ECO 2061 2008 Midterm

1 a)
$$U = \int_{t=0}^{\infty} e^{-\rho t} \frac{(c(t)A(t))^{1-\theta}}{1-\theta} \frac{L(0)e^{nt}}{H} dt.$$

$$= \int_{t=0}^{\infty} e^{-\rho t} (A(0)e^{gt})^{1-\theta} \frac{c(t)^{1-\theta}}{1-\theta} \frac{L(0)e^{nt}}{H} dt$$

$$= B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt \quad \text{where } B = \frac{A(0)^{1-\theta} L(0)}{H} \;\&\; \beta = \rho - n - (1-\theta)g$$

(2 pts)

b) $\mathcal{L} = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt + \lambda \left[k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt - \int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt \right]$

1 pt →

3 pts

so $\frac{d\mathcal{L}}{dc(t)} = B e^{-\beta t} c(t)^{-\theta} - \lambda e^{-R(t)} e^{(n+g)t} = 0 \quad \forall t$

so $\lambda e^{-R(t)} e^{(n+g)t} = B e^{-\beta t} c(t)^{-\theta}$

or $\ln \lambda - R(t) + (n+g)t = \ln B - \beta t - \theta \ln c(t)$ (*)

so taking the derivative of (*) with respect to t gives

$$\underbrace{-r(t) + (n+g) = -\beta - \theta \frac{\dot{c}(t)}{c(t)} \quad \text{or} \quad \frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta}}_{1 \text{ pt.}}$$

c) i.) befor the tax change households return on capital is $r(t) = f'(k(t))$ (which is given by the firm's first order condition for capital.)

1.c ii) after the tax change, firms still are willing to pay $f'(k(t))$ to rent capital. However when the gov't taxes this income, households only recieve $(1-\tau_1) f'(k(t))$ per unit

iii) before the change we have

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta} \quad \text{so} \quad \dot{c}_{OLD} = 0$$

$$\Rightarrow f'(k^*_{OLD}) = \rho + \theta g$$

after the change

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\tau_1) f'(k(t)) - \rho - \theta g}{\theta} \quad \text{so}$$

$$\dot{c}_{NEW} = 0 \Rightarrow f'(k^*_{NEW}) = \frac{\rho + \theta g}{1-\tau_1}$$

Since $\tau_1 \in (0,1)$ $\quad f'(k^*_{NEW}) > f'(k^*_{OLD})$

$$\Rightarrow k^*_{NEW} < k^*_{OLD} \quad \text{given} \quad f''(k) < 0$$

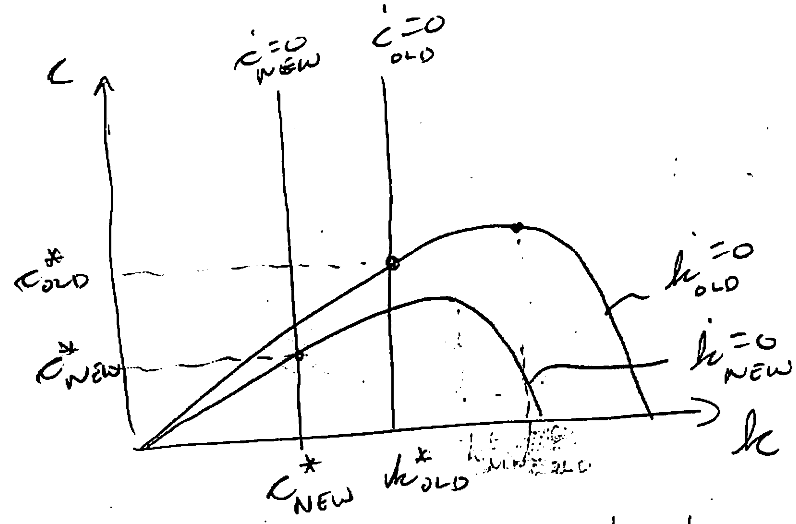
also before the change $\dot{k}_{OLD} = 0 \Rightarrow$

$$c(t) = f(k(t)) - (n+g) k(t)$$

after the change $\dot{k}_{NEW} = 0 \Rightarrow$

$$c(t) = f(k(t)) - (n+g) k(t) - G(t)$$ so $\dot{k} = 0$ curve moves downwards (see graph)

1 iii) cont



4 pts for correct derivation of $\dot{c}=0$ curve movement
4 pts for correct derivation of $\dot{k}=0$ curve movement.

1 iv) before t^*, k^* is given by $f'(k^*) = \rho + \theta g$ (2 pts)

c^* is given by $c^* = f(k^*) - (n+g)\,k^*$ (2 pts)

y^* is given by $f(k^*)$ (2 pts)

and k^G is given by $f'(k^G) = (n+g)$ (2 pts)

v) after t^*

k^*_{NEW} is given by $f'(k^*_{NEW}) = \dfrac{\rho + \theta g}{1 - \tau_1}$

c^*_{NEW} is given by $f(k^*_{NEW}) - (n+g)\,k^* - \tau_1 f'(k^*_{NEW})\,k^*_{NEW}$

since $G^* = \tau_1 f'(k^*_{NEW})\,k^*_{NEW}$

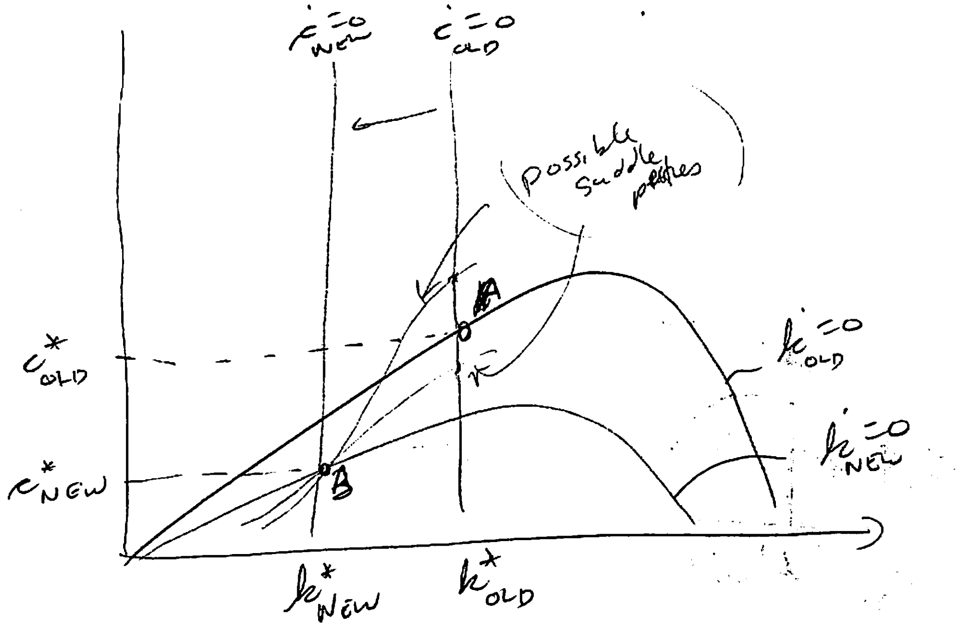
$y^*_{NEW} = f(k^*_{NEW})$

and k^G_{NEW} is given by the 1st order condition for

$$\max_k \; f(k) - (n+g)\,k - \tau_1 f'(k)\,k$$

or $f'(k^G_{NEW}) - (n+g) - \tau_1 f'(k^G_{NEW}) - \tau_1 f''(k^G_{NEW})\,k^G_{NEW} = 0$

1.vii)



at the time of the shock $k(t)$ cant change so $y(t) = f(k(t))$

doesn't change immediately. $c(t)$ either increases or decreases to get the economy on its new saddle path (see above)

after the initial period, $k(t)$ and $c(t)$ decrease as the economy moves along its saddle path to its new steady state. $y(t)$ also decreases as the economy moves towards its new steady state since $\dot{y}(t) = f'(k(t))\,\dot{k}(t)$. Once the economy reaches its new steady state (at point B) all variables stop changing

(4 points for each of the dynamics) $\times$ 3 = 12

1 vii) $\frac{K(t)}{L(t)} = k(t)A(t)$, $\frac{Y(t)}{L(t)} = y(t)A(t)$ and $C(t) = c(t)A(t)$

so $\frac{d\ln\left(\frac{K(t)}{L(t)}\right)}{dt} = \frac{\dot{k}(t)}{k(t)} + g$, $\frac{d\ln\left(\frac{Y(t)}{L(t)}\right)}{dt} = \frac{\dot{y}(t)}{y(t)} + g$

and $\frac{d\ln(C(t))}{dt} = \frac{\dot{c}(t)}{c(t)} + g$

along the balanced growth path $\dot{k}(t) = \dot{y}(t) = \dot{c}(t) = 0$ 3 pts
so the growth rates along the balanced growth path
do not depend on the tax rate. However
when the economy is not on its Balanced growth
rate $\frac{\dot{c}(t)}{c(t)} = \frac{(1-\tau_1)f'(k(t)) - \rho - \theta g}{\theta}$ 6 pts

$$\frac{\dot{k}(t)}{k(t)} = \frac{f(k(t)) - c(t) - (n+g)k(t) - \tau_1 f'(k(t))k(t)}{k(t)}$$

and $\frac{\dot{y}(t)}{y(t)} = \frac{f'(k(t))\,\dot{k}(t)\,k(t)}{y(t)\cdot k(t)} = \alpha_K(t)\cdot\frac{\dot{k}(t)}{k(t)}$

so the growth rates will depend on the tax rate τ_1
as well as on g

viii) Only difference is that the balanced growth path
in the Ramsey-Cass-Koopman model cannot have
k^* exceeding k^G since we require

$$\rho - n - \theta g > 0 \qquad \text{4 pts}$$

2. $$k_{t+1} = \frac{K_{t+1}}{A_{t+1}L_{t+1}}$$

so $$K_{t+1} = K_t + sZK_t^{\alpha}(A_tL_t)^{1-\alpha} - \delta K_t$$

$$\Rightarrow \quad k_{t+1}A_{t+1}L_{t+1} = k_tA_tL_t + sZ(k_tA_tL_t)^{\alpha}(A_tL_t)^{1-\alpha} - \delta k_tA_tL_t$$

$$\Leftrightarrow \quad k_{t+1} = k_t(1-\delta)\frac{A_tL_t}{A_{t+1}L_{t+1}} + sZk_t^{\alpha}\frac{A_tL_t}{A_{t+1}L_{t+1}}$$

$$= k_t(1-\delta)\frac{e^{\bar{A}+gt}\bar{L}}{e^{\bar{A}+g(t+1)}\bar{L}} + sZk_t^{\alpha}\left(\frac{e^{\bar{A}+gt}\bar{L}}{e^{\bar{A}+g(t+1)}\bar{L}}\right)$$

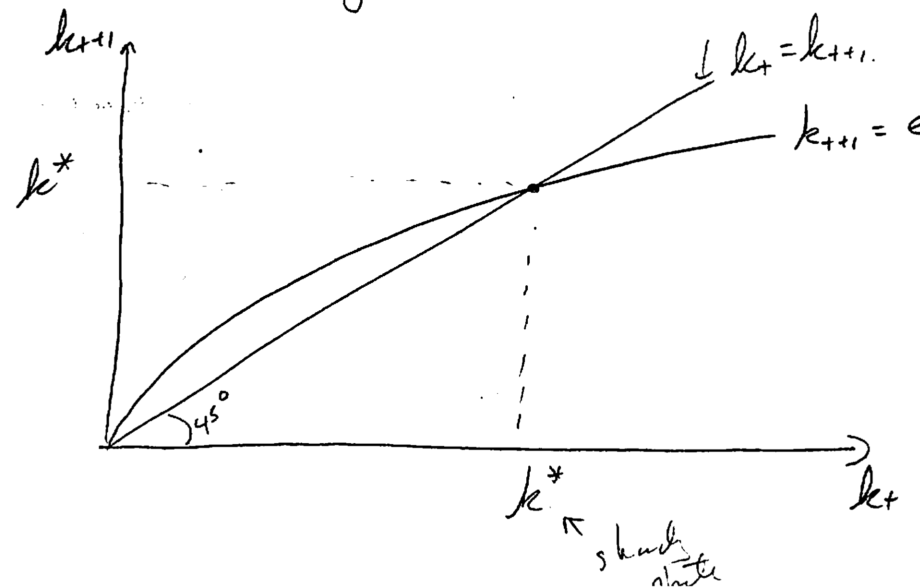
$$= k_t(1-\delta)e^{-g} + sZk_t^{\alpha}e^{-g}$$

since $$\frac{\partial k_{t+1}}{\partial k_t} = (1-\delta)e^{-g} + sZ\alpha k_t^{\alpha-1}e^{-g} > 0$$

$$\frac{\partial^2 k_{t+1}}{\partial k_t^2} = sZ\alpha(\alpha-1)e^{-g}k_t^{\alpha-2} < 0$$

and $\lim_{k_t \to 0} \frac{\partial k_{t+1}}{\partial k_t} = \infty$ and $\lim_{k_t\to\infty} \frac{\partial k_{t+1}}{\partial k_t} = (1-\delta)e^{-g} < 1$ since $\delta \in (0,1)$ & $g > 0$

we know the graph looks like



and the function will cross the 45° line once when $k_t > 0$ so steady state exists

2 b) cont. The steady state is when

$k^* = k_{t+1} = k_t$ so

$$k^* = e^{-g} k^*(1-\delta) + e^{-g} s z (k^*)^{\alpha}$$

so $$1 = e^{-g}(1-\delta) + e^{-g} s z (k^*)^{\alpha-1}$$

so $$\left(\frac{1-e^{-g}(1-\delta)}{e^{-g} s z}\right)^{\frac{1}{\alpha-1}} = k^*$$

then $y^* = z(k^*)^{\alpha}$ and since

$$\begin{aligned} Y_t &= C_t + I_t \\ &= C_t + K_{t+1} - (1-\delta)K_t \\ &= C_t + sY_t \end{aligned}$$

then $C_t = (1-s)Y_t$ and

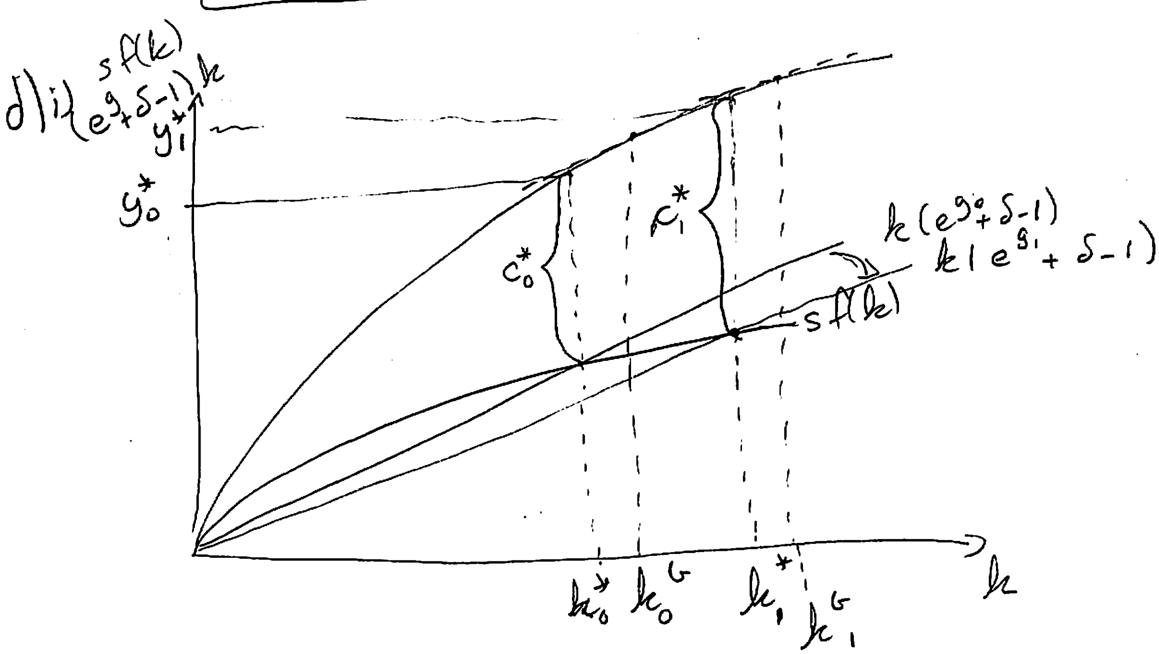
$$c^* = (1-s)y^* = (1-s)f(k^*)$$

c) to find k^G we need to max c^*

$c^* = y^* - i^* = f(k^*) + k^*((1-\delta) - e^{g})$

so $$\boxed{f'(k^G) = -(1-\delta-e^{g})}$$

d) i)



$g_1 < g_0$

2 d i) con't when g falls from g_0 to g_1

The break-even curve rotates downwards from the origin

As a result k^* increases from k_0^* to k_1^*

y_0^* increases to y_1^* since $k_0^* < k_1^*$ & $f'(k) > 0$

and since $c(t) = (1-s) f(k(t))$ c^* increases from c_0^* to c_1^*. Finally since $f''(k) < 0$ and

$$f'(k_0^G) = (e^{g_0} + \delta - 1) > e^{g_1} + \delta - 1 = f'(k_1^G)$$

$$k_0^G < k_1^G$$

d.ii)
$$\frac{\partial k^*}{\partial g} = \frac{\partial \left(\frac{e^g - 1 + \delta}{sz}\right)^{\frac{1}{\alpha-1}}}{\partial g} = \frac{1}{\alpha - 1}\left(\frac{e^g - 1 + \delta}{sz}\right)^{\frac{1}{\alpha-1} - 1} \cdot \frac{e^g}{sz} < 0$$

since $\alpha \in (0,1)$

$$\frac{\partial y^*}{\partial g} = f'(k^*)\frac{\partial k^*}{\partial g} < 0 \quad \text{since } f'(k) > 0 \text{ \& } \frac{\partial k^*}{\partial g} < 0$$

$$\frac{\partial c^*}{\partial g} = (1-s) f'(k^*)\frac{\partial k^*}{\partial g} < 0 \quad \text{since } s \in (0,1),\ f'(k) > 0 \text{ \& } \frac{\partial k^*}{\partial g} < 0$$

$\frac{\partial k^G}{\partial g}$ can be gotten by totally differentiating the equation $f'(k^G) = e^g + \delta - 1$

as
$$f''(k^G)\frac{\partial k^G}{\partial g} = e^g \quad \text{or} \quad \frac{\partial k^G}{\partial g} = \frac{e^g}{f''(k^G)} < 0$$

since $f''(k^G) < 0$

(3 pts each) $\times 4 = 12$.

2 e. i) when Both $K(t)$ and $L(t)$ change, it is uncertaint as to what happens to $k(t) = \frac{K(t)}{A(t)L(t)}$; so there are 3 cases..

if $\frac{K(t)}{L(t)}$ stay the same (case 1), $k(t)$ is unchanges so $y(t) = f(k(t))$ is unchanged

if $\frac{K(t)}{L(t)}$ increases (case 2), $k(t)$ rises so $y(t) = f(k(t))$ rises since $f'(k(t)) > 0$

if $\frac{K(t)}{L(t)}$ decreases (case 3), $k(t)$ falls so $y(t) = f(k(t))$ falls

(2 pts ~~for~~ for each case) $* 3 = 6$

ii) since $\dot{y}(t) = f'(k(t))\dot{k}(t)$

in case 1 since $\dot{k}(t) = 0$, $\dot{y}(t) = 0$ (i.e. we remain at the steady state level)

(2pts)

In case 2, $k(t) > k^*$ so $\dot{k}(t) < 0$ since actual investment is less than break even investment, so $\dot{y}(t) < 0$ (ie $y(t)$ decreases while $k(t) > k^*$) (3 pts)

In case 3, $k(t) < k^*$ so $\dot{k}(t) > 0$ since [illegible] $> n+g+$ $s f(k(t)) > (e^g + \delta - 1)k(t)$, so $\dot{y}(t) > 0$ (ie., $y(t)$ increases while $k(t) < k^*$) (3 pts)

e iii) In all cases y^* is the same as it was before the shock since the shock doesn't effect α, s, Z, δ or g.

3 a) $\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[(C_t + G_t)^{1/2} \right] + \lambda_t \left[(1-\tau_t) r_t K_t + \omega_t + T_t - K_{t+1} + K_t(1-\delta) - C_t \right] \right\}$

so

$$\frac{\partial \mathcal{L}}{\partial C_t} = E_t \left[\beta^t \frac{1}{2} (C_t + G_t)^{-1/2} - \lambda_t \right] = 0 \quad \text{(A)}$$

$\leftarrow \forall t$ & all states of the world

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = E_t \left[-\lambda_t + \lambda_{t+1} \left((1-\tau_{t+1}) r_{t+1} + (1-\delta) \right) \right] = 0 \quad \text{(B)}$$

& we have

$$(1-\tau_t) r_t K_t + \omega_t + T_t - K_{t+1} + K_t(1-\delta) - C_t = 0 \quad \text{(C)}$$

$$\text{(A)} \Rightarrow \lambda_t = \frac{\beta^t (C_t + G_t)^{-1/2}}{2}$$

$$\text{(B)} \Rightarrow E_t \left(\lambda_{t+1} \left[(1-\tau_{t+1}) r_{t+1} + (1-\delta) \right] \right) = \lambda_t$$

so

$$\frac{\beta^t (C_t + G_t)^{-1/2}}{2} = E_t \left(\frac{\beta^{t+1} (C_{t+1} + G_{t+1})^{-1/2}}{2} \left[(1-\tau_{t+1}) r_{t+1} + 1 - \delta \right] \right)$$

or

$$(C_t + G_t)^{-1/2} = \beta E_t \left\{ (C_{t+1} + G_{t+1})^{-1/2} \left((1-\tau_{t+1}) r_{t+1} + 1 - \delta \right) \right\} \quad \circledast$$

~~which [illegible] marginal utility [illegible]~~

b) when marginal taxes are zero and gov't purchases are constant we have $\circledast$ become

$$(C_t + G)^{-1/2} = \beta E_t \left\{ (C_{t+1} + G)^{-1/2} (r_{t+1} + 1 - \delta) \right\}$$

3b) cont.

Using the fact $E_t(XY) = E_t(X)\,E_t(Y) + Cov_t(X,Y)$

we have

$$(C_t + G)^{-1/2} = \beta \Big\{ E_t\left((C_{t+1}+G)^{-1/2}\right) E_t\left(r_{t+1} + 1 - \delta\right) + Cov_t\left((C_{t+1}+G)^{-1/2},\ r_{t+1} + 1 - \delta\right) \Big\}$$

let $\bar{C}_t$ be the current level of C when $Cov_t(C_{t+1}+G, r_{t+1}) > 0$

and $\underline{C}_t$ be " " " " when $Cov_t(C_{t+1}+G, r_{t+1}) < 0$.

when $Cov_t(C_{t+1}+G, r_{t+1}) > 0$ it follows that $Cov_t\left((C_{t+1}+G)^{-1/2}, r_{t+1}+1-\delta\right) < 0$ and when $Cov_t(C_{t+1}+G, r_{t+1}) < 0$, $Cov_t\left((C_{t+1}+G)^{-1/2}, r_{t+1}+1-\delta\right) > 0$

so

$$(\bar{C}_t + G)^{-1/2} - (\underline{C}_t + G)^{-1/2} < 0$$

so $(\bar{C}_t + G)^{-1/2} < (\underline{C}_t + G)^{-1/2}$

so $\underline{C}_t + G < \bar{C}_t + G$ or $\underline{C}_t < \bar{C}_t$

8 pts for derivation & answer

Intuition: when the cov between $C_{t+1}+G$ & r_{t+1} is positive the asset pays off more at a time individuals already have high consumption $\therefore$ saving more in t to consume more in t+1, is not as attractive as

(7pts)

3b) con't.

It would be in the case where r_{t+1} is high when C_{t+1} is low. As a result individuals will choose to consume ~~relatively~~ more in period t. when $Cov_t(C_{t+1}+G, r_{t+1}) > 0$.

3 a) Intuition for (*)
(5 pts)

giving up an extra unit of consumption in period t is worth $\left(\frac{1}{C_t+G_t}\right)^{1/2}$ on the marginal (i.e., the marginal utility of consumption = marginal cost of saving). ~~Notice~~ The marginal benefit of saving this extra unit is worth since:

$$\beta E_t\left(\left(\frac{1}{C_{t+1}+G}\right)^{1/2}(1-\delta+(1-\tau_{t+1})r_{t+1})\right)$$

the savings will yield $1-\delta+(1-\tau_{t+1})r_{t+1}$ in the next period which will be valued as $\left(\frac{1}{C_{t+1}+G}\right)^{1/2}\cdot(1-\delta+(1-\tau_{t+1})r_{t+1})$ in the next period

However since, in period t, we do not value tomorrows consumption as much as todays, we must discount this value by β, and we must weight these values by the probabilities that they occur since we are uncertain about the future. ($\therefore$ we get $\beta E_t\left(\left(\frac{1}{C_{t+1}+G}\right)^{1/2}(1-\delta+(1-\tau_{t+1})r_{t+1})\right)$)

$\therefore$ the equation says marginal cost of savings = Expected marginal benefit.

3 c) They are perfect substitutes since individuals are indifferent between receiving an additional unit of Consumption or an addition unit of G.

(5 pts)

d) $G_t \leq -T_t + \tau_t r_t K_t$

(5 pts)

e) Ricardian Equivalence will not hold because there are marginal taxes on investment income. Therefore the path of consumption is not independent of the way government expenditures are financed

(8 pts)

∴ ricardian equivalence doesn't hold since ricardian equivalence implies that only the quantity of gov't purchases - not the choice of financing - affects the economy

3 f.) The Social planner's problem is

$$\max_{\{C_t, K_{t+1}\}_0^\infty} E_0 \sum_{t=0}^{\infty} \beta^t [(C_t + G_t)^{1/2}]$$

subject to the resource constraint

5pts

$$G_t + K_{t+1} - (1-\delta)K_t + C_t \leq Y_t = F(K_t, A_t L_t)$$

given K_0

so

$$\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [(C_t + G_t)^{1/2}] + \lambda_t (F(K_t, A_t L_t) - G_t - K_{t+1} + (1-\delta)K_t - C_t) \right\}$$

5pts

F.O.N.C.s

$$\frac{\partial \mathcal{L}}{\partial C_t} = E_t \left\{ \beta^t \frac{1}{2} (C_t + G_t)^{-1/2} - \lambda_t \right\} = 0$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = E_t \left\{ -\lambda_t + \left[(1-\delta) \lambda_{t+1} + \lambda_{t+1} F_K (K_{t+1}, A_{t+1} L_{t+1}) \right] \right\} = 0$$

& $F(K_t, A_t L_t) = G_t + K_{t+1} - (1-\delta) K_t + C_t$

for all t & all states of the world

3 g) $Y_t = Z K_t^{\alpha} (A_t L_t)^{1-\alpha}$

In a competitive economy firms take prices as given & maximize profits

$$\max_{K_t, L_t} Z K_t^{\alpha} (A_t L_t)^{1-\alpha} - r_t K_t - w_t L_t \equiv \Pi_t$$

so

$$\frac{\partial \Pi_t}{\partial K_t} = \alpha Z K_t^{\alpha-1} (A_t L_t)^{1-\alpha} - r_t = 0$$

$$\frac{\partial \Pi_t}{\partial L_t} = (1-\alpha) Z K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} - w_t = 0$$

in eq'm. $L_t = 1$ (Labour demand = labour supply)

and $G_t = \tau_t r_t K_t - T_t$

the household's budget constraint can therefore be written as

$$K_{t+1} - (1-\delta) K_t + C_t = r_t K_t + w_t - [\tau_t r_t K_t - T_t]$$

$$= r_t K_t + w_t - G_t$$

or $G_t + C_t + K_{t+1} - (1-\delta) K_t = r_t K_t + w_t$. (*)

3 g) cont.

further more since

$$w_t = (1-\alpha) Z K_t^{\alpha} A_t^{1-\alpha} = (1-\alpha) Y_t \qquad (\text{since } L_t = 1)$$

and $$r_t = \alpha Z_t K_t^{\alpha-1} (A_t L_t)^{1-\alpha} = \alpha \frac{Y_t}{K_t}$$

we have

$$w_t + r_t K_t = (1-\alpha) Y_t + \alpha Y_t = Y_t$$

so combining this equation with (*) gives our result

$$G_t + C_t + K_{t+1} - (1-\delta) K_t = Y_t.$$