Solutions to Midterm

ECO 325 F (2009)

1. (a) C.R.T.S. implies $cF(K(t), A(t)L(t)) = F(cK(t), cA(t)L(t))$ for all $c \geq 0$ (*) (2 pts)

$$cF(K(t), A(t)L(t)) = c\left[K(t)^{\alpha_1}(A(t)L(t))^{\alpha_2}\right]^{\alpha_3} + c\alpha_4 A(t)L(t)$$

also
$$F(cK(t), cA(t)L(t)) = \left[(cK(t))^{\alpha_1}(cA(t)L(t))^{\alpha_2}\right]^{\alpha_3} + \alpha_4 c A(t)L(t)$$

$$= \left[c^{\alpha_1} K(t)^{\alpha_1} c^{\alpha_2} (A(t)L(t))^{\alpha_2}\right]^{\alpha_3} + c\alpha_4 A(t)L(t)$$

$$= c^{(\alpha_1+\alpha_2)\alpha_3}\left[K(t)^{\alpha_1}(A(t)L(t))^{\alpha_2}\right]^{\alpha_3} + c\alpha_4 A(t)L(t)$$

Since C.R.T.S. implies (*), for the function to exhibit C.R.T.S. the following must hold.

$\boxed{(\alpha_1+\alpha_2)\alpha_3 = 1}$ (2 pts) and there are $\boxed{\text{no restrictions on } \alpha_4}$ (2 pts)

b) The intensive form of the production function is given by
$$y(t) = \frac{K(t)^{.34}(A(t)L(t))^{.66} + 2A(t)L(t)}{A(t)L(t)} = \left(\frac{K(t)}{A(t)L(t)}\right)^{.34} + 2$$

$$= k(t)^{.34} + 2 = f(k(t))$$

2 pts for knowing def'n $y(t) = \dfrac{F(K(t), A(t)L(t))}{A(t)L(t)}$

4 pts for correct derivation.

c) (i) $f'(k(t)) = (.34)\, k(t)^{-.66}$, when $k(t) > 0$, then $f'(k(t))$ is positive $\therefore$ the marginal product of capital per unit of effective labour is positive

c) ii) $f''(k(t)) = (0.34)(-0.66)\, k(t)^{-1.66} < 0$

when $k(t) > 0$. $\therefore$ the intensive form of the production function is consistent with diminishing marginal returns on capital per unit of effective labour.

iii) The inada conditions are

$$\lim_{k(t)\to 0} f'(k(t)) = \infty \quad \text{and} \quad \lim_{k(t)\to\infty} f'(k(t)) = 0$$

so $$\lim_{k(t)\to 0} (.34)\, k(t)^{-.66} = \infty \quad \text{and} \quad \lim_{k(t)\to\infty} (0.34)\, k(t)^{-.66} = 0$$

$\therefore$ the intensive form of the production function is consisted with the Inada conditions

2. (a) $\ln Y(t) = \alpha \ln K(t) + (1-\alpha)\ln L(t) + (1-\alpha)\ln A(t)$

6 pts $$\frac{d \ln Y(t)}{dt} = \frac{\dot{Y}(t)}{Y(t)} = \alpha \frac{\dot{K}(t)}{K(t)} + (1-\alpha)\frac{\dot{L}(t)}{L(t)} + (1-\alpha)\frac{\dot{A}(t)}{A(t)}$$

then since the growth rate of output per person is given by

only final eqn. 3 pts

$\frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)}$, we have

$$\frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} = \alpha \frac{\dot{K}(t)}{K(t)} - \alpha \frac{\dot{L}(t)}{L(t)} + (1-\alpha)\frac{\dot{A}(t)}{A(t)}$$

2 b) $\alpha_K = \dfrac{\partial Y(t)}{\partial K(t)} \cdot \dfrac{K(t)}{Y(t)}$ by def'n. (2 pts)

so $\alpha_K = \dfrac{\alpha K(t)^{\alpha-1} (A(t)L(t))^{1-\alpha} \cdot K(t)}{K(t)^{\alpha} (A(t)L(t))^{1-\alpha}} = \alpha$ Q.E.D

(2 pts for derivation)

c) $\alpha_L = \dfrac{\partial Y(t)}{\partial L(t)} \cdot \dfrac{L(t)}{Y(t)} = \dfrac{(1-\alpha) K(t)^{\alpha} L(t)^{-\alpha} A(t)^{1-\alpha} L(t)}{K(t)^{\alpha} (A(t)L(t))^{1-\alpha}}$

(4 pts)

$= (1-\alpha)$

Q.E.D.

so $\alpha_K + \alpha_L = \alpha + 1 - \alpha = 1$

d) Given $\dfrac{\dot{Y}(t)}{Y(t)} - \dfrac{\dot{L}(t)}{L(t)} = 5\%$, $\alpha = \frac{1}{3}$

(4 pts) and $\dfrac{\dot{K}(t)}{K(t)} - \dfrac{\dot{L}(t)}{L(t)} = 3\%$ so the solow residual

2 pts

$$R(t) = (1-\alpha)\frac{\dot{A}(t)}{A(t)} = \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} - \left(\frac{1}{3}\right)\left(\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}\right)$$

$$\frac{\dot{Y}}{Y} = \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L} + (1-\alpha)\frac{\dot{A}}{A} \qquad = 5\% - \left(\frac{1}{3}\right)(3\%) = 4\%$$

(2 pts)

e) The solow residual tells us what component of growth in output is attributable to changes in knowledge (2 pts)

3. (a) $$U = \int_{t=0}^{\infty} e^{-\rho t}\left(C(t)^{\theta}+1\right)\frac{L(t)}{H}\,dt$$

$$= \int_{t=0}^{\infty} e^{-\rho t}\left(\left(c(t)A(t)\right)^{\theta}+1\right)\frac{L(0)e^{nt}}{H}\,dt = \int_{t=0}^{\infty} e^{-\rho t}\frac{c(t)^{\theta}\left(A(0)e^{gt}\right)^{\theta}L(0)e^{nt}}{H}\,dt + \int_{t=0}^{\infty} e^{-\rho t}\frac{L(0)e^{nt}}{H}\,dt$$

1 pt

1 pt

$$= \frac{A(0)^{\theta}L(0)}{H}\int_{t=0}^{\infty} e^{-(\rho-n-\theta g)t}c(t)^{\theta}\,dt + \frac{L(0)}{H}\int_{t=0}^{\infty} e^{-(\rho-n)t}\,dt$$

2 pts

$$= -\frac{L(0)}{H(\rho-n)}e^{-(\rho-n)t}\Big|_0^{\infty} = \frac{L(0)}{H(\rho-n)} \text{ finite}$$

4 pts for derivation

b) $\rho-n-\theta g$ must be greater than 0 to ensure lifetime utility is bounded (i.e., its value $< \infty$)

3 pts

c) $$\mathcal{L} = \frac{A(0)^{\theta}L(0)}{H}\int_{t=0}^{\infty} e^{-(\rho-n-\theta g)t}c(t)^{\theta}\,dt + \frac{L(0)}{H}\int_{t=0}^{\infty} e^{-(\rho-n)t}\,dt$$

$$+\lambda\left[k(0) + \int_{t=0}^{\infty} e^{-R(t)}w(t)e^{(n+g)t}\,dt - \int_{t=0}^{\infty} e^{-R(t)}c(t)e^{(n+g)t}\,dt\right]$$

2 pts

$$\frac{\partial\mathcal{L}}{\partial c(t)} = \frac{A(0)^{\theta}L(0)}{H}e^{-(\rho-n-\theta g)t}\theta c(t)^{\theta-1} - \lambda e^{-R(t)}e^{(n+g)t} = 0$$

for all t.

2 pts

3 c) cont. $\frac{\partial \mathcal{L}}{\partial c(t)} = 0 \iff \lambda e^{-R(t)} e^{(n+g)t} = \frac{A(0)^{\theta} L(0)}{H} e^{-(\rho-n-\theta g)t} \theta c(t)^{\theta-1}$

so

$$\ln \lambda - R(t) + (n+g)t = \ln\left(\frac{A(0)^{\theta} L(0)}{H}\theta\right) - (\rho-n-\theta g)t + (\theta-1)\ln c(t)$$

since this holds for all date t, take the derivative w.r.t. t to get

2 pts $$-\frac{\partial R(t)}{\partial t} + (n+g) = -\rho + n + \theta g + (\theta-1)\frac{\dot{c}(t)}{c(t)}$$

Since $\frac{\partial R(t)}{\partial t} = r(t)$, we get $\frac{\dot{c}(t)}{c(t)} = \frac{r(t) + (\theta-1)g - \rho}{(1-\theta)}$ $\quad r(t) - \rho - (1-\theta)g$

4 pts d) $$\max_{K(t),\,L(t)} F(K(t), A(t)L(t)) - r(t)K(t) - w(t)A(t)L(t)$$

1 pt ↑ 1 pt ↑

e) the problem in d can be written as

$$\max_{K(t),\,L(t)} A(t)L(t)\, f\left(\frac{K(t)}{A(t)L(t)}\right) - r(t)K(t) - w(t)A(t)L(t)$$

then the first order conditions are

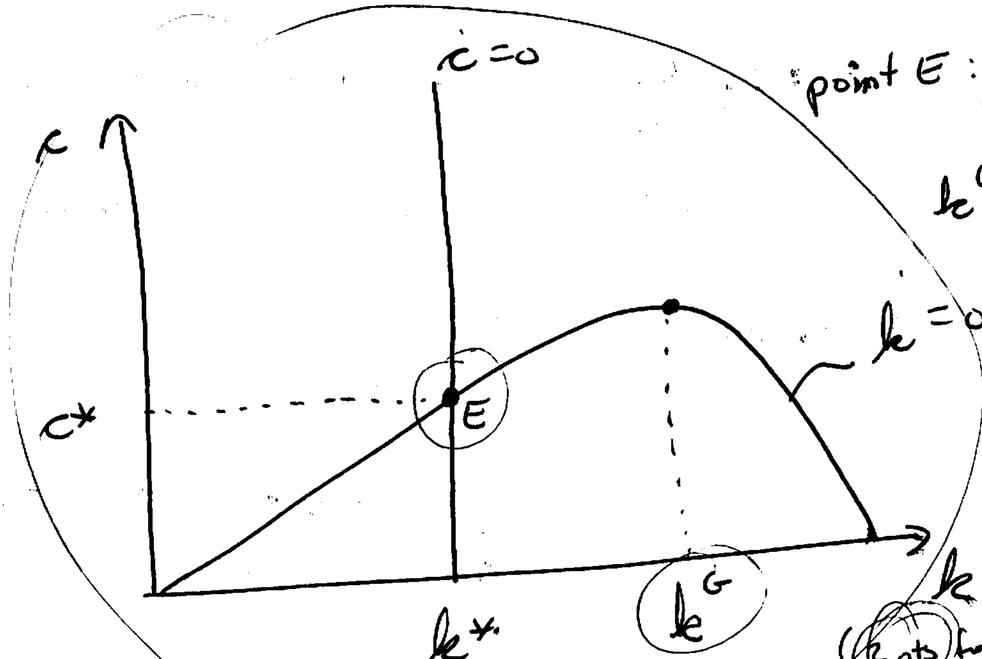
$K(t)$: $\quad A(t)L(t)\, f'\left(\frac{K(t)}{A(t)L(t)}\right)\cdot\frac{1}{A(t)L(t)} - r(t) = 0$ 3pts for this derivation

$\iff f'(k(t)) = r(t)$

$L(t)$: $\quad A(t)\, f\left(\frac{K(t)}{A(t)L(t)}\right) + A(t)L(t)\, f'\left(\frac{K(t)}{A(t)L(t)}\right)\left(\frac{-K(t)}{A(t)L(t)^2}\right) - w(t)A(t) = 0$ 3pts for this derivation which can be rewritten as:

$$w(t) = f(k(t)) - k(t)f'(k(t)) \quad \text{as desired.}$$

4 a)



point E: balanced growth path (2pts)

k^G = golden rule level of k. (2pts)

(2pts for properly drawn graph)

b)

(i) let $\delta_0 < \delta_1$, $\quad \dfrac{\dot{c}_{OLD}(t)}{c(t)} = \dfrac{r(t) - \rho - \theta g}{\theta}$ where $r(t) = f'(k(t)) - \delta_0$

as $\dot{c}_{OLD}(t) = 0 \Rightarrow f'(k^*_{OLD}) - \delta_0 - \rho - \theta g = 0$ or $f'(k^*_{OLD}) = \delta_0 + \rho + \theta g$

also $\dot{c}_{NEW}(t) = 0 \Rightarrow f'(k^*_{NEW}) - \delta_1 - \rho - \theta g = 0$ or $f'(k^*_{NEW}) = \delta_1 + \rho + \theta g$.

since $\delta_0 < \delta_1$, $f'(k^*_{OLD}) < f'(k^*_{NEW})$ so $k^*_{OLD} > k^*_{NEW}$ ~~since~~
(since $f''(k) < 0$) $\therefore$ $\dot{c} = 0$ curve. shifts in

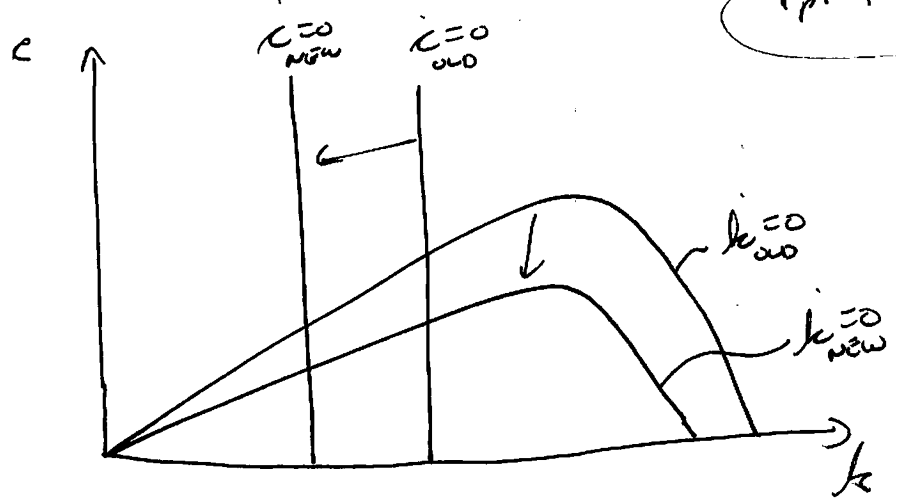
((1 pt for derivation), (1 pt for correct shift))

$\dot{k}_{OLD} = 0$ is given by $0 = f(k(t)) - c(t) - (n+g+\delta_0)k(t)$
$\iff c(t) = f(k(t)) - (n+g+\delta_0)k(t)$

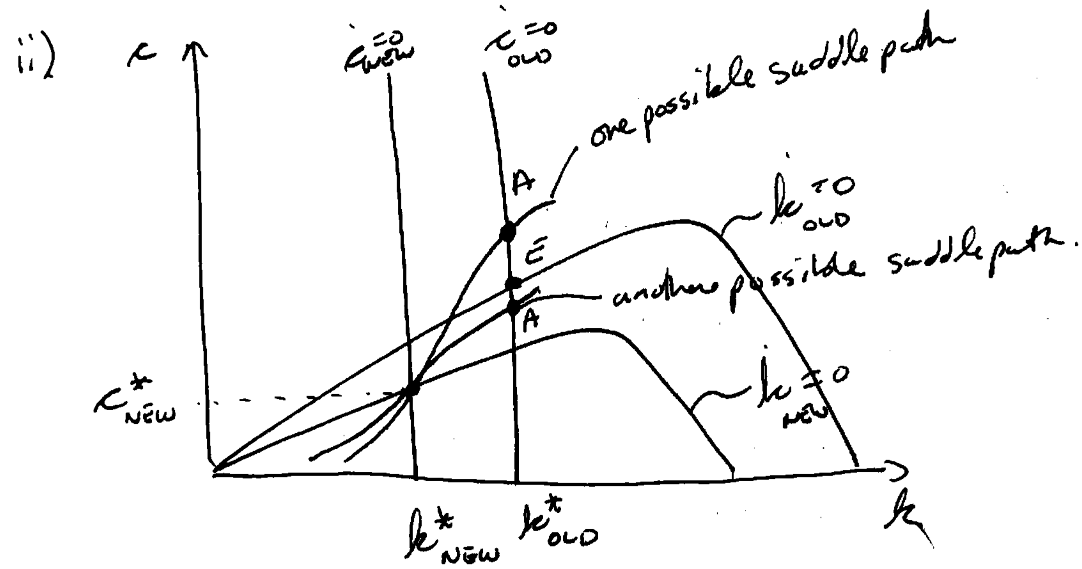
$\dot{k}_{NEW} = 0$ is given by $0 = f(k(t)) - c(t) - (n+g+\delta_1)k(t)$
$\iff c(t) = f(k(t)) - (n+g+\delta_1)k(t)$

4 b) i) can't. since $\delta_1 > \delta_0$, the $\dot{k}=0$ curve moves downwards (see graph)

(1 pt for derivation
1 pt for correct movement)



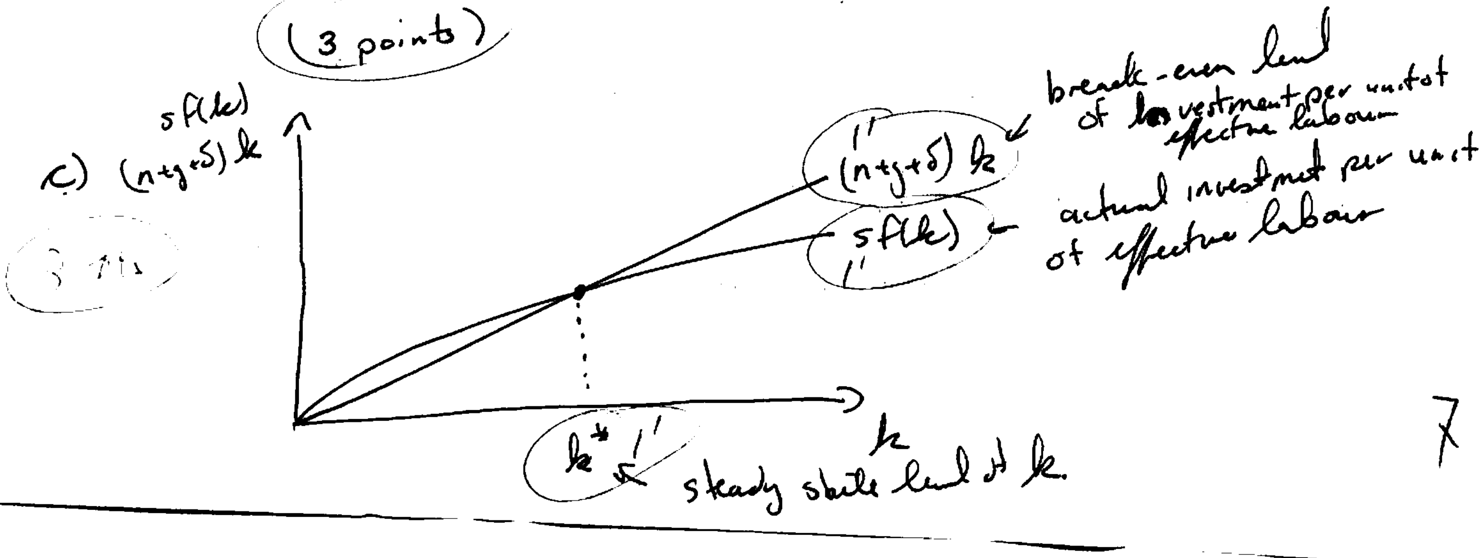
ii)



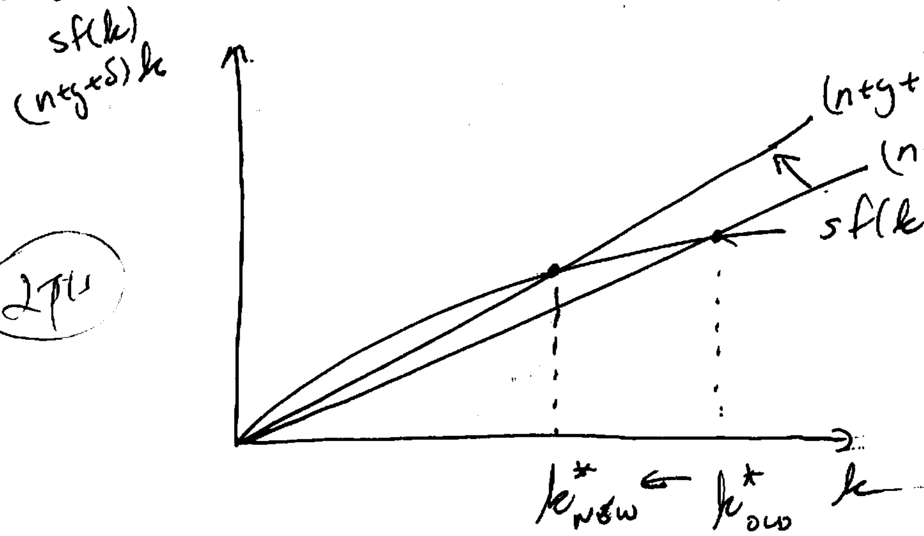
at the time of this change c will either rise, fall or stay the same - it depends on where the saddle path cuts the old $\dot{c}=0$ curve (see points A on the graph above)

(3 points)

c)



d) (i) $\delta\uparrow$: raises the slope of the break-even curve



$\delta_1 > \delta_0$

2 pts

1 pt for proper movement of k^*

1 pt for properly labelled graph

Note k^* is a fcn of δ

(ii) $sf(k^*) = (n+g+\delta)k^*$ so $sf'(k^*)\frac{\partial k^*}{\partial \delta} = k^* + (n+g+\delta)\frac{\partial k^*}{\partial \delta}$

2 pts

or $\frac{\partial k^*}{\partial \delta} = \frac{k^*}{sf'(k^*) - (n+g+\delta)} < 0$ since $sf'(k^*) < (n+g+\delta)$

1 pt for sign, 1 pt for derivation

iii) at the time of the change $c(t) = (1-s)f(k(t))$ and since $k(t)$ does not instantaneously respond, c(t) does not immediately change (3 pts)

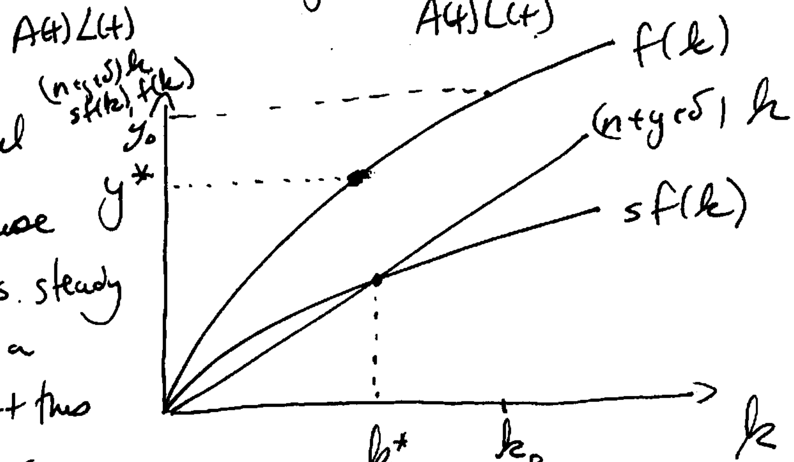
e) i) A one time unexpected decrease in the lvl of knowledge in the economy has no impact on n, g, δ or s in the Solow model and has no effect on δ, n, g, ρ or θ in the Ramsey-Cass Koopman model $\therefore$ this shock has no impact on the Balanced growth path in either model (3pts)

e) i) The golden rule level of k is given by $f'(k^G) = (n+g+\delta)$, which is unchanged in either model. However, to get the per person level of k^G we need to multiply by $A(t)$, which initially drops to its new value $k^G * A(t)$ & then grows at rate g thereafter. (3 pts)

1 pt

2 pts.

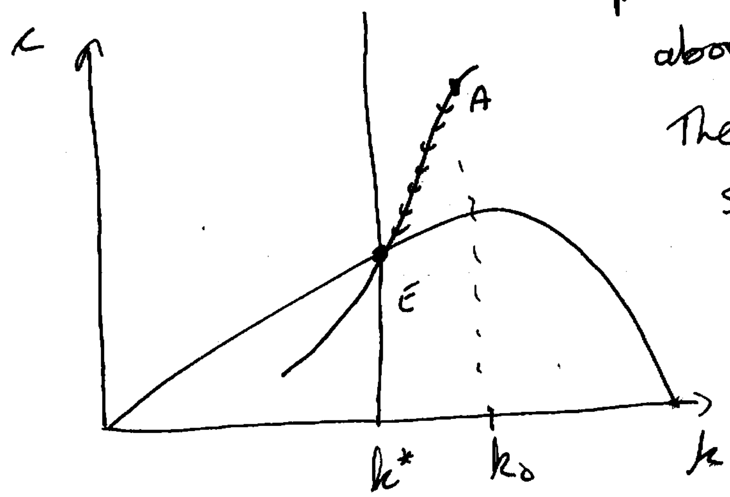
ii) $k(t) = \dfrac{K(t)}{A(t)L(t)}$ and $y(t) = \dfrac{Y(t)}{A(t)L(t)}$



In the solow model over time a decrease in A cause k to rise above its steady state level, k^*, to a point like k_0. At this point $\dot{k}(t) < 0$, so k decreases back towards k^*. When $k(t)$ reaches k^* the economy is again on its balanced growth path and stops changing. Also since $y(t) = f(k(t))$, $\dot{y}(t) = f'(k(t)) \cdot \dot{k}(t)$, so $y(t)$ initially rises to y_0, then decreases, as $k(t)$ decreases, back to its original steady state level y^*. When $k(t)$ reaches k^*, $y(t)$ reaches y^* and $y(t)$ no longer changes.

2 pts for Δ in k, 2 pts for Δ in y.

In the Ramsey - Cass - Koopman model we see a similar pattern. when A drops, $k(t)$ rises above its steady state value, k^*.



The economy then moves along its saddle path from point A, back to point E and as it does $k(t)$ falls. when the economy is again at point E, $k(t)$ is back at its original level k^*, and no longer changes (unless another shock hits). Similarly since $y(t) = f(k(t))$, $\dot{y}(t) = f'(k(t))\,\dot{k}(t)$, so $y(t)$ initially increases to $y_0 = f(k_0)$ then decreases as $k(t)$ falls from k_0 to k^*, and when k reaches its steady state value k^*, y^* returns to $y^* = f(k^*)$ and also no longer changes.

(3 pts for Δk, 2 pts for Δ in y)