Suggested Solutions

Problem Set #1

ECO 325

1. a) $\ln Y = A + \alpha \ln K + \beta \ln L \iff Y = e^A K^\alpha L^\beta$

so we have $Y = F(K, L)$

$F(cK, cL) = e^A (cK)^\alpha (cL)^\beta = c^{\alpha+\beta} e^A K^\alpha L^\beta = c^{\alpha+\beta} F(K,L)$

$c\,F(K,L) = c\,e^A K^\alpha L^\beta$

for arbitrary values of α and β $F(cK, cL) \neq cF(K,L)$

$\therefore$ this function is not always constant-returns to scale (CRTS)

but if $c^{\alpha+\beta} = c$, $F(cK, cL) = cF(K,L)$

so if $\alpha + \beta = 1$, the production function is CRTS

b) $Y = A\min(K, L) = F(K, L)$

$F(cK, cL) = A\min(cK, cL) = Ac\min(K, L) = c\,F(K, L)$

$\therefore$ this function always satisfies CRTS

1 c) $\ln Y = A + B \ln(\alpha_0 K^\rho + \alpha_1 L^\rho)$

$\Rightarrow Y = e^A (\alpha_0 K^\rho + \alpha_1 L^\rho)^B = F(K,L)$

$$F(cK, cL) = e^A (\alpha_0 (cK)^\rho + \alpha_1 (cL)^\rho)^B$$
$$= e^A c^{\rho B} (\alpha_0 K^\rho + \alpha_1 L^\rho)^B$$
$$= c^{\rho B} F(K,L)$$

so $F(cK, cL) \neq c F(K,L)$ for all values of ρ and B

but if $c^{\rho B} F(K,L) = cF(K,L)$ the function is CRTS

$\therefore$ if $c^{\rho B} = c$, the function is CRTS

$\Rightarrow$ if $\rho B = 1$, the function is CRTS

d) $\ln Y = A + \alpha_0 \ln K + \alpha_1 \ln L + \frac{1}{2}\left(\alpha_3 (\ln K)^2 + 2\alpha_4 \ln K \ln L + \alpha_5 (\ln L)^2\right)$

$= \ln(F(K,L))$

Notice that an alternative way of finding what values of $\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_5$ make this function CRTS involves rewritting the definition of CRTS

$$F(cK, cL) = cF(K,L) \Leftrightarrow \ln(F(cK, cL)) = \ln c + \ln F(K,L)$$

For this example using this new form will be easier.

1 d) cont.

$$\ln(F(cK, cL)) = A + \alpha_0 \ln(cK) + \alpha_1 \ln(cL) + \frac{1}{2}\left\{ \alpha_3 (\ln(cK))^2 + 2\alpha_4 \ln(cK)\ln(cL) + \alpha_5(\ln(cL))^2 \right\}$$

$$= A + \alpha_0 \ln c + \alpha_0 \ln K + \alpha_1 \ln c + \alpha_1 \ln L + \frac{1}{2}\left\{ \alpha_3(\ln c + \ln K)^2 + 2\alpha_4(\ln c + \ln K)(\ln c + \ln L) + \alpha_5(\ln c + \ln L)^2 \right\}$$

$$= A + \alpha_0 \ln K + \alpha_1 \ln L + (\alpha_0 + \alpha_1)\ln c + \frac{1}{2}\left\{ \begin{array}{l} \alpha_3(\ln c)^2 + 2\alpha_3 \ln c \ln K + \alpha_3(\ln K)^2 \\ + 2\alpha_4(\ln c)^2 + 2\alpha_4 \ln c \ln L + 2\alpha_4 \ln K \ln c + 2\alpha_4 \ln K \ln L \\ + \alpha_5(\ln c)^2 + 2\alpha_5 \ln c \ln L + \alpha_5(\ln L)^2 \end{array} \right\}$$

$$= \ln(F(K,L)) + (\alpha_0 + \alpha_1)\ln c + \frac{1}{2}\left\{ \begin{array}{l} (\alpha_3 + 2\alpha_4 + \alpha_5)(\ln c)^2 \\ + (2\alpha_3 + 2\alpha_4)\ln c \ln K \\ + (2\alpha_4 + 2\alpha_5)\ln c \ln L \end{array} \right\}$$

which does not generally equal $\ln(F(K,L)) + \ln c$

To find the conditions for which the function is CRTS Notice if

$$(\alpha_0 + \alpha_1)\ln c + \frac{1}{2}\left\{ \begin{array}{l} (\alpha_3 + 2\alpha_4 + \alpha_5)(\ln c)^2 \\ + (2\alpha_3 + 2\alpha_4)\ln c \ln K \\ + (2\alpha_4 + 2\alpha_5)\ln c \ln L \end{array} \right\} = \ln c$$

then the function is CRTS

2c. $F(K,L) = G(K,L) + \alpha_0 L$

First prove if $G(K,L)$ is homogeneous of degree 1 then $F(K,L)$ is homogeneous of degree 1

if $G(cK,cL) = c\,G(K,L)$ then $F(cK,cL) = G(cK,cL) + \alpha_0 cL$

$= c\,G(K,L) + c\alpha_0 L = c\,(G(K,L) + \alpha_0 L) = c\,F(K,L)$

$\therefore$ if $G(K,L)$ is homogeneous of degree 1, $F(K,L)$ is homogeneous of degree 1

Next Show that if $F(K,L)$ is homogeneous of degree 1, $G(K,L)$ is also homogeneous of degree 1

Assume $F(cK,cL) = c\,F(K,L)$

$G(K,L) = F(K,L) - \alpha_0 L$ so

$$
\begin{aligned}
G(cK,cL) &= F(cK,cL) - \alpha_0 cL \\
&= cF(K,L) - c\alpha_0 L \\
&= c\,(F(K,L) - \alpha_0 L) \\
&= c\,G(K,L)
\end{aligned}
$$

$\therefore$ $G(K,L)$ is homogeneous of degree 1 if $F(K,L)$ is homogeneous of degree 1

$\therefore$ $F(K,L)$ is homogeneous of degree 1 if and only if $G(K,L)$ is homogeneous of degree 1.

3. (Romer 1.1)

a) $Z(t) = X(t)\,Y(t)$

$\Rightarrow \ln Z(t) = \ln X(t) + \ln Y(t)$

so $\dfrac{\partial \ln Z(t)}{\partial t} = \dfrac{\partial \ln (X(t))}{\partial t} + \dfrac{\partial \ln Y(t)}{\partial t}$

$\Leftrightarrow \dfrac{\dot Z(t)}{Z(t)} = \dfrac{\dot X(t)}{X(t)} + \dfrac{\dot Y(t)}{Y(t)}$

alternatively

$\dfrac{\partial Z(t)}{\partial t} = \dfrac{\partial X(t)}{\partial t} Y(t) + X(t) \dfrac{\partial Y(t)}{\partial t}$

$\Leftrightarrow \dot Z(t) = \dot X(t)\,Y(t) + X(t)\,\dot Y(t)$

so $\dfrac{\dot Z(t)}{Z(t)} = \dfrac{\dot X(t)\,Y(t) + X(t)\,\dot Y(t)}{Z(t)} = \dfrac{\dot X(t)\,Y(t) + X(t)\,\dot Y(t)}{X(t)\,Y(t)}$

$= \dfrac{\dot X(t)}{X(t)} + \dfrac{\dot Y(t)}{Y(t)}$

b) $Z(t) = X(t) / Y(t)$

$\Rightarrow \ln Z(t) = \ln X(t) - \ln Y(t)$

so $\dfrac{\partial \ln Z(t)}{\partial t} = \dfrac{\partial \ln X(t)}{\partial t} - \dfrac{\partial \ln Y(t)}{\partial t}$

$\Leftrightarrow \dfrac{\dot Z(t)}{Z(t)} = \dfrac{\dot X(t)}{X(t)} - \dfrac{\dot Y(t)}{Y(t)}$

alternatively

$$\frac{\partial Z(t)}{\partial t} = \frac{\partial X(t)}{\partial t} \cdot \frac{1}{Y(t)} + X(t)\left(\frac{-1}{Y(t)^2}\right)\frac{\partial Y(t)}{\partial t}$$

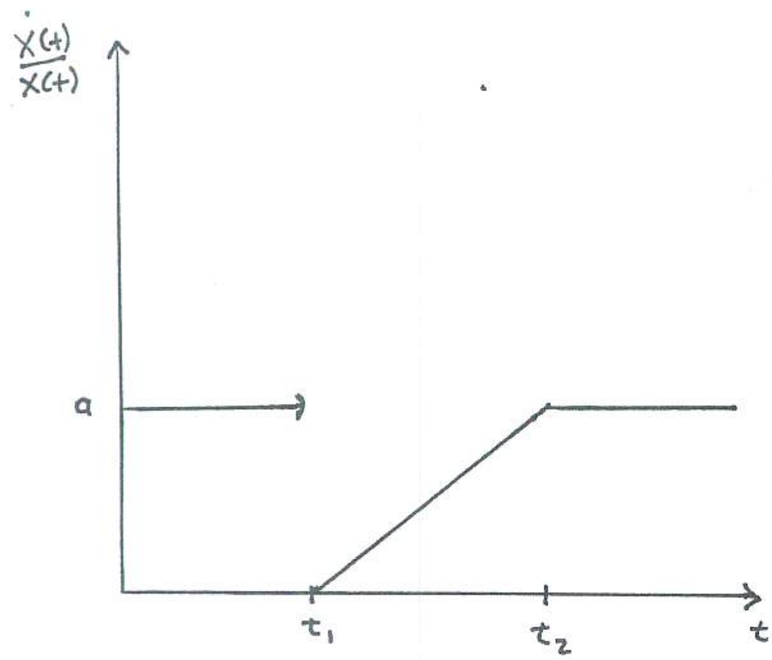
$$\Rightarrow \dot{Z}(t) = \frac{\dot{X}(t)}{Y(t)} - \frac{X(t)}{Y(t)} \cdot \frac{\dot{Y}(t)}{Y(t)}$$

so $$\frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{Y(t)\,Z(t)} - \frac{X(t)}{Y(t)}\,\frac{\dot{Y}(t)}{Y(t)}\,\frac{1}{Z(t)}$$

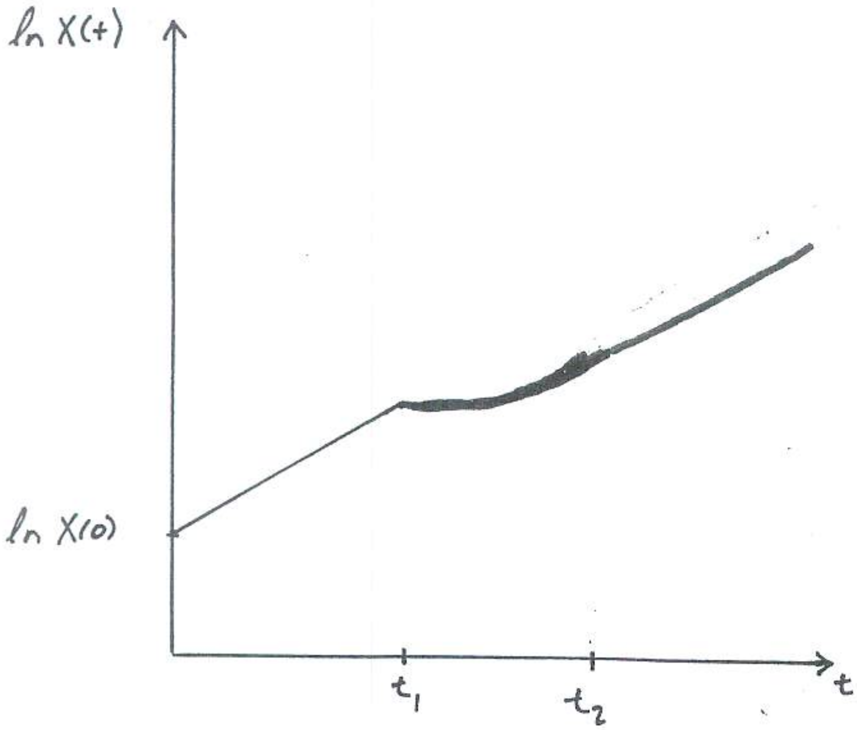
$$= \frac{\dot{X}(t)}{X(t)} - \frac{\dot{Y}(t)}{Y(t)}$$

Romer 1.2)

a)



b)



$$X(t) = \begin{cases} X(0)e^{at} & \text{for } t \in [0, t_1) \\ X(t_1)e^{\frac{a}{2}\frac{(t-t_1)^2}{(t_2-t_1)}} & \text{for } t \in [t_1, t_2] \\ X(t_2)e^{at} & \text{for } t \in (t_2, \infty) \end{cases}$$

to see that this has the growth rates described in Question 1.2. we examine the function's growth rates

for $t \in [0, t_1)$

$$\ln X(t) = \ln X(0) + at$$

$$\Rightarrow \frac{\partial \ln(X(t))}{\partial t} = a \iff \frac{\dot{X}(t)}{X(t)} = a$$

for $t \in [t_1, t_2]$

$$\ln(X(t)) = \ln X(t_1) + \frac{a(t-t_1)^2}{2(t_2-t_1)}$$

$$\Rightarrow \frac{\partial \ln(X(t))}{\partial t} = a\frac{(t-t_1)}{t_2-t_1}$$

then when $t = t_1$ growth rate is 0 and when $t = t_2$ the growth rate is a

for $t \in (t_2, \infty)$

$$\ln(X(t)) = \ln X(t_2) + at \Rightarrow \frac{\partial \ln(X(t))}{\partial t} = a \iff \frac{\dot{X}(t)}{X(t)} = a$$