where \( \Theta \) is the unknown parameter vector. Rosenblatt (1952) showed that if the data are drawn from the density \( f(\cdot) \), then the series \( \{\tilde{u}_i\}_{i=1}^{T} \), defined from

\[
\tilde{u}_t = \int_{-\infty}^{R_t} f(v|\Phi_{t-1}, \Theta) dv,
\]

will be iid \( U(0, 1) \). Therefore, testing whether \( \tilde{u}_i \sim \text{iid } U(0, 1) \) is a test for correct model specification.

The first test that we consider is a Pearson goodness-of-fit test for \( \tilde{u}_i \), suggested by Vlaar and Palm (1993). Under the null hypothesis of a correctly specified model,

\[
\sum_{i=1}^{g} \frac{(n_i - E n_i)^2}{E n_i} \sim \chi^2(g - 1),
\]

where \( g \) is the number of equally spaced groups and \( n_i \) is the number of observations of \( \tilde{u}_i \) that occur in group \( i \). After \( g \) is chosen, \( n_i = \sum_{t=1}^{T} I_{it} \), where

\[
I_{it} = \begin{cases} 1 & \text{if } (i-1)/g < u_t \leq i/g \\ 0 & \text{otherwise} \end{cases}
\]

for \( 1 \leq i \leq g \).

To focus on the dynamics of the conditional distribution, Palm and Vlaar (1997) and Berkowitz (1999) recommended applying an inverse normal transformation to the data \( \{\tilde{u}_i\}_{i=1}^{T} \) to test whether the transformed data \( \{\tilde{v}_i\}_{i=1}^{T} \) are independent standard normal. Under the null hypothesis of a correctly specified model, \( \tilde{v}_i \sim \text{iid } N(0, 1) \). Let \( \tilde{v}_i = F^{-1}(\tilde{u}_i) \), where \( F^{-1}(\cdot) \) is the inverse of the standard normal cumulative distribution function. We consider a likelihood ratio (LR) test for

\[
a_0 = a_1 = \cdots = a_5 = 0, \sigma = 1, \text{ in the regression}
\]

\[
\tilde{v}_i = a_0 + a_1 \tilde{v}_{i-1} + \cdots + a_5 \tilde{v}_{i-5} + \sigma v_i,
\]

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
against an unrestricted alternative hypothesis that maintains only that \( \eta \) is normally distributed. Under the null hypothesis, the LR statistic is distributed as \( \chi^2(7) \).

The final test is based on work of Diebold, Gunther, and Tay (1998) and evaluates graphically whether \( \tilde{u}_t \sim \text{iid } U(0, 1) \). This is done through a histogram of \( \{\tilde{u}_t\}_{t=1}^T \) and estimates of the autocorrelation functions of successive powers of \( \tilde{u}_t \). For a correctly specified model, \( \tilde{u}_t \) will have the density of the uniform distribution, whereas the powers of \( \tilde{u}_t \) should display no evidence of autocorrelation. Like the Pearson goodness-of-fit test, the density estimate of \( \tilde{u}_t \) provides a measure of a model’s ability to capture the unconditional distribution of returns, whereas the estimates of the autocorrelation function of powers of \( \tilde{u}_t \) assess the adequacy of the conditional distribution. Note that all of the tests in this section ignore parameter uncertainty, and in practice the tests are applied to models with an estimated parameter vector.

4. DATA

The data consist of daily close and intraday high and low for the DJIA. Returns are defined as 100 times the first difference in the logarithm of the close of the DJIA index from October 1, 1928 to January 11, 2000. The dataset contains a total of 18,947 observations.

Table 1 reports summary statistics for the sample for 1928–1984. Standard errors robust to heteroscedasticity are given in parentheses. Although the skewness coefficient is not significantly different from 0, the deviation from normality is apparent in the excessive kurtosis, which is 18.5198. Evidence of time dependence is found using the modified Ljung–Box (LB) statistic (West and Cho 1995), which is robust to heteroscedasticity and reported for autocorrelations up to 15 lags in the last column of Table 1. The modified LB statistics show strong serial correlations in both the levels and the squares of the return series. This is consistent with the results of Brock, Lakonishok, and LeBaron (1992), who found that the serial correlations in DJIA returns are significant but unstable and depend on the sample period.

Figure 1 illustrates the statistical properties of the data series. Volatility clustering is clear; high volatility in one period tends to be followed by high volatility in the next period. The GARCH variance structure is appropriate for modeling this phenomenon.

<table>
<thead>
<tr>
<th>Table 1. Summary Statistics, 1928–1984</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>( R_i )</td>
</tr>
<tr>
<td>(</td>
</tr>
<tr>
<td>( R_i )</td>
</tr>
<tr>
<td>(</td>
</tr>
<tr>
<td>( R_i^2 )</td>
</tr>
<tr>
<td>(</td>
</tr>
</tbody>
</table>

NOTE: Summary statistics for daily DJIA returns from Oct. 1, 1928 to Dec. 31, 1984. Q(15) are modified Ljung–Box statistics robust to heteroscedasticity for serial correlation with 15 lags. Standard errors robust to heteroscedasticity are in parentheses, and \( p \) values are in square brackets.

5. AN APPLICATION TO STOCK MARKET RETURNS

5.1 Estimation Results

Estimation of all models was conducted using data from 1928–1984; the remaining data from 1985–2000 are reserved for out-of-sample analysis. For all models, \( \tau = 20 \) was selected as the truncation point for the distribution determining the number of jumps, based on the selection method discussed in Section 2. Table 2 presents the estimation results for the GARCH-constant jump intensity model over various subsamples, and Table 3 presents the results for all models over the full in-sample period. Misspecification tests based on the modified LB statistic are reported for autocorrelation in the squared
standardized residuals ($Q_i^2$) and the jump intensity residuals ($\xi_i$) for 15 lags at the bottom of each table. Table 4 reports additional specification tests. We found it necessary to use an AR(2) model to capture autocorrelation in the conditional mean of stock returns for all models. Similarly, all models use a GARCH(1,1) and, where appropriate, an ARJI(1,1) specification which LB statistics suggest is adequate for the in-sample period. In what follows, we use ARJI to denote the ARJI(1,1) model.

Estimates for the constant jump intensity model over the different sample periods, 1928–1950, 1951–1969, and 1970–1984 are given in Table 2. This model imposes the restrictions $\lambda_i = \lambda$, $\theta_i = \theta$, and $\delta_i = \delta$ in the ARJI model. These results provide evidence of changing jump dynamics over time. For example, the jump intensity parameter $\lambda$ is .1512 for the full sample (reported in Table 3 under Constant) but varies substantially across different subsamples. $\lambda$ is .1116 in 1928–1950, and increases to 1.6742 in 1951–1969. The estimates for 1951–1969 indicate that the jump component is more important, whereas the GARCH effects diminish compared to results for other periods. Both $\theta$ and $\delta$ in Table 2 exhibit instability over the subsamples. $\delta$ is estimated as high as 1.4624 and as low as .2711.

Evidence of time variation in $\lambda$ is also supported by the LB statistics for $\xi$. Recall that $\xi$ is the measurable shock constructed by the econometrician using the ex post filter. In a correctly specified model, $\xi$ should not display any systematic behavior; otherwise, it could be exploited to improve

### Table 2. Estimates of the Constant Intensity Jump Model for Different Sample Periods

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.0753</td>
<td>0.2138</td>
<td>-0.005</td>
</tr>
<tr>
<td></td>
<td>(0.0097)</td>
<td>(0.0331)</td>
<td>(0.0142)</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>0.0940</td>
<td>0.1848</td>
<td>0.1623</td>
</tr>
<tr>
<td></td>
<td>(0.0125)</td>
<td>(0.0154)</td>
<td>(0.0166)</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>-0.0267</td>
<td>-0.0818</td>
<td>-0.0166</td>
</tr>
<tr>
<td></td>
<td>(0.0123)</td>
<td>(0.0151)</td>
<td>(0.0168)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.0009</td>
<td>2.07e-13</td>
<td>0.0072</td>
</tr>
<tr>
<td></td>
<td>(0.0009)</td>
<td>(0.0005)</td>
<td>(0.0024)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.0596</td>
<td>0.1118</td>
<td>0.0402</td>
</tr>
<tr>
<td></td>
<td>(0.0058)</td>
<td>(0.0139)</td>
<td>(0.0056)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>2.913</td>
<td>0.0008</td>
<td>0.9472</td>
</tr>
<tr>
<td></td>
<td>(0.0069)</td>
<td>(0.0264)</td>
<td>(0.0074)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>1.4624</td>
<td>2.711</td>
<td>1.430</td>
</tr>
<tr>
<td></td>
<td>(1.345)</td>
<td>(0.346)</td>
<td>(2.279)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>-6.447</td>
<td>-1.129</td>
<td>0.9892</td>
</tr>
<tr>
<td></td>
<td>(1.147)</td>
<td>(0.184)</td>
<td>(0.7942)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1.116</td>
<td>1.6742</td>
<td>0.1996</td>
</tr>
<tr>
<td></td>
<td>(0.186)</td>
<td>(0.4043)</td>
<td>(0.1066)</td>
</tr>
<tr>
<td>$Q_i^2$</td>
<td>52.72</td>
<td>27.25</td>
<td>19.01</td>
</tr>
<tr>
<td></td>
<td>[0]</td>
<td>[0.03]</td>
<td>[0.21]</td>
</tr>
<tr>
<td>$Q_{\xi}$</td>
<td>23.74</td>
<td>23.66</td>
<td>30.14</td>
</tr>
<tr>
<td></td>
<td>[0.07]</td>
<td>[0.07]</td>
<td>[0.01]</td>
</tr>
<tr>
<td>Log-likelihood</td>
<td>-9498.74</td>
<td>-4213.27</td>
<td>-4814.20</td>
</tr>
</tbody>
</table>

**Note:** Standard errors are in parentheses; $p$ values are in square brackets. $Q_i^2$ is the modified Ljung-Box portmanteau test, robust to heteroscedasticity, for serial correlation in the squared standardized residuals with 15 lags for the respective models. $Q_{\xi}$ is the same test for serial correlation in the jump intensity residuals. The model. We test for dependence in $\xi$, in the constant jump intensity model using the modified LB statistics denoted by $Q_{\xi}$ in Tables 2 and 3. In Table 3, the $Q_{\xi}$ statistic rejects the constant intensity assumption.

### Table 3. Estimates of the ARJI Models, 1928–1984

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\mu$</th>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\omega$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\delta$</th>
<th>$\theta$</th>
<th>$\lambda$</th>
<th>$Q_i^2$</th>
<th>$Q_{\xi}$</th>
<th>$\log$-likelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td>1928–1950</td>
<td>0.0692</td>
<td>0.1389</td>
<td>-0.0561</td>
<td>0.0005</td>
<td>-0.0665</td>
<td>0.9192</td>
<td>0.8744</td>
<td>0.0539</td>
<td>0.3222</td>
<td>18.27</td>
<td>30.96</td>
<td>-18,315.61</td>
</tr>
<tr>
<td>1951–1969</td>
<td>0.0700</td>
<td>0.1331</td>
<td>-0.0528</td>
<td>0.0005</td>
<td>-0.0461</td>
<td>0.9491</td>
<td>0.8777</td>
<td>0.0626</td>
<td>-0.3962</td>
<td>7.54</td>
<td>9.84</td>
<td>-18,276.10</td>
</tr>
<tr>
<td>1970–1984</td>
<td>0.0713</td>
<td>0.1486</td>
<td>-0.0581</td>
<td>0.0020</td>
<td>-0.0665</td>
<td>0.9564</td>
<td>0.9566</td>
<td>0.0760</td>
<td>-0.2518</td>
<td>14.70</td>
<td>15.60</td>
<td>-18,232.45</td>
</tr>
</tbody>
</table>

**Note:** Standard errors are in parentheses; $p$ values are in square brackets. $Q_i^2$ is the modified Ljung-Box portmanteau test, robust to heteroscedasticity, for serial correlation in the squared standardized residuals with 15 lags for the respective models. $Q_{\xi}$ is the same test for serial correlation in the jump intensity residuals. Constant is the constant jump intensity, constant jump size model.
Table 3 reports a series of model estimates for the simplest constant intensity jump model, the ARJI model with a constant jump size distribution, and the fully dynamic jump models ARJI-$h_i$ and ARJI-$R_{t-1}^2$. The log-likelihood for the constant intensity jump model is $-18,315.61$, which represents an increase of 356.0 compared to a plain AR(2)-GARCH(1,1) model (estimates not reported) with no jumps. This suggests that jumps in the DJIA may be important. Moreover, the ARJI parameterization, which allows the conditional jump intensity to vary over time, provides an improvement in the likelihood over that obtained in the constant intensity model. The (LR) test of constant jump intensity ($\gamma = 0$) against the ARJI specification is 79.02. This test is nonstandard, because under the null hypothesis, the constant jump intensity is asymptotically unidentified. (Under the null hypothesis, if $|\rho| < 1$, then $\lambda_0 = \lambda_0/(1 - \rho)$. ) Although methods such as those of Davies (1987) and Hansen (1996) could be used to obtain a $p$ value, the magnitude of the test statistic suggests that the ARJI parameterization provides a significant statistical improvement over the constant jump intensity model. In addition, the ARJI model captures the autocorrelation in $\xi_t$ found in the constant intensity model. The $p$ values for the $Q_{\xi_t}$ statistic are .01 for the constant intensity model and .83 for the ARJI model.

The $\rho$ parameter in the ARJI model, estimated to be .9153 with an asymptotic standard error of .02, provides a measure of the persistence in the conditional jump intensity. This suggests that a high probability of many (few) jumps today tends to be followed by a high probability of many (few) jumps tomorrow. However, unconditionally, jumps are infrequent. The unconditional jump intensity [defined in (9)] is only .1547, which is very close to the $\lambda$ reported in the constant intensity model in Table 3. $\gamma$ measures the sensitivity of $\lambda_t$ to the past shock, $\xi_{t-1}$. A unit increase in $\xi_{t-1}$ results in a dampened effect (only .5) on next period’s jump intensity.

Figure 1 displays returns, the conditional intensity, and the conditional standard deviation for the ARJI specification. This figure shows considerable variation in the conditional jump intensity. To illustrate, consider the 1940s, during which the daily jump intensity ranged from only .03 up to 2.12. This indicates that in the 1940s there were periods where almost no jumps were expected (.03) and periods where several jumps (2.12) were expected.

What effect does time variation in $\lambda_t$ have on the distribution of the number of jumps? Figure 2 provides two illustrations of the Poisson distribution governed by $\lambda_t$ for two dates in our sample period against the Poisson distribution with the constant intensity assumption $\lambda_t = \lambda$. This figure shows that small changes in $\lambda_t$ can have important effects on the Poisson distribution. It also shows that the risk associated with the realization of jumps in the constant-intensity model is considerably less than in the ARJI model.

Figure 3 displays the predictive content of $\lambda_t$ in forecasting a jump around the 1929 crash. In-sample, this model suggests that at least one jump was expected days before the crash on October 28. A 95% confidence interval for the effect of one jump on returns is $(-2.7, 1.91)$. In fact, two or three jumps were not unlikely, because their probability of occurring just days before the crash (October 24) was .22 and .08.

The last two columns of Table 3 report estimates for two models that extend the jump dynamics in the ARJI specification. Both the ARJI-$R_{t-1}^2$ and ARJI-$h_i$ models allow the conditional mean and conditional variance of the jump size distribution to be a function of past returns. As measured by LR tests, both models provide significant improvement over the simpler ARJI specification. Also note that these extensions do not appear to alter the dynamics found in conditional intensity. For example, $\lambda_t$ is still very persistent, and $\rho$ is .91 in the ARJI-$R_{t-1}^2$ model and .84 in the ARJI-$h_i$ model.

Our estimates of these final models provide evidence that the jump direction is asymmetric and sensitive to the state of the stock market. In both the ARJI-$R_{t-1}^2$ and ARJI-$h_i$ specifi-
cations, $\eta_2$ is significantly negative, which implies that after a stock market downturn, the direction of a jump in the next period is more likely to be positive than negative.

Given the large dataset used in this study, it is not surprising that the Pearson goodness-of-fit test statistics in Table 4 identify problems in all models, although the $p$ value for the ARJI-$h$ formulation is only 0.031. However, the addition of jumps to a plain GARCH(1,1) model does result in a dramatic improvement in the test values. (A GARCH(1,1) model with normal innovations has a $p$ value of 6.3e-51.) The jump models may not do a good job fitting the unconditional distribution of returns, but the LR test introduced in Section 3 and reported in Table 4 indicates that the addition of jump dynamics improves the specification of the conditional distribution in all models as compared to the constant-intensity specification.

5.2 Out-of-Sample Analysis

This section discusses and evaluates the models in the out-of-sample period, 1985–2000. In all cases, model parameters were initially set at the values in Table 3; then, after every 250 observations, the model parameters are updated by estimation, which includes the most recently available data.

The last row of Table 4 presents Pearson goodness-of-fit test statistics for the out-of-sample predictive density of the models. Similar to the in-sample results, the ARJI-$h$ model performs the best among the group with the ARJI model ranking second, with a $p$ value of 0.003. Figures 4, 5, and 6 display the results from the Diebold et al. (1998) evaluation of the forecast density for a GARCH(1,1) model with normal innovations and ARJI and ARJI-$h$ models. Recall that under a correctly specified model, $u_t$ derived from the integral transformation should be iid $U(0,1)$ and display no autocorrelation. Panel (a) of Figures 4, 5 and 6 present an estimated density of $u_t$. The benchmark GARCH(1,1) model has obvious problems matching the unconditional features of the data. Both the ARJI and ARJI-$h$ models provide successive improvements compared to the GARCH density. Estimates of the autocorrelations of powers of $u_t$ are found in panels (b)–(e) of each figure. All models perform well in capturing the dynamics in the conditional forecast density, and there is some marginal

![Figure 4](image_url)

Figure 4. GARCH(1,1), Estimates of the Density of $u_t$: (a) and Autocorrelation Functions of Powers of $u_t$: (b) $(u_t - \bar{u}_t)$; (c) $(u_t - \bar{u}_t)^2$; (d) $(u_t - \bar{u}_t)^3$; (e) $(u_t - \bar{u}_t)^4$. The horizontal lines superimposed on the histogram (a) are approximate 95% confidence intervals for the individual bin heights under the null that $u_t$ is iid $U(0,1)$. The horizontal lines superimposed on the correlograms (b)–(e) are approximate 95% confidence intervals for the individual bin heights under the null that $u_t$ is iid $U(0,1)$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
improvement from adding the jump dynamics to the GARCH model.

Out-of-sample one-step-ahead forecasts of volatility are evaluated against the Parkinson (1980) range statistic. This is the intraday logarithm of the ratio of the high to the low transaction prices and is a measure of the daily latent volatility. Traditional measures of volatility, such as squared returns, are very noisy and thus practically useless as a measure of forecasting performance. The range has been used in many studies, and the simulation results of Andersen and Bollerslev (1998, footnote 20) show it to be a very efficient estimator compared to daily squared returns.

A good model should not only explain the variation of the data series within the sample, but also accurately forecast the series out of the sample. Table 5 shows the $R^2$ from a linear regression of the range on a constant and the one-period-ahead predicted standard deviation from the models. The first row of Table 5 covers the period 1985–2000, and the second row covers the period 1990–2000. A benchmark GARCH(1,1) is included for comparison purposes. The GARCH(1,1), constant-intensity, and ARJI models provide very similar forecasts. However, both the ARJI-$R^2_{t-1}$ and ARJI-$h_t$ formulations perform much better than the models without jump size dynamics. Because the jump models are designed to explain extreme market movements, the ranking of the models in Table 5 for 1985–2000 may be sensitive to the number of significant stock market downturns in this period. The results for the 1990–2000 period provide a check on this, because there are only 2 days with $|R_t| > 5.0$ ($R_t = -7.4$ on 27/10/97 and $R_t = -6.5$, on 31/8/98) in this shorter period.

<table>
<thead>
<tr>
<th>Sample period</th>
<th>GARCH(1,1)</th>
<th>Constant</th>
<th>ARJI</th>
<th>ARJI-$h_t$</th>
<th>ARJI-$R^2_{t-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1985–2000</td>
<td>0.269</td>
<td>0.262</td>
<td>0.264</td>
<td>0.307</td>
<td>0.392</td>
</tr>
<tr>
<td>1990–2000</td>
<td>0.334</td>
<td>0.334</td>
<td>0.334</td>
<td>0.359</td>
<td>0.364</td>
</tr>
</tbody>
</table>

NOTE: This table shows the $R^2$ from a regression of the range statistic on a constant and the out-of-sample, one-period-ahead forecast of the conditional standard deviation for the respective models. The second column contains results for a GARCH(1,1) model estimated assuming normal innovations.

Figure 5. ARJI, Estimates of the Density of $u_t$, (a) and Autocorrelation Functions of Powers of $u_t$: (b) $(u_t - \bar{u})$; (c) $(u_t - \bar{u})^2$; (d) $(u_t - \bar{u})^3$; (e) $(u_t - \bar{u})^4$. The horizontal lines superimposed on the histogram (a) are approximate 95% confidence intervals for the individual bin heights under the null that $u_t$ is iid $U(0,1)$. The horizontal lines superimposed on the correlograms (b)–(e) are approximate 95% confidence intervals for the individual bin heights under the null that $u_t$ is iid $U(0,1)$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
as compared to 6 days in 1985–1990. The results in Table 5 reverse our ranking of the ARJI-$_{R_{t-1}}$ and ARJI-$_h$ models compared to the in-sample log-likelihood values and statistical tests given in Table 4.

Estimation of our dynamic jump model remains tractable by assuming that the updating scheme for the conditional jump intensity can be projected onto past observables. As a result, ex post inference regarding the jump distribution is directly available from estimation. Figure 7 displays the ex ante and ex post probabilities of at least one jump occurring, calculated using Eqs. (2) and (6) for the ARJI-$_{R_{t-1}}$ model. The prediction of this model can be contrasted with the constant-intensity specification, which predicts a jump with constant probability .140 for the whole subsample.

Figure 8 presents further evidence supporting the hypothesis that the ARJI model may contain predictive information for stock market crashes. The graph in Figure 8(a) plots the market returns during October 1987 and the negative 25% crash on October 19. The predicted conditional average number of jumps rises from less than .2 at the start of October 1987 to 1.37 on the day of the crash. Recall that this is an out-of-sample prediction and that $\lambda_t$ is based on the previous day’s information set. The value 1.37 represents a more than eightfold increase in the conditional intensity over its unconditional value of .1547.

6. DISCUSSION

All models of conditional jump dynamics estimated on the DJA returns show significant persistence in the conditional intensity. This indicates that the risk associated with jumps in stock market returns is systematic and may be important for derivative pricing. Furthermore, the jump dynamics do not reduce the GARCH effects in any of the models. The GARCH parameters across all models, including a no-jump AR(2)-GARCH(1,1) model (not reported here), are very similar. The ARJI family of models that we have investigated appear to explain dynamics not captured in the constant intensity specification.

Evidence of persistence in the jump intensity can also be found in the analysis of Lin, Knight, and Satchell (1999) applied to intraday equity returns. These authors proposed a
pure jump diffusion model that allows the intensity to depend on the past conditional variance. Similar to our specification, this model can capture time dependence in the conditional jump intensity; however, the log-likelihood function does not have a closed-form solution.

In Section 5 we reported a robust result with respect to a switch in the jump direction after a stock market decrease. Both models with jump size dynamics have a significant $\eta > 0$. Using the estimates from Table 3 for the ARJI-$R^t_{t-1}$ model, any decrease in the market of 2.5% or more implies a positive conditional mean in the next period’s jump size distribution. Therefore, after a stock market crash, the likelihood of a jump in the next period does not necessarily decrease, but the likelihood of a negative jump decreases and the likelihood of a positive jump increases. This asymmetry in the jump direction is also seen in the out-of-sample data period. For instance, there are 45 times in which returns decrease by 2.5% or more. Of the 45 times that $R_{t-1} < -2.5$, in 31 cases $R_t > 0$ on the next trading day. Conditional on $R_{t-1} < -2.5$, the sample averages of $R_t$, $\theta_t$, and $\lambda_t$ (the next-period values) are .79, .31, and .90. This means that after a market downturn of $-2.5\%$ or more, a positive jump in returns is very likely on the next trading day.

Previous research (Bates 2000; Chernov et al. 1999) has investigated a possible relationship between jump intensity and a stochastic volatility specification of volatility. In our model, the analog relationship is between the conditional jump intensity parameterization and the GARCH specification. In general, our results were mixed. After permitting the variance of the jump size distribution to be a function of the GARCH variance (ARJI-$\lambda_t$), we found no evidence (estimates not reported here) that the GARCH process affects the conditional intensity specification for our dataset. However, we did find that the GARCH variance was positive and significant in affecting $\lambda_t$ in the ARJI-$R^t_{t-1}$ model.

This study documents evidence of significant conditional dynamics in the distribution governing the number of jumps and jump size. An important topic from a risk management perspective is the prediction of extreme volatility. Our model captures this through jumps. In particular, we present evidence that before both the 1929 (in-sample) and 1987 (out-of-sample) stock market crashes, the conditional expected number of jumps increased. This suggests that time series data alone may contain predictive content that jump models such as those explored in this article could exploit in forecasting future market downturns. One way that jump predictability may work is that proportionately, jumps may become more important just before a crash. Thus the GARCH or SV volatility component of the conditional variance may become less important in describing the total volatility while jumps become more important. These issues will be explored in future work.

7. CONCLUSION

This article has proposed an ARJI model to capture jump dynamics in stock market returns. Our model extends the GARCH-constant jump intensity model with a time-varying
jump intensity and jump size distribution. The article proposes a simple filter to provide inference regarding the number of jumps. Jump intensity is modeled as a parsimonious ARMA structure driven by an ex post measure of the jump probability.

We find significant time variation in the conditional jump intensity and the jump size distribution in our application to daily stock market returns. Modeling jump dynamics in this article reveals several improvements compared to the basic GARCH-constant jump intensity model. First, the time-varying jump intensity provides good forecasts of stock market volatility. Second, incorporating the autoregressive jump intensity exploits additional structure ignored in a constant-intensity model. Third, the ARJI model has significantly higher ex ante probabilities regarding jumps on the days of stock market downturns.

We find that time series dynamics in the jump size distribution are important in providing a good characterization of the data. The most favorable model based on an in-sample and out-of-sample analysis combines the autoregressive jump intensity with a time-varying jump size distribution.

ACKNOWLEDGMENTS

The authors thank Jeff Wooldridge, an associate editor, and two anonymous referees for their very helpful comments and suggestions. The authors are also grateful for comments from Adolf Buse, Ray Chou, Stephen Gordon, Matthew Higgins, Bob Korkie, Tom McCurdy and seminar participants at the Midwest Econometrics Meetings, Chicago 2000; North American Summer Econometrics Society Meetings 2001, University of Alberta, and the University of Toronto. The second author thanks the Social Sciences and Humanities Research Council of Canada for financial support.

[Received September 2001. Revised November 2001.]

REFERENCES


