# 1. MATH REVIEW FOR ECO 325H1F

The majority of the math tools that you will need during this course can be broken down into the basic categories of algebra and differential calculus. These notes are not meant to be a complete list of the mathematics that you will encounter in the course. Instead, they are meant to refresh your memory of the techniques by giving you some simple examples.

## 1.1. Algebra

Basically, you will need to know how to solve a system of equations and how to expand equations.

**Example 1.1.** *Given ax+b=c. Solve for x.*

$ax + b = c \rightarrow ax = c - b \rightarrow x = \frac{c-b}{a}$

**Example 1.2.** *Expand the following equation:*

$$x^a(x^b + c)$$

$x^a(x^b + c) = x^a x^b + x^a c = x^{a+b} + x^a c$

**1.2. Differential Calculus**

Let a, b, and c be constants.

**Example 1.3.** *If* $f(x) = ax$, *then* $f'(x) = \frac{\partial f(x)}{\partial x} = \frac{\partial ax}{\partial x} = a\frac{\partial x}{\partial x} = a$

**Example 1.4.** *If* $f(x) = ax^b$, *then* $f'(x) = \frac{\partial f(x)}{\partial x} = \frac{\partial ax^b}{\partial x} = a\frac{\partial x^b}{\partial x} = abx^{b-1}$

**Example 1.5.** *If* $f(x) = x^3$, *then* $f'(x) = \frac{\partial f(x)}{\partial x} = \frac{\partial x^3}{\partial x} = 3x^2$

**Example 1.6.** *If* $f(x) = \frac{1}{x^a} = x^{-a}$, *then* $f'(x) = \frac{\partial f(x)}{\partial x} = \frac{\partial x^{-a}}{\partial x} = -ax^{-a-1}$

**Example 1.7.** *If* $f(x) = e^{ax}$, *then* $f'(x) = \frac{\partial f(x)}{\partial x} = \frac{\partial e^{ax}}{\partial x} = e^{ax}\frac{\partial(ax)}{\partial x} = e^{ax}a$

**Example 1.8.** *If* $f(x) = e^{2x}$, *then* $f'(x) = \frac{\partial f(x)}{\partial x} = \frac{\partial e^{2x}}{\partial x} = e^{2x}\frac{\partial(2x)}{\partial x} = e^{2x}2$

**Example 1.9.** *If* $f(x) = e^{x^b}$, *then* $f'(x) = \frac{\partial f(x)}{\partial x} = \frac{\partial e^{x^b}}{\partial x} = e^{x^b}\frac{\partial(x^b)}{\partial x} = e^{x^b}bx^{b-1}$

**Example 1.10.** *If* $f(x) = e^{x^2}$, *then* $f'(x) = \frac{\partial f(x)}{\partial x} = \frac{\partial e^{x^2}}{\partial x} = e^{x^2}\frac{\partial(x^2)}{\partial x} = e^{x^2}2x$

**Example 1.11.** *If* $f(x) = b\ln(x)$, *then* $f'(x) = \frac{\partial f(x)}{\partial x} = \frac{\partial b\ln(x)}{\partial x} = b\frac{\partial \ln(x)}{\partial x} = b\frac{1}{x}$

**1.2.1. The Chain Rule**

If $f(x) = g(x)h(x)$ where $g(x)$ and $h(x)$ are functions of $x$, $f'(x) = \frac{\partial(g(x)h(x))}{\partial x} = \frac{\partial g(x)}{\partial x}h(x) + \frac{\partial h(x)}{\partial x}g(x) = g'(x)h(x) + g(x)h'(x)$

**Example 1.12.** *If* $f(x) = x^6(x+3), g(x) = x^6$ *and* $h(x) = (x+3)$. *As a result,* $g'(x) = 6x^5$ *and* $h'(x) = 1$. *Therefore,* $f'(x) = 6x^5(x+3) + x^6 1 = 6x^6 + 18x^5 + x^6 = 7x^6 + 18x^5$

If $f(x) = g(h(x)), f'(x) = g'(h(x))h'(x)$

**Example 1.13.** *Let* $f(x) = g(h(x))$ *where* $g(y) = 2y$ *and* $h(x) = x^3$. *Then* $g'(y) = 2$ *and* $h'(x) = 3x^2$. *Therefore,* $f'(x) = 2(3x^2) = 6x^2$

Over the semester you will see the following type of notation:

$$Y(t) \text{ which denotes output as a function of time}$$
$$\text{so } \frac{\partial Y(t)}{\partial t} = Y'(t) = \dot{Y}(t) \text{ in your book's notation}$$

This derivative describes how output changes over time.

We can use this derivative to find the growth rate of a variable.

In discrete time (i.e., a case where we can count a distinct number of time periods), the growth rate of output between periods t and t+1 is:

$$\frac{Y_{t+1} - Y_t}{Y_t} \times 100\%$$

In continuous time, $\overset{\bullet}{Y}(t)$ is analogous to $Y_{t+1} - Y_t$ so the growth rate is:

$$\frac{\overset{\bullet}{Y}(t)}{Y(t)}$$

### 1.3. Unconstrained Maximization

Problem:

$$\max_x f(x)$$

The first order necessary condition (often abbreviated FONC) for this problem is:

$$f'(x) = 0$$

The sufficient condition for a local maximization problem is:

$$f''(x) < 0$$

Note for a minimization problem the FONC is the same, but the sufficient condition for a local minimum is:

$$f''(x) > 0$$

Another important fact to remember is that when f(x) is a concave everywhere, the local maximum is also the global maximum.

Problem:

$$\max_{x(t)} f(x(t))$$

where $f(x(t))$ is a concave function. The first order necessary condition for this problem is:

$$f'(x(t)) = 0$$

and the sufficient condition is:

$$f''(x(t)) < 0$$

## 1.4. Constrained Maximization

Problem:

$$\max_{x} f(x)$$

$$\text{subject to } h(x) \geq a \text{ where the value of a does not depend on x}$$

The Lagrangian for this problem is

$$\mathcal{L} = f(x) + \lambda(h(x) - a)$$

where $\lambda$ is the lagrange multiplier.

The FONCs for an interior solution are:

$\frac{\partial \mathcal{L}}{\partial x} = f'(x) + \lambda h'(x) = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = h(x) - a = 0$

Problem:

$$\max_{x(t)} f(x(t))$$

$$\text{subject to } h(x(t)) \geq a \text{ where the value of a does not depend on x}$$

The Lagrangian for this problem is

$$\mathcal{L} = f(x(t)) + \lambda(h(x(t)) - a)$$

where $\lambda$ is the lagrange multiplier.

The FONCs for an interior solution are:

$\frac{\partial \mathcal{L}}{\partial x(t)} = f'(x(t)) + \lambda h'(x(t)) = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = h(x(t)) - a = 0$

### 1.5. Multivariable Maximization

Example:

$$f(x, y) = x^b(y + 3)$$

$f_x(x, y) = f_1(x, y) = \frac{\partial f(x,y)}{\partial x} = \frac{\partial x^b(y+3)}{\partial x} = \frac{\partial x^b}{\partial x}(y + 3) = bx^{b-1}(y + 3)$

$f_y(x, y) = f_2(x, y) = \frac{\partial f(x,y)}{\partial y} = \frac{\partial x^b(y+3)}{\partial y} = x^b \frac{\partial (y+3)}{\partial x} = x^b$

Example:

$$f(x, y) = ye^x$$

$f_x(x, y) = f_1(x, y) = \frac{\partial f(x,y)}{\partial x} = \frac{\partial ye^x}{\partial x} = y\frac{\partial e^x}{\partial x} = ye^x$

$f_y(x, y) = f_2(x, y) = \frac{\partial f(x,y)}{\partial y} = \frac{\partial ye^x}{\partial y} = \frac{\partial y}{\partial x}e^x = e^x$

Remember:

$f_{xx}(x, y) = \frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial^2 f(x,y)}{\partial x \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial x}\right)$

$f_{yy}(x, y) = \frac{\partial^2 f(x,y)}{\partial y^2} = \frac{\partial^2 f(x,y)}{\partial y \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial y}\right)$

$f_{xy}(x, y) = f_{yx}(x, y) = \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right)$

**1.6. Multivariate Unconstrained Optimization**

Problem:

$$\max_{x,y} f(x,y) + g(x) + h(y)$$

FONCs for this problem are:

$f_x(x,y) + g'(x) = 0$

$\mathrm{f}_y(x,y) + h'(y) = 0$

The second order sufficient conditions are that:

$$\begin{aligned} f_{xx}(x,y) + g''(x) &< 0 \\ f_{yy}(x,y) + h''(y) &< 0 \\ f_{xx}f_{yy} - (f_{xy})^2 &> 0 \end{aligned}$$

**1.7. Multivariate Constrained Optimization**

Problem:

$$\max_{x,y} f(x,y) + g(x) + h(y)$$

$$\text{subject to } k(x,y) + m(x) + n(y) = a$$

The Lagrangian for this problem is:

$$\mathcal{L} = f(x,y) + g(x) + h(y) + \lambda(k(x,y) + m(x) + n(y) - a)$$

FONCs for this problem are:

$f_x(x,y) + g'(x) + \lambda(k_x(x,y) + m'(x)) = 0$

$f_y(x,y) + h'(y) + \lambda(k_y(x,y) + n'(y)) = 0$

The second order sufficient conditions is:

$$\begin{vmatrix} 0 & a & b \\ a & c & d \\ b & d & e \end{vmatrix} > 0$$

where

$$\begin{aligned} a &= k_x(x,y) + m'(x) \\ b &= k_y(x,y) + n'(y) \\ c &= f_{xx}(x,y) + g''(x) + \lambda(k_{xx}(x,y) + m''(x)) \\ d &= f_{xy}(x,y) + \lambda(k_{xy}(x,y)) \\ e &= f_{yy}(x,y) + h''(y) + \lambda(k_{yy}(x,y) + n''(y)) \end{aligned}$$

evaluated at the points satisfying the FONCs. This conditions says that the determinant of the matrix must be positive. In other words:

$$0(ce - d^2) - a(ae - db) + b(ad - cb) > 0$$