2901H,S

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Single Agent Dynamics: Problem Set 3
(Due: March 28, 2006)

You may work alone or in groups of two or three, but you need to write up the PS individually. If you choose to work in groups, please inform me of your group members. You should also send me a copy of the computer codes used in your solution.

Rust (1987): Machine Replacement

Consider a capital replacement problem similar to that in Rust (1987). Firms each use one machine to produce output in each period. These machines age, becoming more likely to breakdown, and in each time period the firms have the option of replacing the machines. Let a_t be the age of your machine at time t and let the expected current period profits from using a machine of age a_t be given by:

$$\Pi(a_t,\ i_t,\ \epsilon_{0t},\ \epsilon_{1t}) = \begin{cases} \theta_1 a_t + \epsilon_{0t} & \text{if } i_t = 0 \\ R + \epsilon_{1t} & \text{if } i_t = 1 \end{cases}$$

where $i_t = 1$ if the firm decides to replace the machine at t, R is the net cost of a new machine, and the ϵ_t's are time specifc shocks to the utilities from replacing and not replacing. Assume that these ϵ_t's are i.i.d. logit errors.

Lets assume a very simple state evolution equation:

$$a_{t+1} = \begin{cases} \min\{5, a_t + 1_g\} \text{ if } & i_t = 0 \\ 1 & \text{if } i_t = 1 \end{cases}$$

In words, if the firm decides not to replace, the machine ages by one year (up to a maximum of 5 years - after 5 years machines don't age). If the firm replaces in the current year, the age next year is 1. Note that there are thus only 5 possible values of a_t - 1,2,3,4, and 5.

1. Write down the sequence problem for a firm maximizing the EDV of future profits (assume an infinite horizon).

2. Write down Bellman's equation (the functional equation) for the value function of the firm. Use the "alternative-specific" value functions method from in class, i.e., define $\bar{V}_0(a_t)$ and $\bar{V}_1(a_t)$, where $\bar{V}_0(a_t)$ is the firm's value at time t if it chooses not to replace the machine and $\bar{V}_1(a_t)$ is the firm's value at time t if it replaces the machine. The functional equation should be a mapping from this pair of equations to itself. (For the parts that follow, remember from class that there are two ways to write the functional equation, and that one of them will be easier to use than the other.)

3. On the computer, write a procedure that solves this dynamic programming problem given values of the parameters (θ_1, R). Assume that $\beta = 0.9$. Your procedure should iterate the contraction mapping on the two alternative-specific value functions (two five-vectors) until the V functions don't change very much. Remember that given the logit error assumption there is an analytic solution to the expectation of the max in these equations and that Euler's constant is approximately 0.5775. (Note: for reference my function is about 15 lines of code.)

4. Solve the model for the parameters ($\theta_1 = -1, R = -3$) . Suppose $a_t = 2$. Will the firm replace its machine in period t? For what value of $\epsilon_{0t} - \epsilon_{1t}$ is the firm indifferent between replacing its machine or not? What is the probability (to an econometrician who doesn't observe the ϵ's) that this firm will replace its machine? What is the EDV of future profits for a firm at state $\{a_t = 4,\ \epsilon_{0t} = 1,\ \epsilon_{1t} = -1.5\}$? (The constant term has been normalized out so this EDV could be negative.)

5. This dataset accompanying this question has just two columns - a_t and i_t. Consider this as cross-sectional data - i.e. there is only one data point per firm. Estimate (θ_1, R) using maximum likelihood. The best way to do this is to write a function that returns $-ln(L)$, where L is the likelihood function and then use a package (e.g. gauss or matlab) to minimize this function. Be

sure to the compute standard errors. The function itself should follow these steps (for reference my likelihood function is four lines of code):

(a) Start with arbitrary $(\theta_1,\ R)$, i.e., these should be the arguments of the function.

(b) Solve the dynamic programming problem given these parameters, i.e., compute the functions $\bar{V}_0(a_t)$ and $\bar{V}_1(a_t)$ using your procedure from part 3.

(c) Using $\bar{V}_0(a_t)$ and $\bar{V}_1(a_t)$, compute the probability of replacement for each possible a_t.

(d) Compute the likelihood of each observation using the above probabilities. Form the log likelihood function by summing the logs of the likelihoods across observations (i.e. assume that each data point is an independent observation).

(e) Return -ln(L). The minus is if you are using a minimization (rather than maximization) procedure, which is typical.