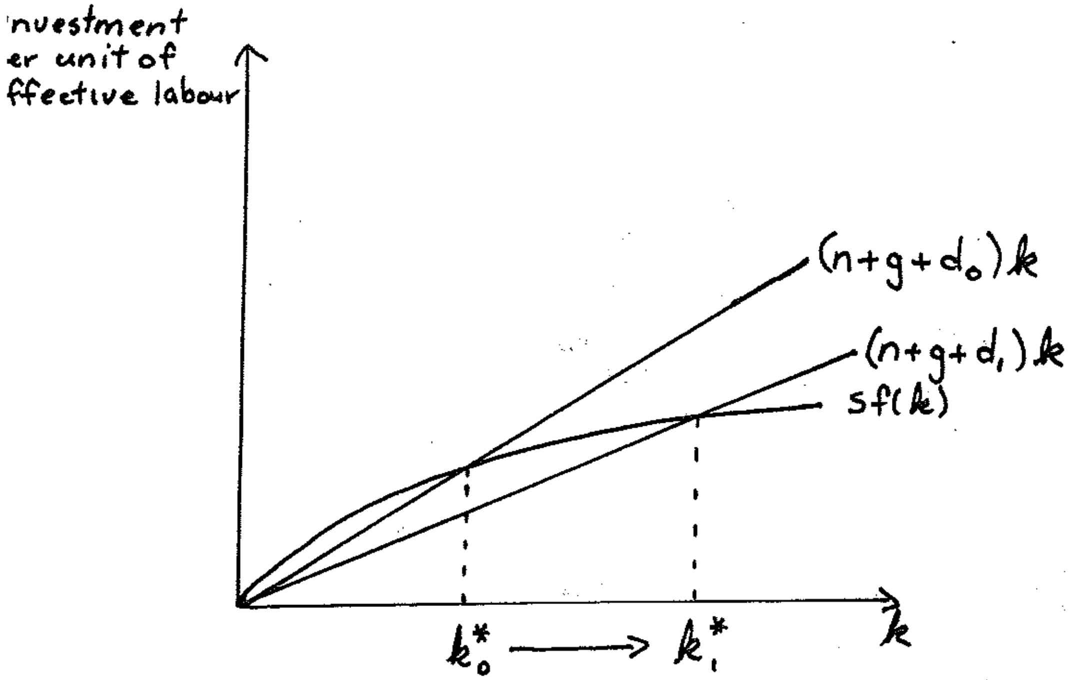
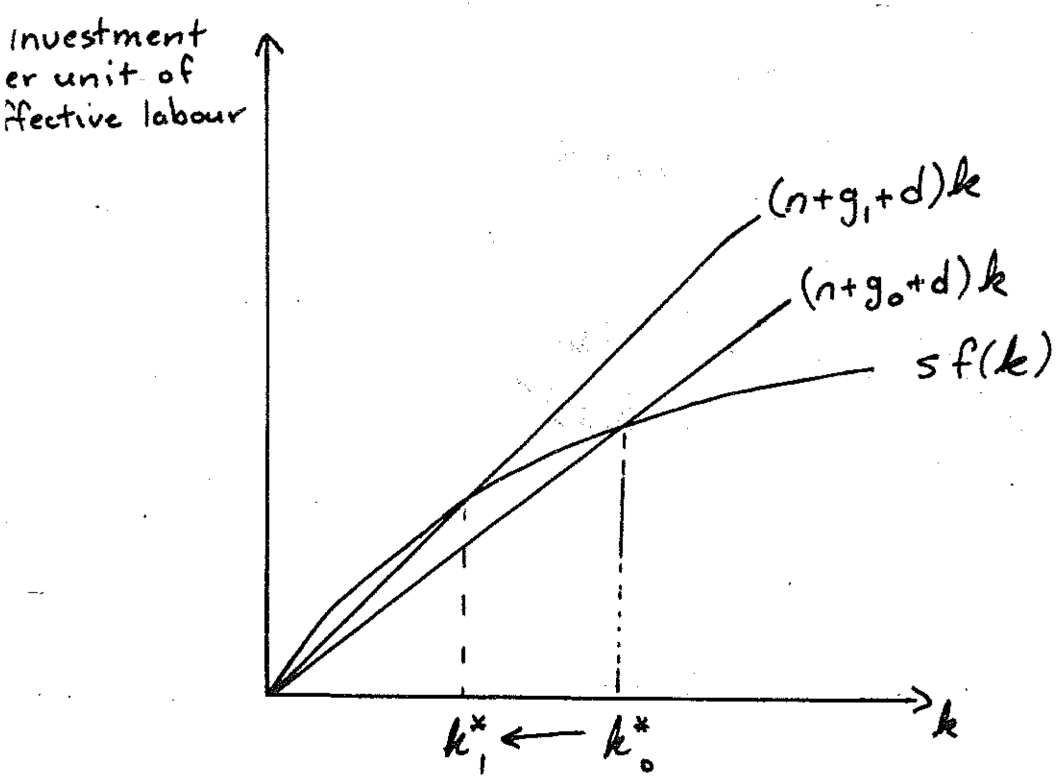
ECO 325S - Suggested Solutions Problem Set #2
(Romer 1.3)

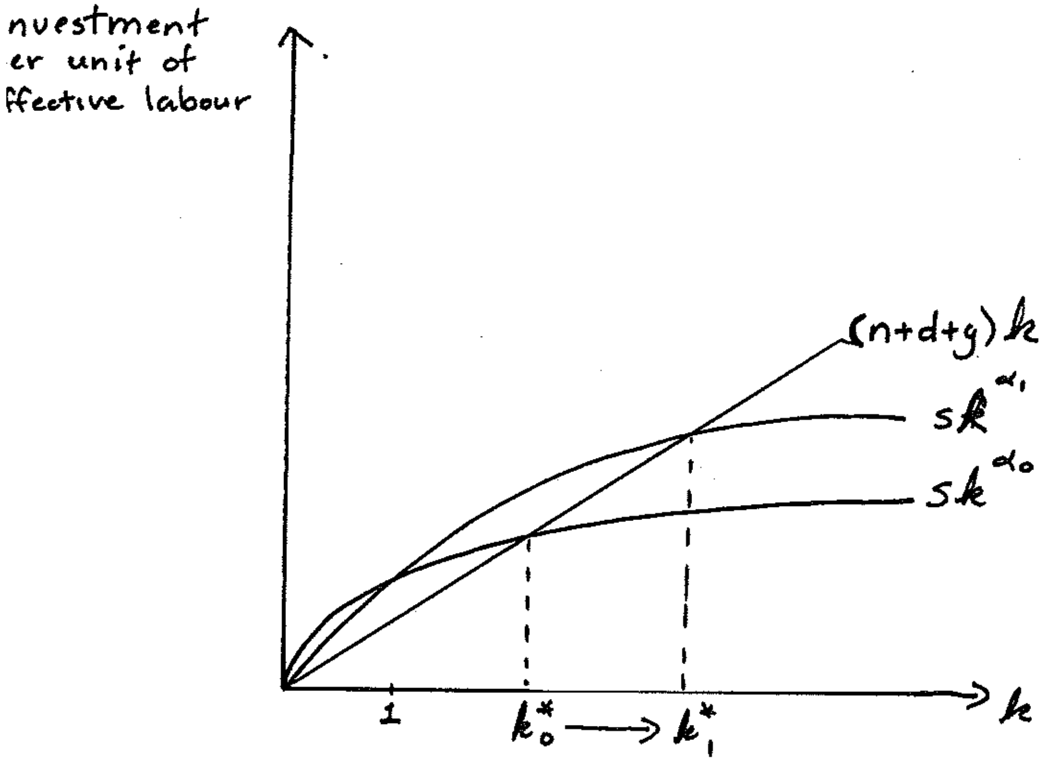
a) let $d_0 > d_1$



b) let $g_0 < g_1$



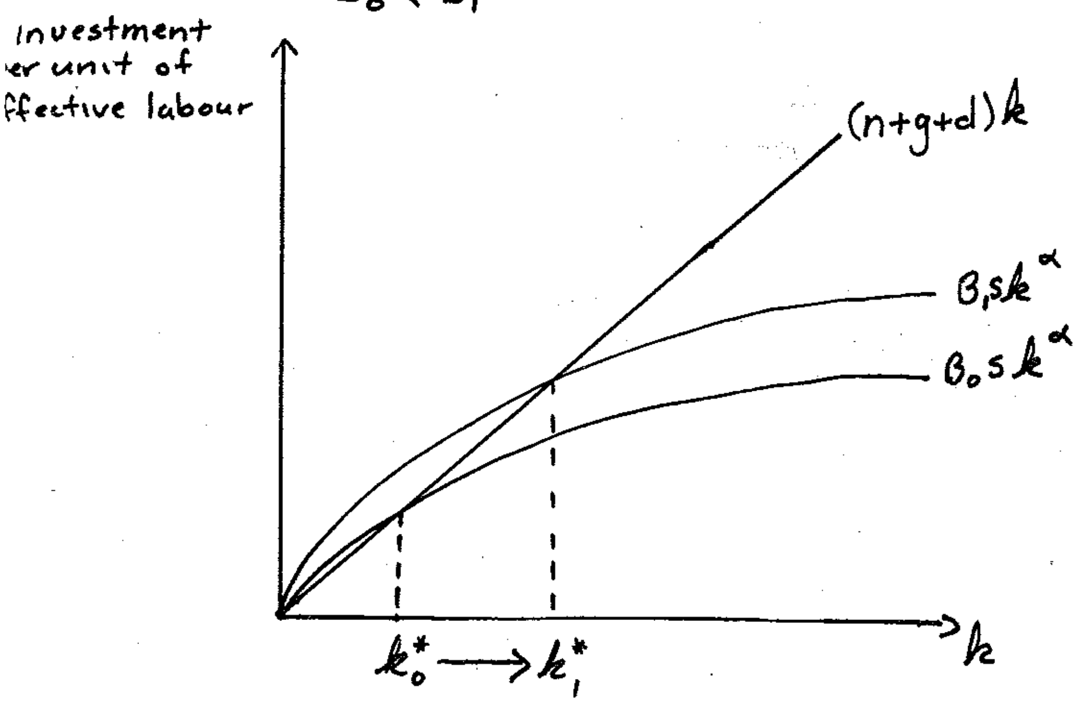
c) $\alpha_0 < \alpha_1$



d) Now the production function is $BK^{\alpha}(AL)^{1-\alpha}$

so the intensive form of the production function is Bk^{α}

let $B_0 < B_1$



2: Romer 1.4

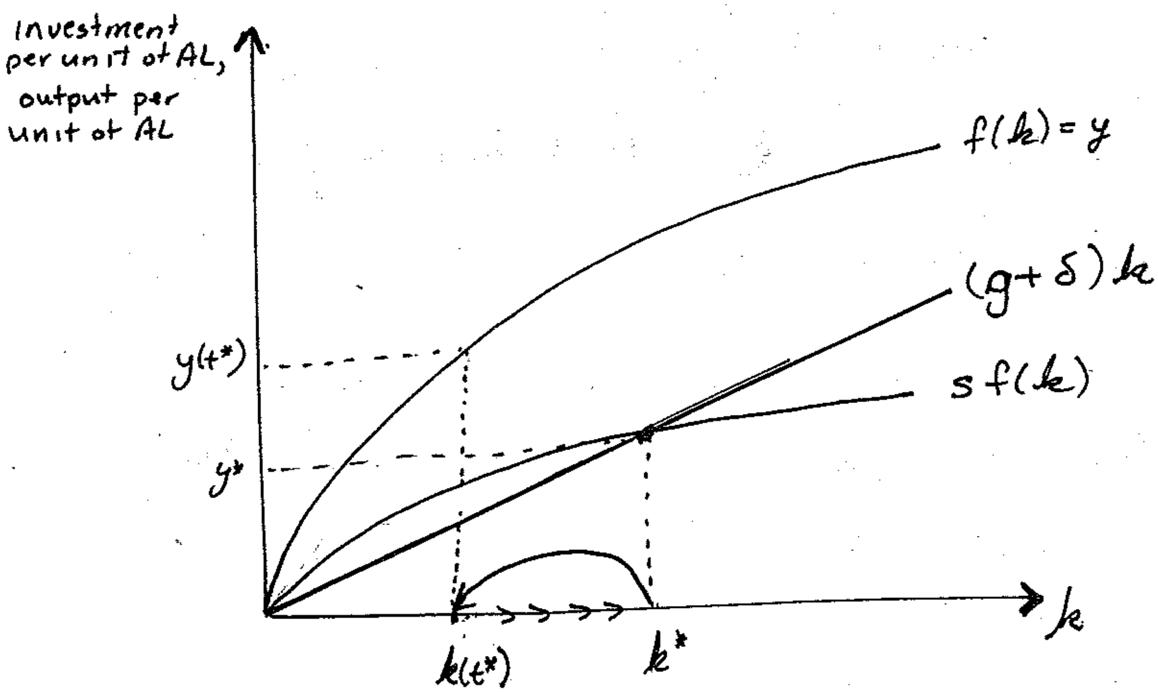
a) Assume $n = 0$ so $L(t) = L(0)$ before the one time jump in the number of workers at time t^*

after time t^* $L(t) = L(t^*) > L(0)$

before time t^* we were on the balanced growth path so $K(t) = k^* A(t) L(0)$ for $t < t^*$

at the time of the jump $k(t)$ decreases because of the jump in L from $L(0)$ to $L(t^*)$. (k^* to $k(t^*)$ on graph)

Since $y(t) = f(k(t))$ and $f'(k(t)) > 0$, the decrease in $k(t)$ at the time of the jump causes $y(t)$ to fall.



2 cont.

b) After the initial change from k^* to $k(t^*)$, $k(t)$ starts to increase back towards k^* since actual investment exceeds break-even investment. Since $\dot{k}(t) > 0$ and $y(t) = f(k(t))$ where $f'(k(t)) > 0$, $\dot{y}(t) = f'(k(t))\,\dot{k}(t) > 0$ so output per unit of effective labour increases

c) Once $k(t)$ has again reached k^*, it will stop growing because actual investment equals break even investment at this point. (the one time jump in the number of workers does not affect the steady state levels)

when the economy again reaches its steady state level of $k(t)$, k^*, output per unit of effective labour will reach its level of $y^* = f(k^*)$.

The steady state levels are the same as they were before.

3. (Romer 1.5)

a) for a Cobb-Douglas production function, the intensive form of the production function is

$$y = f(k) = k^{\alpha} \qquad 0 < \alpha < 1$$

in steady state $\dot{k} = 0 \Rightarrow sf(k^*) = (n+g+\delta)k^*$

$$\Leftrightarrow s(k^*)^{\alpha} = (n+g+\delta)\,k^* \Leftrightarrow \left(\frac{s}{n+g+\delta}\right) = (k^*)^{1-\alpha}$$

$$\Leftrightarrow k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$y^* = f(k^*) = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}, \quad c^* = (1-s)f(k^*) = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

b) the golden rule level of capital is the steady state level of capital per unit of effective labour that maximizes steady state consumption

ie. $$\max_{k} \; f(k) - (n+g+\delta)k$$

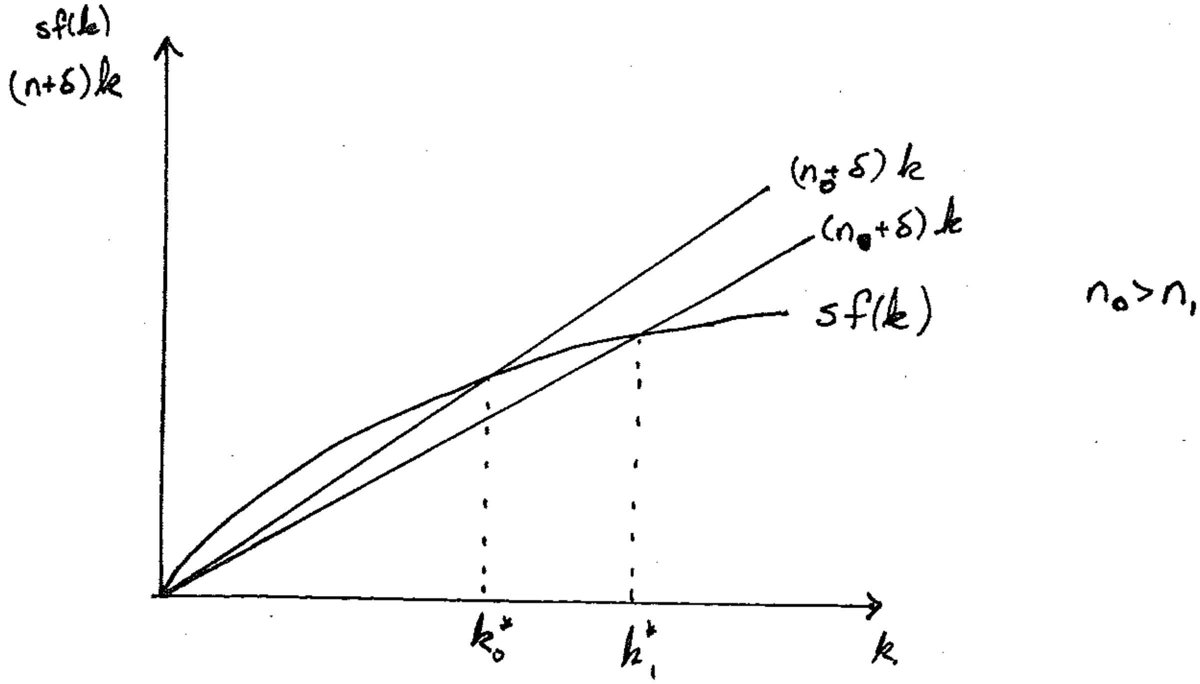
F.O.N.C. $$f'(k) - (n+g+\delta) = 0$$

so $$\alpha (k^G)^{\alpha-1} = (n+g+\delta)$$

$$\Rightarrow \boxed{k^G = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}}$$ in the Cobb-Douglas case.

c) $k^G = k^*$ if and only if $s = \alpha$. *

4. (Romer 1.6)



Notice if $g=0$ $\quad A(t) = A(0)$ which says that Knowledge is just a constant.

Along the balanced growth path

$$\frac{K(t)}{L(t)} = \frac{k^* A(t) L(t)}{L(t)} = k^* A(0) \quad \text{in this case.}$$

we Know that as n goes from n_0 to n_1
the steady state level of k changes from k_0^* to k_1^*
where $k_0^* < k_1^*$

then along the balanced growth path where $n = n_1$ we have

$$\left(\frac{K(t)}{L(t)}\right)_{new} = k_1^* A(0) > k_0^* A(0) = \left(\frac{K(t)}{L(t)}\right)_{OLD}$$ along the balanced growth path when $n = n_0$

so the balanced growth path of capital per worker increases

4 a) cont.

since $y^* = f(k^*)$ and $f'(k) > 0$

$$y_1^* = f(k_1^*) > f(k_0^*) = y_0^*$$

then along the balanced growth path

$$\frac{Y(t)}{L(t)} = \frac{y^* A(t) L(t)}{L(t)} = y^* A(0)$$

so since $y_0^* < y_1^*$

$$\left(\frac{Y(t)}{L(t)}\right)_{new} = y_1^* A(0) > y_0^* A(0) = \left(\frac{Y(t)}{L(t)}\right)_{OLD}$$ along the balanced growth path when $n = n_0$

since $c^* = (1-s) f(k^*)$ and $f'(k) > 0$

$$c_1^* = (1-s) f(k_1^*) > (1-s) f(k_0^*) = c_0^*$$

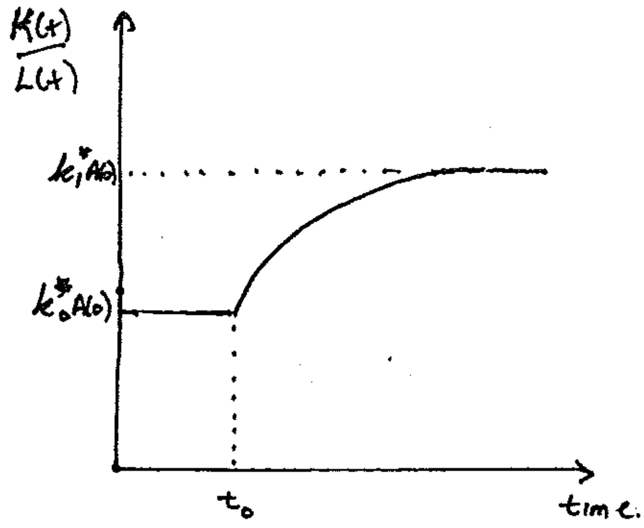
$$\frac{C(t)}{L(t)} = \frac{c^* A(t) L(t)}{L(t)} = c^* A(0)$$ along the balanced growth paths

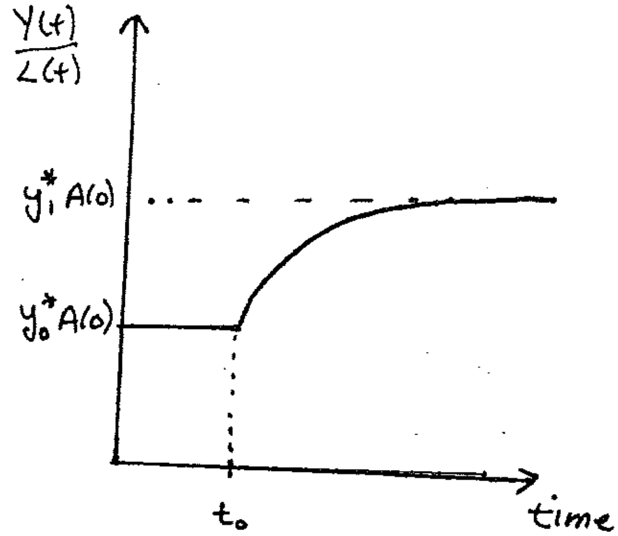
then since $c_0^* A(0) < c_1^* A(0)$

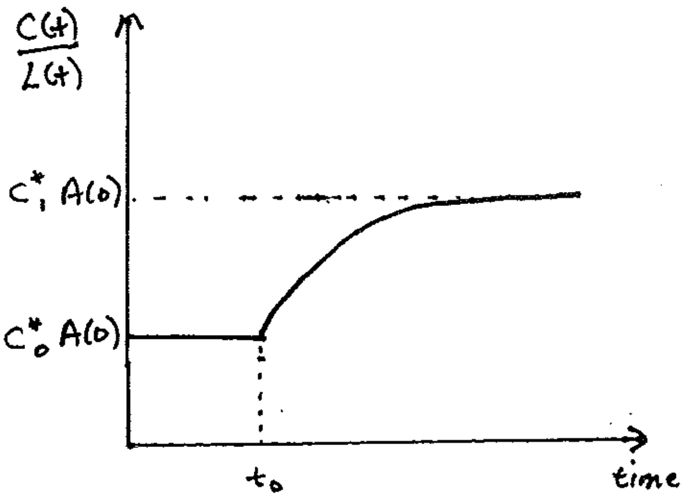
$$\left(\frac{C(t)}{L(t)}\right)_{new} = c_1^* A(0)$$ along the balanced growth path

$$> c_0^* A(0) = \left(\frac{C(t)}{L(t)}\right)_{OLD}$$ along the balanced growth path when $n = n_0$

4 a) cont'd







here t_0 is the time that n changes from n_0 to n_1, then since $k_0^* < k_1^*$ after time t_0 $\dot{k} > 0$ so k increases toward k_1^* in this case since $\dot{k} > 0$

$\dot{y} = f'(k)\dot{k} > 0$ and

$\dot{c} = (1-s)f'(k)\dot{k} > 0$

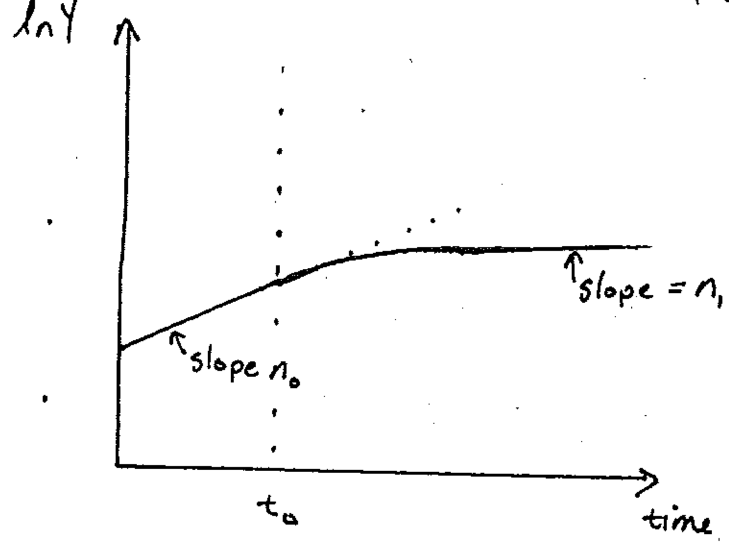
b) By definition $Y(t) = A(0)L(t)y(t)$. The growth rate of output

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{L}(t)}{L(t)} + \frac{\dot{y}(t)}{y(t)} = n$$ along the balanced growth path

when $n = n_0$ $\quad \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{L}(t)}{L(t)} = n_0$ before t_0

between t_0 and the time that k reaches k_1^*

$\frac{\dot{Y}(t)}{Y(t)} = n_1 + \frac{\dot{y}(t)}{y(t)} > n_1$ since $y(t)$ is growing after k reaches k_1^* $\quad \frac{\dot{Y}(t)}{Y(t)} = n_1$



5. (Romer 1.9)

Let $\quad w = \frac{\partial F(K, AL)}{\partial L} \quad$ and $\quad r = \frac{\partial F(K, AL)}{\partial K} - \delta$

a) Note $\quad F(K, AL) = AL\,F(\frac{K}{AL}, 1) = AL\,f(k) \quad$ where $k = \frac{K}{AL}$

then $\quad \frac{\partial F(K, AL)}{\partial L} = \frac{\partial\, AL f(k)}{\partial L} = A\,f(k) + AL\,f'(k)\,\frac{\partial k}{\partial L}$

$$= Af(k) + AL\,f'(k)\left(\frac{-K}{AL^2}\right)$$

$$= Af(k) + A\,f'(k)(-k)$$

$$= A\left[f(k) - k f'(k)\right]$$

so $\quad w = A\left[f(k) - kf'(k)\right]$

5 b)

$$\delta + r = \frac{\partial F(K, AL)}{\partial K} = \frac{\partial ALf(k)}{\partial K} = ALf'(k)\frac{\partial k}{\partial K} = ALf'(k)\frac{1}{AL} = f'(k)$$

then

$$r \cdot K + w \cdot L = f'(k) \cdot K + AL[f(k) - kf'(k)] - \delta K$$

$$= f'(k)K + ALf(k) - ALkf'(k) - \delta K$$

$$= \cancel{f'(k)K} + AL\,F\left(\tfrac{K}{AL}, 1\right) - \cancel{Kf'(k)} - \delta K$$

$$= F(K, AL) - \delta K$$

c)

$$\frac{\dot r(t)}{r(t)} = \frac{\dfrac{\partial\left(f'(k(t)) - \delta\right)}{\partial t}}{f'(k(t)) - \delta} = \frac{f''(k(t))\,\dot k(t)}{f'(k(t)) - \delta}$$

but along the balanced growth path $\dot k(t) = 0$

and $k(t) = k^*$

$\therefore \quad \dfrac{\dot r(t)}{r(t)} = 0$ so $r(t)$ doesn't grow along the balanced growth path

$$\frac{\dot w(t)}{w(t)} = \frac{\dfrac{\partial\left[A(t)\left(f(k(t)) - k(t)f'(k(t))\right)\right]}{\partial t}}{A(t)\left(f(k(t)) - k(t)f'(k(t))\right)}$$

$$= \frac{\dot A(t)\left(f(k(t)) - k(t)f'(k(t))\right) + A(t)\left[f'(k(t))\dot k(t) - \dot k(t)f'(k(t)) - k(t)f''(k(t))\dot k(t)\right]}{A(t)\left(f(k(t)) - k(t)f'(k(t))\right)}$$

5 c) cont.

$$\frac{\dot{w}(t)}{w(t)} = \frac{\dot{A}(t)\left(\frac{w(t)}{A(t)}\right) + A(t)\left[-k(t) f''(k(t))\, \dot{k}(t)\right]}{w(t)}$$

$$= \frac{\dot{A}(t)}{A(t)}$$ in steady state (along the balanced growth path) since $\dot{k}(t) = 0$

so

$$\frac{\dot{w}(t)}{w(t)} = g.$$ along the balanced growth path.

What about the shares

$$\frac{rK}{Y} \quad \text{and} \quad \frac{wL}{Y}$$

so

$$\frac{\dot{\left(\frac{rK}{Y}\right)}}{\frac{rK}{Y}} = \frac{\dot{r}}{r} + \frac{\dot{K}}{K} - \frac{\dot{Y}}{Y} = 0 + (n+g) - (n+g) = 0$$

$$\frac{\dot{\left(\frac{wL}{Y}\right)}}{\frac{wL}{Y}} = \frac{\dot{w}}{w} + \frac{\dot{L}}{L} - \frac{\dot{Y}}{Y} = g + n - (g+n) = 0$$

$\therefore$ the growth rates of the shares are also equal to zero

Remember along the balanced growth path

$K(t) = k^* A(t) L(t)$ so $\frac{\dot{K}(t)}{K(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} = g + n$

and $Y(t) = y^* A(t) L(t)$ so $\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} = g + n$

5 d) from part c) we know

$$\frac{\dot{r}(t)}{r(t)} = \frac{f''(k(t))\,\dot{k}(t)}{r(t)} < 0 \qquad \text{since } \dot{k}(t) > 0,\ r(t) > 0 \text{ and } f''(k) < 0$$

so it is growing at a rate less than its rate on the balanced growth path

$$\frac{\dot{w}(t)}{w(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{A(t)}{w(t)}\left[-k(t)\, f''(k(t))\, \dot{k}(t)\right]$$

$$> g \qquad \text{since } \frac{A(t)}{w(t)} > 0,\ \dot{k}(t) > 0 \text{ and } f''(k(t)) < 0$$

So the growth rate of wages is greater when $k(t) < k^*$.