Question # 1

1. CRTS $\Rightarrow$ $cF(K(t), A(t)L(t)) = F(cK(t), cA(t)L(t))$ $\forall c$

so

$cF(K(t), A(t)L(t)) = cK(t)^{\alpha_1}(A(t)L(t))^{1-\alpha_1} - \alpha_2 c A(t)^{\alpha_3} L(t)$

$F(cK(t), cA(t)L(t)) = (cK(t))^{\alpha_1}(cA(t)L(t))^{1-\alpha_1} - \alpha_2 A(t)^{\alpha_3 - 1} c A(t)L(t)$

$= c K(t)^{\alpha_1}(A(t)L(t))^{1-\alpha} - \alpha_2 A(t)^{\alpha_3 - 1} c A(t)L(t)$

No restriction on α_1 or α_2

but $\alpha_3 = 1$

b) $\dfrac{\boxed{K(t)^{.34}(A(t)L(t))^{.66}} - 2A(t)L(t)}{A(t)L(t)} = \dfrac{F(K(t), A(t)L(t))}{A(t)L(t)}$

so $f(k(t)) = k(t)^{.34} - 2$

(c) (i) $f'(k(t)) = (.34)\, k(t)^{-.66} > 0$ since $k(t) > 0$

$\therefore$ marginal product of $k(t)$ is positive.

(ii) $f''(k(t)) = (-.66)(.34)\, k(t)^{-1.66} < 0$

$\therefore$ diminishing returns holds

(iii) $\lim_{k(t)\to 0} f'(k(t)) = .34 \lim_{k(t)\to 0} k(t)^{-.66} = \infty$

(iii) $\lim_{k(t)\to\infty} f'(k(t)) = .34 \lim_{k(t)\to\infty} k(t)^{-.66} = 0$

$\therefore$ Inada conditions hold

2. (a)

$$\ln Y(t) = \alpha \ln K(t) + \gamma \ln(A(t)) + \gamma \ln L(t)$$

so $\frac{\dot{Y}(t)}{Y(t)} = \frac{\partial \ln Y(t)}{\partial t} = \alpha \frac{\dot{K}(t)}{K(t)} + \gamma \frac{\dot{A}(t)}{A(t)} + \gamma \frac{\dot{L}(t)}{L(t)}$ ✓

(b) $\frac{\partial Y(t)}{\partial L(t)} \cdot \frac{L(t)}{Y(t)} = \gamma K(t)^{\alpha} A(t)^{\gamma} L(t)^{\gamma-1} \cdot \frac{L(t)}{Y(t)}$

$= \gamma$ ← 2 points)

but $\alpha_A = .55 = \gamma K(t)^{\alpha} A(t)^{\gamma-1} L(t)^{\gamma} \cdot \frac{A(t)}{Y(t)}$

$= \gamma$

(2 points)

so $\frac{\partial Y(t)}{\partial L(t)} \cdot \frac{L(t)}{Y(t)} = .55$

c) $\alpha_K = \frac{\partial Y(t)}{\partial K(t)} \frac{K(t)}{L(t)} = \alpha$

$\alpha_L = \gamma$

so $\alpha + \gamma = \alpha_K + \alpha_L = 1$
when this occures we have Constant returns to scale

moreover this implies that when labour and capital are paid their marginal products profits are zero in eq'm

i.e., $w = \frac{\partial Y}{\partial L}$, $r = \frac{\partial Y}{\partial K}$

$$\Rightarrow \frac{w \cdot L}{Y} + \frac{rK}{Y} = 1$$

$$\Rightarrow Y - wL - rK = 0$$

(d) $$R(t) = \frac{\dot{Y}(t)}{Y(t)} - \alpha_K \frac{\dot{K}(t)}{K(t)} - \alpha_L \frac{\dot{L}(t)}{L(t)}$$

$$= \frac{\dot{Y}(t)}{Y(t)} - (1-\alpha_L)\frac{\dot{K}(t)}{K(t)} - \alpha_L \frac{\dot{L}(t)}{L(t)}$$

$$= \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} - (1-\alpha_L)\frac{\dot{K}(t)}{K(t)} + (1-\alpha_L)\frac{\dot{L}(t)}{L(t)}$$

$$= 3\% - (1-.55)3\% + (1-.55)2\%$$

$$= (.55)3\% + (.45)2\%$$

1.65 + .90 = 2.55.

3.

a)
$$\int_{t=0}^{\infty} e^{-R(t)} c(t) A(t) \frac{L(t)}{H} dt \le k(0) \frac{A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} w(t) \frac{A(t)\, L(t)}{H} dt$$

$$\Leftrightarrow \int_{t=0}^{\infty} e^{-R(t)} c(t) \frac{A(0)L(0)}{H} e^{(n+g)t} dt \le k(0) \frac{A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} w(t) \frac{A(0)L(0)}{H} e^{(n+g)t} dt$$

so
$$\boxed{\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt \le k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt.}$$

(b)
$$\mathcal{L} = B \int_{t=0}^{\infty} e^{-\beta t} \left(\frac{c(t)^{\Delta} - 1}{\Delta} \right) dt + \lambda \left[k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt - \int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt \right]$$

$$\frac{\partial \mathcal{L}}{\partial c(t)} = B e^{-\beta t} c(t)^{\Delta - 1} - \lambda e^{-R(t)} e^{(n+g)t} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt - \int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt = 0$$

(c) firms maximize profits. Choose $K(t)$ & $L(t)$

$$\max_{\{K(t), L(t)\}} K(t)^{\alpha} (A(t)L(t))^{1-\alpha} - w(t) A(t) L(t) - r(t) K(t)$$

First order conditions:

$$K(t) : \quad \alpha K(t)^{\alpha - 1} (A(t)L(t))^{1-\alpha} - r(t) = 0$$

3.6 cont

$$\beta e^{-\beta t} c(t)^{\Delta-1} = \lambda e^{-R(t)} e^{(n+g)t}$$

so

$$\ln B - \beta t + (\Delta-1)\ln c(t) = \ln\lambda - R(t) + (n+g)t$$

since this holds for all t we can take the derivative with respect to t to get:

$$-\beta + (\Delta-1)\frac{\dot{c}(t)}{c(t)} = -r(t) + (n+g)$$

so

$$\frac{\dot{c}(t)}{c(t)} = \frac{-(\rho-n-\Delta g) + r(t) - n - g}{1-\Delta}$$

$$= \frac{r(t) - \rho - g(1-\Delta)}{1-\Delta}$$

this reduces to

$$\alpha k(t)^{\alpha-1} = r(t) \qquad \text{since} \quad k(t) = \frac{K(t)}{A(t)L(t)}$$

$f(k(t)) = k(t)^{\alpha}$ here so

$$f'(k(t)) = \alpha k(t)^{\alpha-1}$$

$$\therefore \quad f'(k(t)) = r(t)$$

$L(t)$: $(1-\alpha) K(t)^{\alpha} A(t)^{1-\alpha} L(t)^{-\alpha} - w(t) A(t) = 0$

so $(1-\alpha) K(t)^{\alpha} (A(t)L(t))^{-\alpha} = w(t)$

or $(1-\alpha) k(t)^{\alpha} = w(t)$

so $k(t)^{\alpha} - \alpha k(t)^{\alpha-1} k(t) = w(t)$

so $f(k(t)) - f'(k(t)) k(t) = w(t)$ as required.

d) No, k^* cannot exceed k^G.

since

$$f'(k^G) = (n+g)$$

$$f'(k^*) = r(t) = \rho + (1-\Delta)g$$

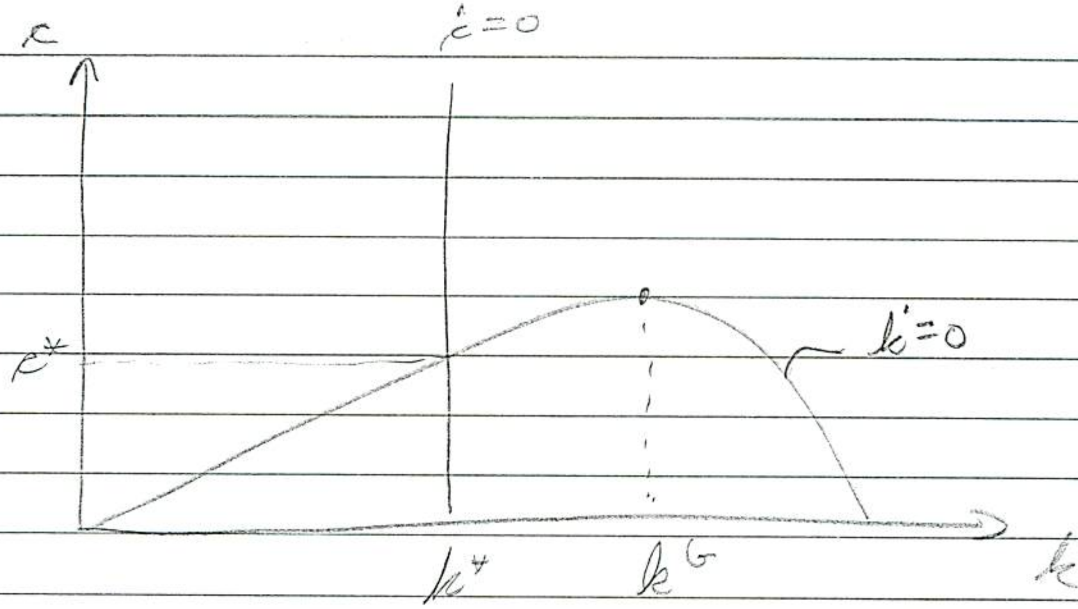
$$f'(k^G) - f'(k^*) = n + g - \rho - g + \Delta g = -(\rho - n - \Delta g) < 0$$

since $\beta > 0$

Since $f''(k) < 0$ we know $f'(k^G) - f'(k^*) < 0 \Rightarrow k^G > k^*$

4.

(a)



k^* = steady state level of k

k^G = golden rule level of k

5

(b) i) $\dot{k}_{OLD}=0$ is given by

$c(t) = f(k(t)) - (n+g)k(t)$

4

$\dot{k}_{NEW}=0$ is given by $c(t) = f(k(t)) - (n+g+\delta)k(t)$

$\therefore$ $\dot{k}(t)=0$ curve moves down (as seen in the diagram on next page)

Initially $\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$ ~~so~~

$\dot{c}_{OLD}=0 \Rightarrow f'(k^*_{OLD}) = \rho + \theta g$

$\dot{c}_{NEW}=0 \Rightarrow \frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta} = 0$

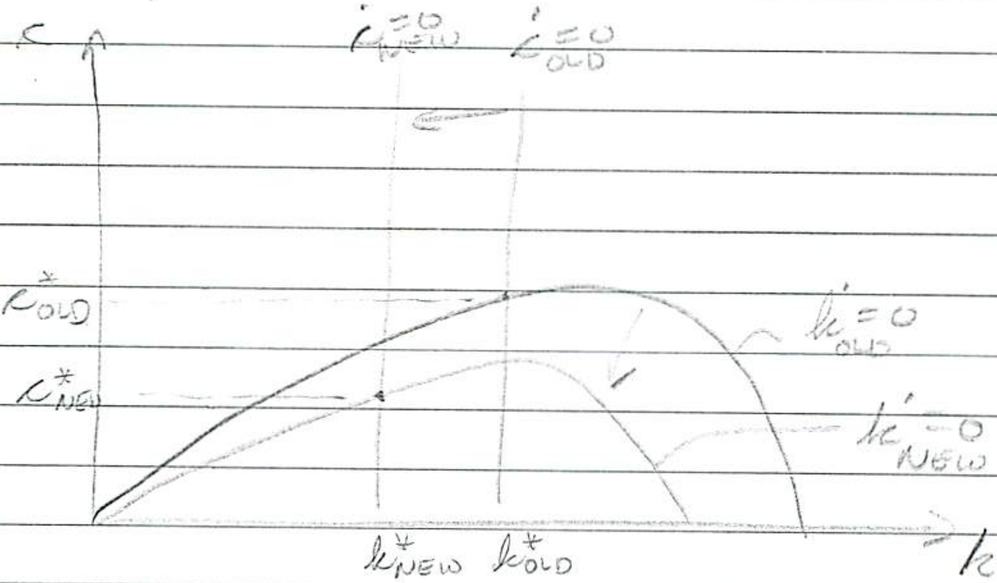
so $f'(k^*_{NEW}) = \delta + \rho + \theta g$

Since

$f'(k^*_{OLD}) < f'(k^*_{NEW})$ (since $\delta > 0$)

we know $k^*_{OLD} > k^*_{NEW}$ since $f''(k) < 0$

as the $\dot{c} = 0$ curve shifts inwards



2 points for each correct shift & justification.

(ii) initially c may increase or decrease (or stay the same)

depending on where the saddle path crosses the

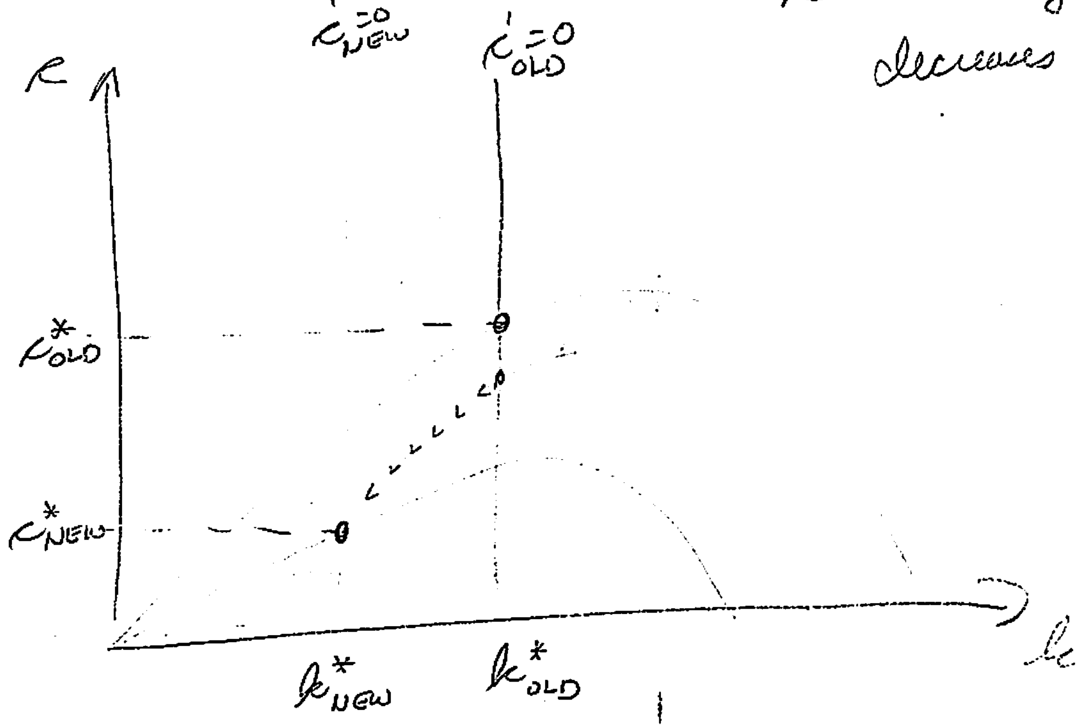
3 $\dot{c}_{OLD} = 0$ curve. However after the initial

move c will decrease (towards its new steady state value) as the economy

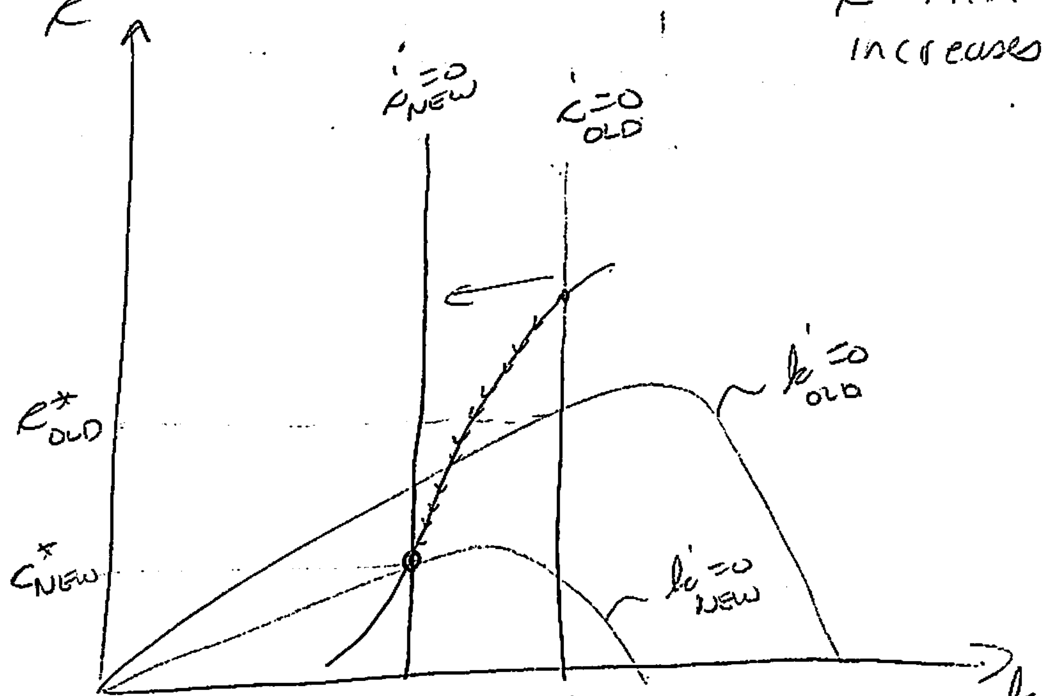
moves along the balanced growth path

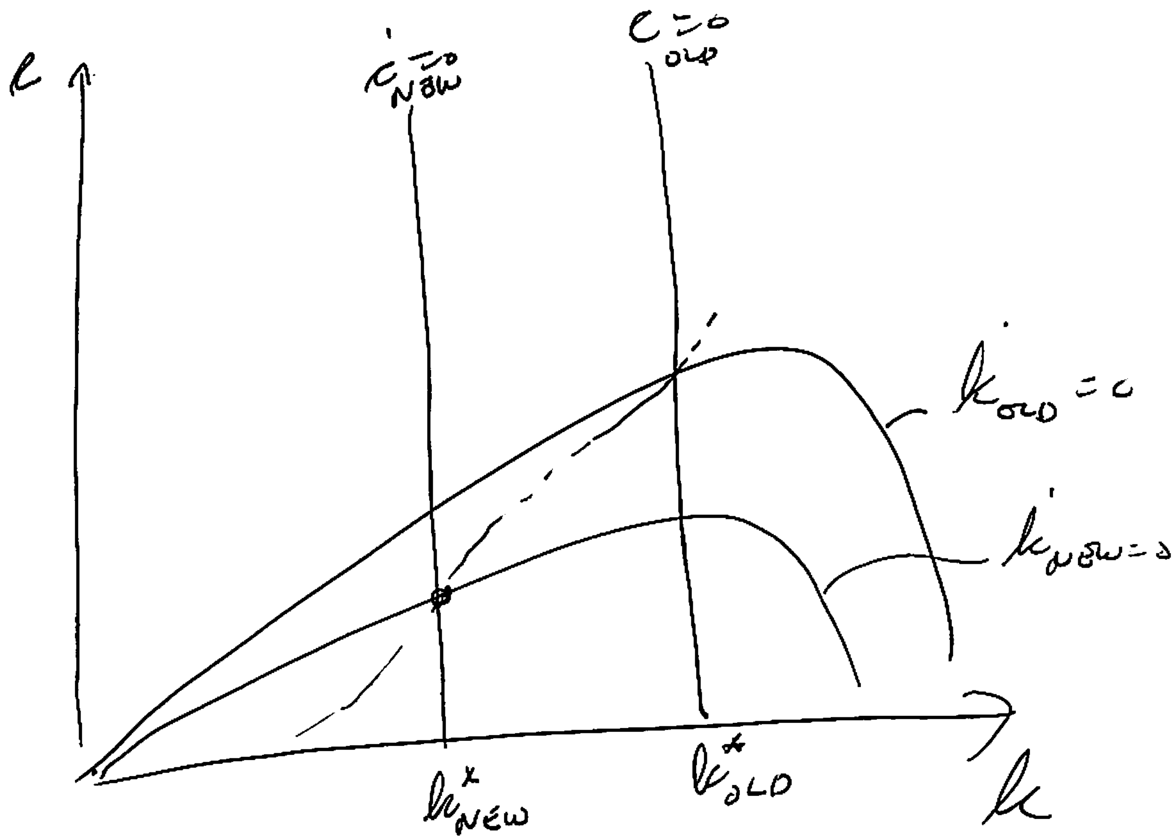
←

case where c initially decreases



case where c initially increases




c
ċ = 0 NEW
ċ = 0 OLD
k̇ OLD = 0
k̇ NEW = 0
k*NEW
k*OLD
k

NBER

NATIONAL BUREAU OF ECONOMIC RESEARCH

4(c)

If Δ in d is temporary. Then $\dot{k}=0$ and $\dot{c}=0$ would eventually shift back to original places

~~There would be no change in~~ the steady state values are the same as the original values and ~~of~~ the growth rates of variables along the balanced growth path. since (n & g don't change)

8

(d) since $\frac{K(t)}{A(t)L(t)} = k(t)$

a 1 time decrease in $K(t)$ causes $k(t)$ to fall.
at the time of the decrease
Since $k(t)$ falls so does $y(t)$ since $y(t) = f(k(t))$

after the initial decrease $k(t)$ increases ($\dot{k}(t) > 0$)
back towards its ss level k^* (and

$y(t)$ increases back towards its steady state

level. y^*.

There is no effect on k^* & k^G

since the parameters that determine these values
don't change.

4

Printed on recyclable paper

5. $f(k) = k^\alpha$ here

(a) $s k^{*\alpha} = (n+g+\delta) k^*$

so $k^* = \left(\frac{n+g+\delta}{s}\right)^{\frac{1}{\alpha-1}}$

$y^* = f(k^*) = \left(\frac{n+g+\delta}{s}\right)^{\frac{\alpha}{\alpha-1}}$

$c^* = (1-s) f(k^*) = (1-s)\left(\frac{n+g+\delta}{s}\right)^{\frac{\alpha}{\alpha-1}}$

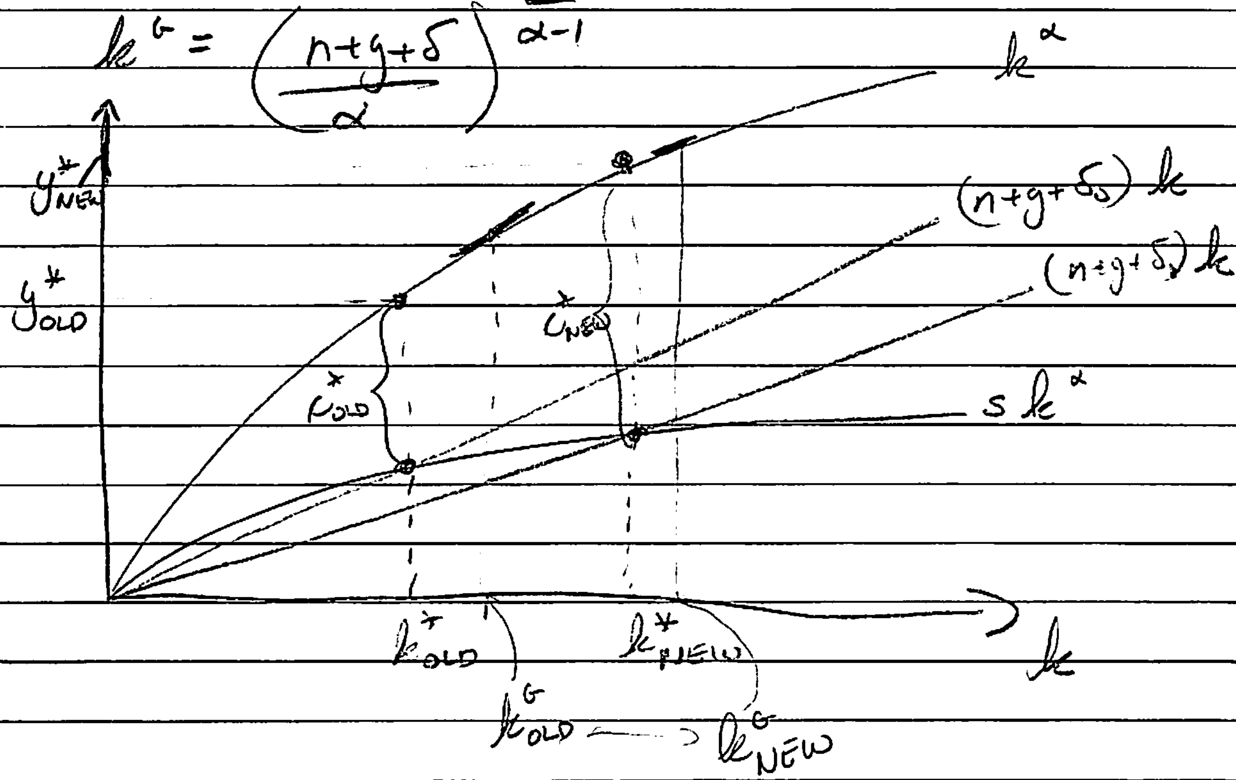
or $c^* = f(k^*) - (n+g+\delta) k^*$

b) $f'(k^G) = n+g+\delta$

so $\alpha (k^G)^{\alpha-1} = (n+g+\delta)$

$k^G = \left(\frac{n+g+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}$

(c)



so k^G increases, k^* increases, y^* increases & c^* increases c (ii)

$$s f(k^*) = (n+g+\delta) k^*$$

so $\frac{\partial k^*}{\partial \delta}$ can be gotten by total differentiation

$$s f'(k^*) \frac{\partial k^*}{\partial \delta} = k^* + (n+g+\delta) \frac{\partial k^*}{\partial \delta}$$

so $$\frac{\partial k^*}{\partial \delta} = \frac{k^*}{[s f'(k^*) - (n+g+\delta)]} < 0$$

so $\delta \downarrow \Rightarrow k^* \uparrow$

alternatively

$$\frac{\partial k^*}{\partial \delta} = \frac{\partial}{\partial \delta} \left[\left[\frac{n+g+\delta}{s} \right]^{\frac{1}{\alpha-1}} \right]$$

$$= \left(\frac{1}{\alpha-1} \right) (n+g+\delta)^{\frac{1}{\alpha-1}-1} \left(\frac{1}{s} \right)^{\frac{1}{\alpha-1}} < 0$$ since $\alpha \in (0,1)$

$$\frac{\partial y^*}{\partial \delta} = f'(k^*) \frac{\partial k^*}{\partial \delta} < 0$$ since $f'(k^*) > 0$ and $\frac{\partial k^*}{\partial \delta} < 0$

$$\frac{\partial c^*}{\partial \delta} = (1-s) f'(k^*) \frac{\partial k^*}{\partial \delta} < 0$$ since $s \in (0,1)$, $\frac{\partial k^*}{\partial \delta} < 0$ and $f'(k^*) > 0$

$$\frac{\partial k^G}{\partial \delta} = \frac{\partial}{\partial \delta} \left[\left[\frac{n+g+\delta}{\alpha} \right]^{\frac{1}{\alpha-1}} \right] = \left(\frac{1}{\alpha-1} \right) (n+g+\delta)^{\frac{1}{\alpha-1}-1} \left(\frac{1}{\alpha} \right)^{\frac{1}{\alpha-1}} < 0$$

since $\alpha \in (0,1)$

(d)

$$(.5) = e^{-\lambda t^*} \quad \text{where}$$

$$\lambda = (1-\alpha_K)(n+g+\delta)$$

$$\alpha_K = \frac{\partial Y}{\partial K} \cdot \frac{K}{Y} = \alpha K^{\alpha-1}(AL)^{1-\alpha} \cdot \frac{K}{Y} = \alpha$$

so $\lambda = \left(1-\frac{1}{3}\right)(.01+.01+.01)$

$$= \frac{2}{3}(.03) = .02$$

so $\boxed{t^* = \frac{-\ln(.5)}{.02}}$