Suggested Solutions - Problem Set # 4

\# 1 (Romer 2.9)

a) if $r(t) = (1-\tau) f'(k(t))$

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\tau) f'(k(t)) - \rho - \theta g}{\theta}$$

so $\dot{c}(t) = 0 \Rightarrow (1-\tau) f'(k^*_{TAX}) = \rho + \theta g$

with No taxes $\dot{c}(t) = 0 \Rightarrow f'(k^*_{NO\,TAX}) = \rho + \theta g$

so $(1-\tau) f'(k^*_{TAX}) = f'(k^*_{NO\,TAX})$

$\Rightarrow \quad f'(k^*_{TAX}) > f'(k^*_{NO\,TAX}) \quad$ since $(1-\tau) < 1$

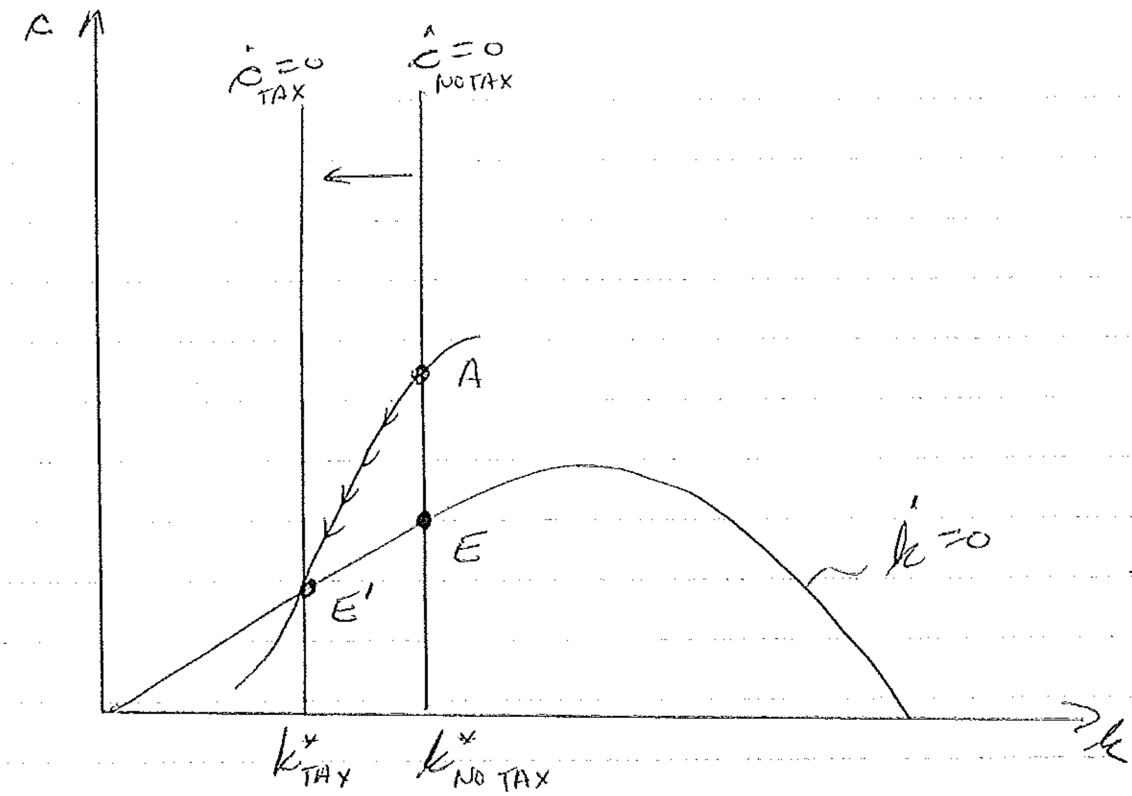
$\Rightarrow \quad k^*_{TAX} < k^*_{NO\,TAX} \quad$ since $f''(k) < 0$

so the $\dot{c} = 0$ locus shifts to the left.

The $\dot{k} = 0$ curve is still

$\dot{k} = f(k) - c - (n+g)k$ since the tax is rebated to the households using lump-sum transfers.

cont.



b) The path the economy takes after the ~~tax~~ tax change is depicted in the previous graph. When taxes increase, initially k remains the same but people consume more (the economy jumps to point A from point E). Here the economy is on the new saddle path and then the economy moves along the saddle path to its new balanced growth path at point E'

c) You can see from the previous graph that both steady state consumption and capital per unit of effective labour (i.e., c^* and k^*) are lower on the new balanced growth path.

d) i) $s = \dfrac{y^* - c^*}{y^*}$ and $c^* = y^* - (n+g)k^*$ where $y^* = f(k^*)$

so $s = \dfrac{(y^* - y^* + (n+g)k^*)}{y^*} = \dfrac{(n+g)k^*}{f(k^*)}$

$$\frac{\partial s}{\partial J} = \frac{f(k^*)(n+g)\frac{\partial k^*}{\partial J} - (n+g)k^* f'(k^*)\frac{\partial k^*}{\partial J}}{(f(k^*))^2}$$

$$= \frac{(n+g)}{f(k^*)}\frac{\partial k^*}{\partial J}\left[1 - \frac{k^* f'(k^*)}{f(k^*)}\right]$$

$$= (n+g)\frac{\partial k^*}{\partial J}\left[1 - \alpha_K(k^*)\right]$$

Since $\alpha_K(k^*)$ is between 0 and 1 in our model $\frac{\partial s}{\partial J}$ has the same sign as $\frac{\partial k^*}{\partial J}$

d i) cont.

using $\dot{c}=0 \Rightarrow f'(k^*_{tax}) = \dfrac{\rho+\theta g}{(1-\tau)}$

we can find the sign of $\dfrac{\partial k^*}{\partial \tau}$.

Using the equation we get

$$f''(k^*_{TAX})\frac{\partial k^*}{\partial \tau} = \frac{(\rho+\theta g)}{(1-\tau)^2}$$

$$\Rightarrow \frac{\partial k^*}{\partial \tau} = \frac{\rho+\theta g}{(1-\tau)^2}\,\frac{1}{f''(k^*_{TAX})} < 0$$

so $\dfrac{\partial s}{\partial \tau} < 0$

ii) In steady state we know all countries have equal values of after tax income $(1-\tau)f'(k)$ (since this must equal $\theta g+\rho$). Therefore there is no incentive for citizens in low-τ, high k^* high savings countries to invest in low savings countries

e) if the government subsidized investment both k^* & c^* would be higher however the original levels of c^* & k^* are the ones that satisfy the social planners problem & therefore maximizes welfare. We know this because the First Welfare Theorem holds - so the decentralized problem has the same solution as the social planner's & the social planner is unaffected by taxes - that is he simply chooses allocations subject to the resource constraint. Therefore, it must be that the losses associated with the inital drop in c when τ goes from 0 to a negative amount outweighs the benefits from being at the higher c along the B.G.P

f) In this case $\dot{k}=0$ now is determined by

$\dot{k} = f(k) - c(t) - G(t) - (n+g)k$

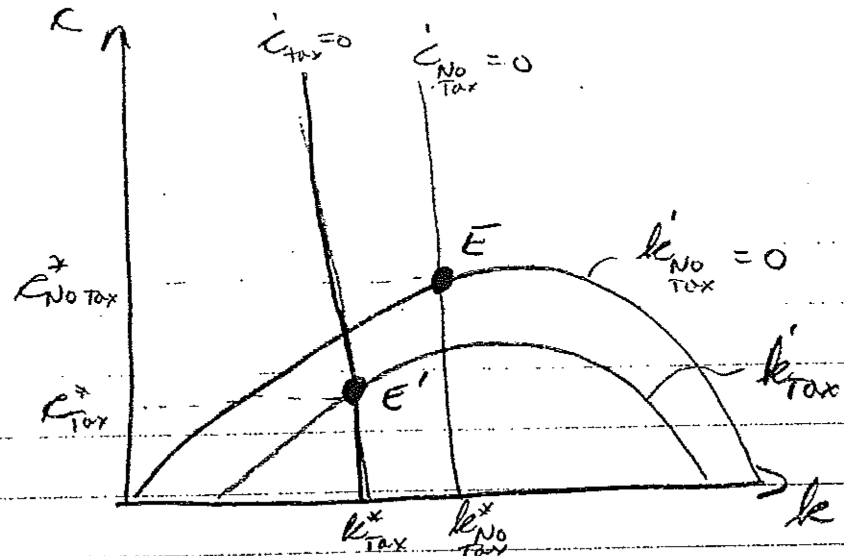
so the $\dot{k}=0$ locus would shift down

since $G(t)$ increases

but $\dot{c}=0$ still shifts left when

taxes are implemented as discussed in a)

so



here both c^* & k^* is lower

but whether c initially increases

depends on whether the saddle path

leading to E' has c greater or less

than the $c^*_{No\ Tax}$ at $k^*_{No\ Tax}$

#2 (Romer 2.12)

$$\mathcal{L} = B \int_{t=0}^{\infty} e^{-\beta t} \frac{[c(t) + G(t)]^{1-\theta}}{1-\theta} dt + \lambda \left\{ k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} w(t) dt - \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} c(t) dt - \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} G(t) dt \right\}$$

$$\frac{\partial \mathcal{L}}{\partial c(t)} = Be^{-\beta t}(c(t)+G(t))^{-\theta} - e^{-R(t)} e^{(n+g)t} = 0$$

$$\Rightarrow \quad Be^{-\beta t}(c(t)+G(t))^{-\theta} = e^{-R(t)} e^{(n+g)t}$$

$$\Leftrightarrow \quad \ln B - \beta t - \theta \ln(c(t)+G(t)) = -R(t) + (n+g)t \quad (\forall t)$$

$$\Rightarrow \quad -\beta - \frac{\theta}{c(t)+G(t)} [\dot{c}(t) + \dot{G}(t)] = -r(t) + (n+g)$$

$$\Leftrightarrow \quad \frac{\dot{c}(t) + \dot{G}(t)}{c(t)+G(t)} = \frac{r(t) - \rho - \theta g}{\theta}$$

Now for simplicity assume $G(t) = G_L > 0$

and the shock to the economy will be that

G_L increases to G_H temporarily and return to

G_L at a date that is known with certainty.

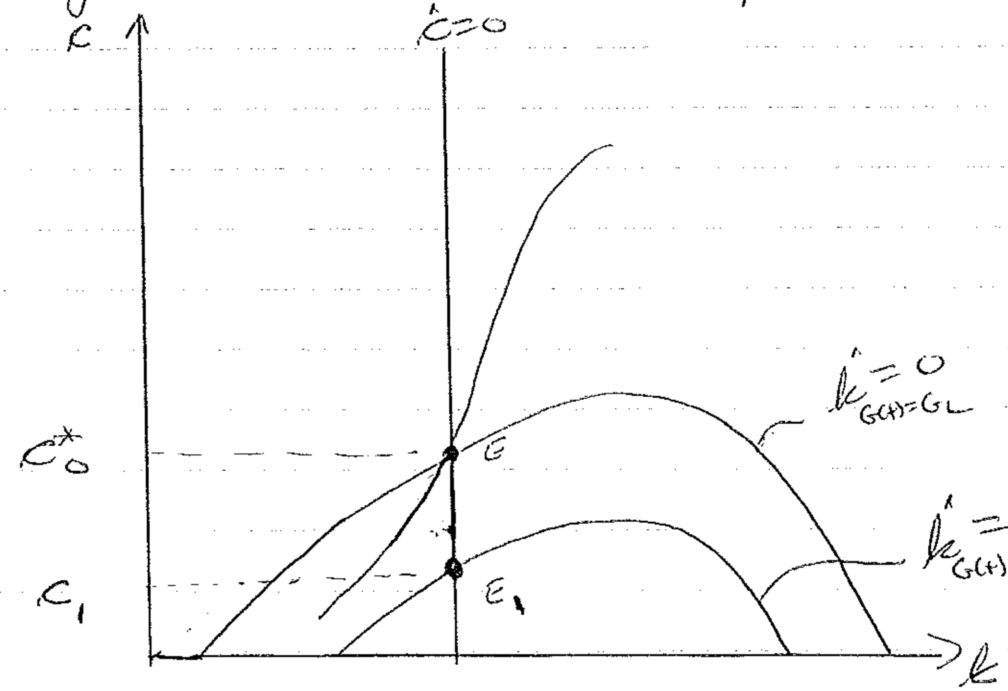
2 cont.

Notice $\dot{c}=0$ is still going to give us

$f'(k^*) = \rho + \theta g$ so a temporary increase in gov't expenditures will not affect the $\dot{c}=0$ curve.

$\dot{k}=0 \Rightarrow c(t) = f(k(t)) - G_L - (n+g)k(t)$ before the shock and after the government has reduced expenditures

$\dot{k}=0 \Rightarrow c(t) = f(k(t)) - G_H - (n+g)k(t)$ when the gov't has raised its expenditures



2 con't again.

The difference between this case and the case discussed in the book is that c can and must jump at time t_1 when G returns to G_L because otherwise there would be a discontinuous jump in marginal utility that cannot be optimal for households (remember risk adverse individuals do not like large movements in variables they care about - before this was $c(t)$, now it is $c(t) + G(t)$)

Therefore at the time the gov't increases its spending by $G_H - G_L$, households will drop there consumption by exactly $G_H - G_L$ and be at point E_1. They will then remain there until time t_1 (the time the gov't switches back to $G(t) = G_L$) when they will increase their consumption

3) $k_{t+1} = \frac{K_{t+1}}{A_{t+1}L_{t+1}}$

$$\frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{K_t + sY_t - \delta K_t}{A_{t+1}L_{t+1}} = \frac{A_t L_t}{A_{t+1}L_{t+1}}\left[\frac{K_t + sY_t - \delta K_t}{A_t L_t}\right]$$

$$= \frac{A_t L_t}{(1+g)A_t(1+n)L_t}\left[k_t + sf(k_t) - \delta k_t\right]$$

$$= \frac{1}{(1+g)(1+n)}\left(k_t + sf(k^*) - \delta k_t\right)$$

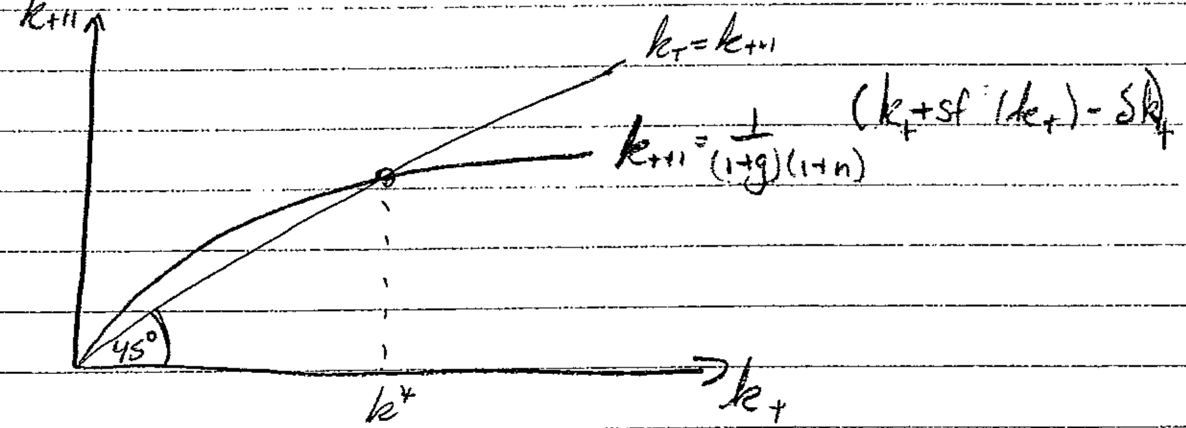
so $k_{t+1} = \frac{1}{(1+g)(1+n)}\left(k_t + sf(k_t) - \delta k_t\right)$

b) $\frac{\partial k_{t+1}}{\partial k_t} = \frac{1}{(1+g)(1+n)}\left(1 + sf'(k_t) - \delta\right)$

so the slope of the line $k_{t+1} = \frac{1}{(1+g)(1+n)}(k_t + sf(k) - \delta k_t)$

is positive but decreasing in k

so



It is clear from this diagram that there is a steady state level of k

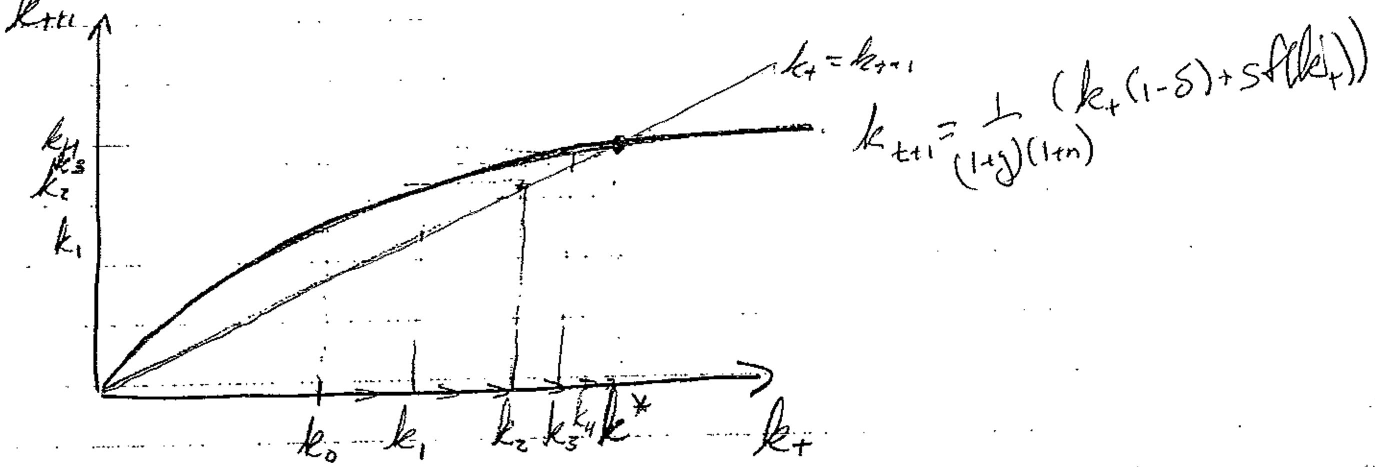
& this occurs when $k_{t+1} = k_t = k^*$ where k^* satisfies:

$$k^* = \frac{1}{(1+g)(1+n)}\left(k^* + sf(k^*) - \delta k^*\right)$$

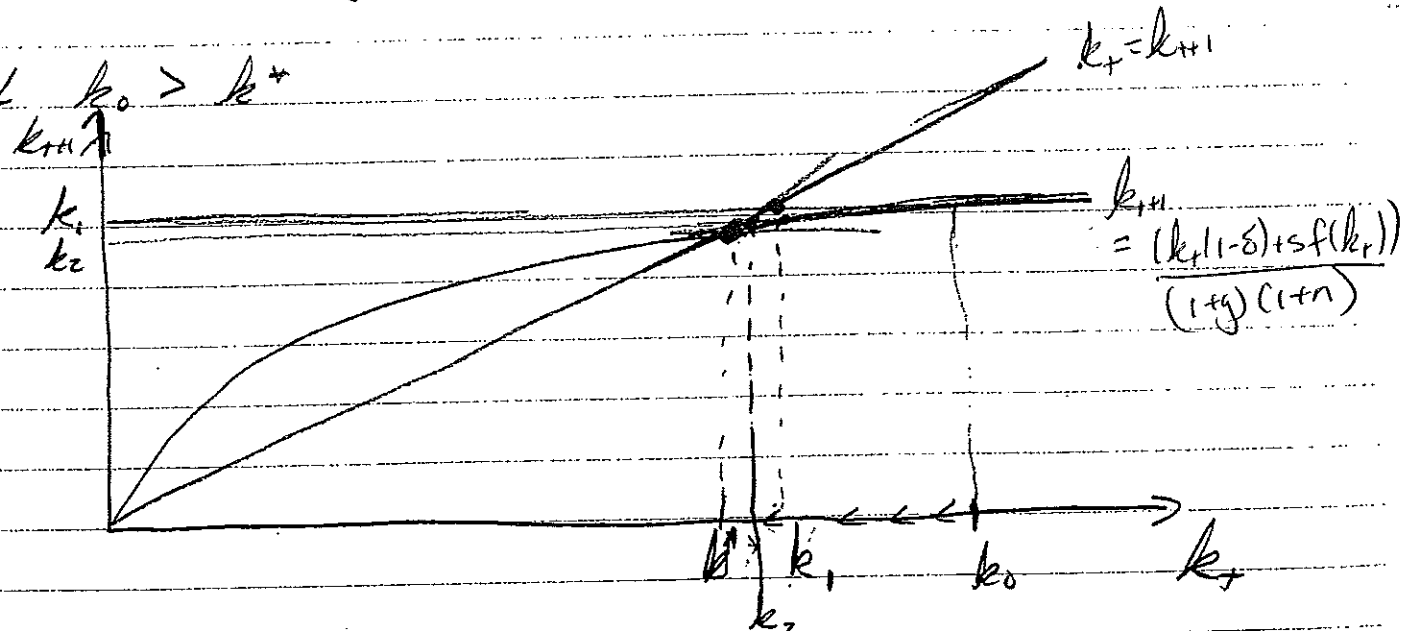
Also this economy will converge to k^*

& you can see this by the dynamics seen below

if $k_0 < k^*$



if $k_0 > k^*$



so again we will converge to k^*

d) i) for the cobb-Douglas Function

$$f(k_t) = k_t^{\alpha}$$

so $$k_{t+1} = \frac{1}{(1+n)(1+g)}\left[k_t + s(k_t)^{\alpha} - \delta k_T\right]$$

ii) $$k^* = \frac{1}{(1+g)(1+n)}\left(k^*(1-\delta) + sk^{*\alpha}\right)$$

$$\Rightarrow k^*\left((1+g)(1+n) - (1-\delta)\right) = sk^{*\alpha}$$

$$\Rightarrow (k^*)^{1-\alpha} = \left[\frac{s}{(1+g)(1+n) - (1-\delta)}\right]$$

$$\Rightarrow k^* = \left[\frac{s}{(1+g)(1+n) - (1-\delta)}\right]^{\frac{1}{1-\alpha}}$$

iii) $$(k_{t+1} - k^*) = \frac{1}{(1+n)(1+g)}\left[1 + s\cdot\alpha(k^*)^{\alpha-1} - \delta\right](k_t - k^*)$$

$$= \frac{1}{(1+n)(1+g)}\left[1-\delta + s\cdot\alpha\underbrace{\left(\frac{(1+g)(1+n)-(1-\delta)}{s}\right)}_{(k^*)^{\alpha-1}}\right](k_t - k^*)$$

$$= \frac{1}{(1+n)(1+g)}\left[1-\delta + \alpha(1+g)(1+n) - \alpha(1-\delta)\right](k_t - k^*)$$

$$= \frac{1}{(1+n)(1+g)}\left[(1-\alpha)(1-\delta) + \alpha(1+g)(1+n)\right](k_t - k^*)$$

then

$$(k_t - k^*) = \left[\frac{(1-\alpha)(1-\delta) + \alpha(1+g)(1+n)}{(1+n)(1+g)}\right]^t (k_0 - k^*)$$

so the economy moves

$$1 - \left[\frac{(1-\alpha)(1-\delta) + \alpha(1+g)(1+n)}{(1+n)(1+g)}\right] \text{ of the way}$$

each period

when $\alpha = \frac{1}{3}$ $n = 1\%$ $g = 2\%$ & $\delta = 3\%$

the rate of convergence:

$$\frac{(1+n)(1+g) - (1-\alpha)(1-\delta) - \alpha(1+g)(1+n)}{(1+n)(1+g)}$$

$$= \frac{(1-\alpha)\left((1+n)(1+g) - (1-\delta)\right)}{(1+n)(1+g)} \simeq 3.9\%$$

Question #4.

$$\max_{\{c(t)\}_0^\infty} \; B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt + B_1 \int_{t=0}^{\infty} e^{-\beta_1 t} V(G_t) dt$$

$$+ \lambda \Big\{ k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} \omega(t) dt - \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} G(t) dt$$

$$- \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} c(t) dt \Big\}$$

F.O.N.C. for consumption per unit of effective labour is:

$$c(t): \quad B e^{-\beta t} c(t)^{-\theta} - \lambda e^{-R(t)} e^{(n+g)t} = 0$$

$$\Leftrightarrow \quad B e^{-\beta t} c(t)^{-\theta} = \lambda e^{-R(t)} e^{(n+g)t}$$

$$\Leftrightarrow \quad \ln B - \beta t - \theta \ln c(t) = \ln \lambda - R(t) + (n+g)t$$

since this holds for all date t we can take the derivative with respect to t to get:

$$-\beta - \frac{\theta}{c(t)} \dot{c}(t) = -r(t) + (n+g)$$

$$\Leftrightarrow \quad \frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta} \{ r(t) - (n+g) - \rho + n + (1-\theta) g \}$$

$$= \frac{r(t) - \rho - \theta g}{\theta}$$