Household behaviour (CASE 1)

- Case 1: households only lives for one period and there is no uncertainty and no initial wealth and only one person in each family and there is no growth
- budget constraint

$$c \leq wl$$

Household's problem

$$\max_{c,l} \ln c + b \ln(1-l) \text{ subject to } c \leq wl$$

the Lagrangian is

$$\mathcal{L} = \ln c + b \ln(1-l) + \lambda(wl - c)$$

FONC

$$\frac{\partial \mathcal{L}}{\partial c} = \frac{1}{c} - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial l} = \frac{-b}{1-l} + \lambda w = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = wl - c = 0$$

combining these equations we get

$$\frac{w}{c} = \frac{b}{1-l}$$

or

$$\frac{w}{wl} = \frac{b}{1-l}$$

so

$$1 - l = bl \rightarrow l = \frac{1}{1+b} < 1$$

- here labour does not depend on w. The income and substitution effects offset each other because utility is logarithmic in consumption and the household has no initial wealth
- as leisure becomes more important to the individual (i.e., as b increases) the equilibrium level of labour decreases
- Case 2: the household lives for 2 periods, there is no growth, no initial wealth, there is only one person in each family, there is no uncertainty about the interest rate or the second period wage
- the lifetime budget constraint is

$$c_0 + \frac{c_1}{1+r} \leq w_0 l_0 + \frac{w_1 l_1}{1+r}$$

so the household's problem is

$$\max_{\{c_0,c_1,l_0,l_1\}} \left\{ \begin{array}{c} \ln c_0 + b\ln(1-l_0) \\ +e^{-\rho}[\ln c_1 + b\ln(1-l_1)] \end{array} \right\}$$

$$s.t.$$

$$0 \leq w_0 l_0 + \frac{w_1 l_1}{1+r} - \left[c_0 + \frac{c_1}{1+r} \right]$$

- the Lagrangian is

$$\mathcal{L} = \left\{ \begin{array}{c} \ln c_0 + b\ln(1-l_0) \\ +e^{-\rho}[\ln c_1 + b\ln(1-l_1)] \end{array} \right\} + \lambda\left[w_0 l_0 + \frac{w_1 l_1}{1+r} - c_0 - \frac{c_1}{1+r} \right]$$

FONC

$$\frac{\partial \mathcal{L}}{\partial c_0} = \frac{1}{c_0} - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{e^{-\rho}}{c_1} - \frac{\lambda}{1+r} = 0$$

$$\frac{\partial \mathcal{L}}{\partial l_0} = \frac{-b}{1-l_0} + \lambda w_0 = 0$$

$$\frac{\partial \mathcal{L}}{\partial l_1} = \frac{-be^{-\rho}}{1-l_1} + \lambda \frac{w_1}{1+r} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \left[w_0 l_0 + \frac{w_1 l_1}{1+r} - c_0 - \frac{c_1}{1+r} \right] = 0$$

then

$$\lambda = \frac{1}{c_0}$$

$$\lambda = \frac{e^{-\rho}}{c_1}(1+r)$$

$$\lambda = \frac{b}{w_0(1-l_0)}$$

$$\lambda = \frac{be^{-\rho}(1+r)}{w_1(1-l_1)}$$

then

$$\frac{b}{w_0(1-l_0)} = \frac{be^{-\rho}(1+r)}{w_1(1-l_1)} \rightarrow$$

$$\frac{1-l_0}{1-l_1} = \frac{w_1}{w_0}\frac{1}{e^{-\rho}(1+r)}$$

so the relative supply of labour in the 2 periods responds to changes in the relative wage

- Observations
- 1) if w_0 rises relative to w_1 then $\frac{w_1}{w_0}$ falls, so $\frac{1-l_0}{1-l_1}$ falls, so it decreases leisure in period 1 relative to period 2 so it relatively increases labour in period 1
- 2) as ρ increases (i.e., the discount rate increases) $e^{-\rho}$ decreases so $\frac{w_1}{w_0}\frac{1}{e^{-\rho}(1+r)}$ increases, which implies that $\frac{1-l_0}{1-l_1}$ increases. This implies that leisure in period 1 increases relative to

leisure in period 2 which implies a relative decrease in period 1 labour. (intuitively this is because we care about now more than the future and we care about leisure in our utility function)

- 3) as r increases, $\frac{w_1}{w_0}\frac{1}{e^{-\rho}(1+r)}$ falls→ $\frac{1-l_0}{1-l_1}$ falls which implies that there is relatively less leisure enjoyed in period 1, which implies that people work relatively more in period 1. Intuitively this occurs because the value of working hard today and saving more has a higher payoff than before
- 4) The elasticity of substitution between leisure in the 2 periods is one i.e.,

$$\frac{\partial(1-l_0)}{\partial(1-l_1)}\frac{(1-l_1)}{(1-l_0)} = \frac{w_1}{w_0}\frac{1}{e^{-\rho}(1+r)}\frac{(1-l_1)}{(1-l_0)} = 1$$

The response of labour supply to the relative wage and the interest rate are known as intertemporal substitution in labour supply (Lucas and Rapping-1969)

- Case 3: No uncertainty, one member per household no uncertainty, no depreciation, infinite time horizon and set H=1
- period t budget constraint

$$K_{t+1} - K_t + c_t + G_t \leq w_t l_t + r_t K_t$$

the Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} e^{-\rho t}[\ln c_t + b\ln(1-l_t)] + \lambda_t(w_t l_t + r_t K_t - K_{t+1} + K_t - c_t - G_t)$$

- FONC

$$\frac{\partial\mathcal{L}}{\partial c_t} = e^{-\rho t}\frac{1}{c_t} - \lambda_t = 0$$

$$\frac{\partial\mathcal{L}}{\partial l_t} = \frac{-be^{-\rho t}}{1-l_t} + \lambda_t w_t = 0$$

$$\frac{\partial\mathcal{L}}{\partial K_{t+1}} = -\lambda_t + \lambda_{t+1}(r_{t+1}+1) = 0$$

$$\frac{\partial\mathcal{L}}{\partial \lambda} = w_t l_t + r_t K_t - K_{t+1} + K_t - c_t = 0$$

then from the first equation we get

$$e^{-\rho t}\frac{1}{c_t} = \lambda_t \text{ and } e^{-\rho(t+1)}\frac{1}{c_{t+1}} = \lambda_{t+1}$$

and from the 3rd equation we get

$$\lambda_t = (1+r_{t+1})\lambda_{t+1}$$

then we get

$$\frac{1}{c_t} = (1+r_{t+1})e^{-\rho}\frac{1}{c_{t+1}}$$

also from equation 2 we get

$$\frac{be^{-\rho t}}{(1-l_t)w_t} = \lambda_t \text{ and } \frac{be^{-\rho(t+1)}}{(1-l_{t+1})w_{t+1}} = \lambda_{t+1}$$

combining this with the 3rd equation we get

$$\frac{be^{-\rho t}}{(1-l_t)w_t} = (1+r_{t+1})\frac{be^{-\rho(t+1)}}{(1-l_{t+1})w_{t+1}}$$

also since

$$e^{-\rho t}\frac{1}{c_t} = \lambda_t \text{ and } \frac{be^{-\rho t}}{(1-l_t)w_t} = \lambda_t$$

it follows that

$$\frac{1}{c_t}w_t = \frac{b}{(1-l_t)}$$

for all t

- Case 4: Adding uncertainty to Case 3
- period t budget constraint

$$K_{t+1} - K_t + c_t + G_t \leq w_t l_t + r_t K_t$$

the Lagrangian is

$$\mathcal{L} = E_0\left\{ \begin{array}{c} \sum_{t=0}^{\infty} e^{-\rho t}[\ln c_t + b\ln(1-l_t)] \\ +\lambda_t(w_t l_t + r_t K_t - K_{t+1} + K_t - c_t - G_t) \end{array} \right\}$$

- FONC

$$\frac{\partial \mathcal{L}}{\partial c_t} = E_t\left(e^{-\rho t}\frac{1}{c_t} - \lambda_t\right) = 0$$
$$\frac{\partial \mathcal{L}}{\partial l_t} = E_t\left(\frac{-be^{-\rho t}}{1-l_t} + \lambda_t w_t\right) = 0$$
$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = E_t(-\lambda_t + \lambda_{t+1}(r_{t+1}+1)) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = E_t(w_t l_t + r_t K_t - K_{t+1} + K_t - c_t) = 0$$

where E_t is the expected value at time t. So here we pick the variable knowing the values of the shocks up to and including time t.

- We are choosing c_t, K_{t+1} and l_t at time t so the value of these variables are known as well as the values of w_t, r_t, K_t and λ_t
- then from the first equation we get

$$e^{-\rho t}\frac{1}{c_t} = \lambda_t \text{ and } e^{-\rho(t+1)}\frac{1}{c_{t+1}} = \lambda_{t+1}$$

and from the 3rd equation we get

$$\lambda_t = E_t[(1+r_{t+1})\lambda_{t+1}]$$

then we get

$$e^{-\rho t}\frac{1}{c_t} = E_t\left[(1+r_{t+1})e^{-\rho(t+1)}\frac{1}{c_{t+1}}\right] \rightarrow$$
$$\frac{1}{c_t} = e^{-\rho}E_t\left[(1+r_{t+1})\frac{1}{c_{t+1}}\right]$$

but remember that

$$E_t(XY) = E_t(X)E_t(Y) + Cov_t(X,Y)$$

so

$$\frac{1}{c_t} = e^{-\rho}\left\{ \begin{array}{c} E_t(1+r_{t+1})E_t(\frac{1}{c_{t+1}}) \\ +Cov_t((1+r_{t+1}),\frac{1}{c_{t+1}}) \end{array} \right\}$$

if r_{t+1} is high when c_{t+1} is also high, then $\frac{1}{c_{t+1}}$ is low so $Cov_t((1+r_{t+1}),\frac{1}{c_{t+1}})$ is negative. This makes the expected marginal benefit lower than if

- $Cov_t((1+r_{t+1}),\frac{1}{c_{t+1}}) = 0.$
- So when $Cov_t((1+r_{t+1}),\frac{1}{c_{t+1}}) < 0,$
- current consumption tends to be higher also from equation 2 we get

$$\frac{be^{-\rho t}}{(1-l_t)w_t} = \lambda_t \text{ and } \frac{be^{-\rho(t+1)}}{(1-l_{t+1})w_{t+1}} = \lambda_{t+1}$$

combining this with the 3rd equation we get

$$\frac{be^{-\rho t}}{(1-l_t)w_t} = E_t\left[(1+r_{t+1})\frac{be^{-\rho(t+1)}}{(1-l_{t+1})w_{t+1}}\right]$$

also since

$$e^{-\rho t}\frac{1}{c_t} = \lambda_t \text{ and } \frac{be^{-\rho t}}{(1-l_t)w_t} = \lambda_t$$

it follows that

$$\frac{1}{c_t}w_t = \frac{b}{(1-l_t)}$$

for all t

- A Special Case of the Model

Simplifying Assumptions:

- Assume that depreciation is 100% i.e., $\delta = 1$
- from the firm's problem we get

$$1+r_t = \alpha\left(\frac{A_tL_t}{K_t}\right)^{1-\alpha} = \alpha\frac{Y_t}{K_t}$$

to solve the model we will rewrite the equations in log-linear form

- The solution focuses on 2 variables: labour supply per person, l, and the fraction of output that is saved,s
- Substitute $(1-s_t)Y_t$ for C_t whenever it appears so

$$c_t = \frac{C_t}{N_t} = (1-s_t)\frac{Y_t}{N_t}$$

Then $\frac{1}{c_t} = e^{-\rho}E_t[(1+r_{t+1})\frac{1}{c_{t+1}}]$ can be rewritten as:

$$-\ln(c_t) = -\rho + \ln E_t\left[(1+r_{t+1})\frac{1}{c_{t+1}}\right] \rightarrow$$

$$-\ln\left((1-s_t)\frac{Y_t}{N_t}\right) = -\rho + \ln E_t\left[\frac{(1+r_{t+1})}{(1-s_{t+1})\frac{Y_{t+1}}{N_{t+1}}}\right] \rightarrow$$

$$-\ln\left((1-s_t)\frac{Y_t}{N_t}\right) = -\rho + \ln E_t\left[\frac{\alpha\frac{Y_{t+1}}{K_{t+1}}}{(1-s_{t+1})\frac{Y_{t+1}}{N_{t+1}}}\right] \rightarrow$$

$$\left\{\begin{array}{c} -\ln(1-s_t) \\ -\ln(Y_t)+\ln N_t \end{array}\right\} = -\rho + \ln E_t\left[\frac{\alpha\frac{1}{K_{t+1}}}{(1-s_{t+1})\frac{1}{N_{t+1}}}\right] \to$$

$$\left\{\begin{array}{c} -\ln(1-s_t) \\ -\ln(Y_t)+\ln N_t \end{array}\right\} = -\rho + \ln E_t\left[\frac{\alpha\frac{1}{s_tY_t}}{(1-s_{t+1})\frac{1}{N_{t+1}}}\right] \to$$

since $s_tY_t = K_{t+1}$

$$\left\{\begin{array}{c} -\ln(1-s_t) \\ -\ln(Y_t)+\ln N_t \end{array}\right\} = -\rho + \ln E_t\left[\frac{\alpha N_{t+1}}{(1-s_{t+1})s_tY_t}\right] \to$$

$$\left\{\begin{array}{c} -\ln(1-s_t) \\ -\ln(Y_t)+\ln N_t \end{array}\right\} = \left\{\begin{array}{c} -\rho + \\ \ln\left\{\left(\frac{\alpha}{s_tY_t}\right)E_t\left[\frac{N_te^n}{(1-s_{t+1})}\right]\right\} \end{array}\right\} \to$$

$$\left\{\begin{array}{c} -\ln(1-s_t) \\ -\ln(Y_t)+\ln N_t \end{array}\right\} = \left\{\begin{array}{c} -\rho+\ln\alpha-\ln s_t-\ln Y_t \\ +\ln\left\{N_te^nE_t\left[\frac{1}{(1-s_{t+1})}\right]\right\} \end{array}\right\} \to$$

$$\left\{\begin{array}{c} -\ln(1-s_t) \\ -\ln(Y_t)+\ln N_t \end{array}\right\} = \left\{\begin{array}{c} -\rho+\ln\alpha-\ln s_t-\ln Y_t \\ +\ln N_t+n \\ +\ln E_t\left[\frac{1}{(1-s_{t+1})}\right] \end{array}\right\} \to$$

$$-\ln(1-s_t) = \left\{\begin{array}{c} -\rho+\ln\alpha-\ln s_t+n \\ +\ln\left\{E_t\left[\frac{1}{(1-s_{t+1})}\right]\right\} \end{array}\right\} \to$$

A and K do not enter into this equation so there is a value of s that is constant that satisfies this condition. To se this, note that if $s = s_t = s_{t+1}$, we get that

$$-\ln(1-s_t) = \left\{\begin{array}{c} -\rho+\ln\alpha-\ln s_t+n \\ +\ln\left\{E_t\left[\frac{1}{(1-s_{t+1})}\right]\right\} \end{array}\right\} \to$$

$$\ln s - \ln(1-s) = -\rho + n + \ln\alpha - \ln(1-s) \to$$

$$s = \alpha e^{n-p}$$

so the savings rate is constant

- next we can look at the equation $\frac{b}{1-l_t} = \frac{w_t}{c_t} \to$

$$\ln b - \ln(1-l_t) = \ln(w_t) - \ln c_t$$

$$\ln b - \ln(1-l_t) = \ln\left((1-\alpha)\frac{Y_t}{l_tN_t}\right) - \ln\left((1-s)\frac{Y_t}{N_t}\right)$$

$$\ln b - \ln(1-l_t) = \ln(1-\alpha) - \ln l_t - \ln(1-s) \to$$

$$\frac{1-s}{1-l_t} = \frac{1-\alpha}{l_tb} \to l_t = \frac{1-\alpha}{(1-s)b+1-\alpha} = l$$

so the labour supplied by individuals is constant

- Therefore we have found that the solution to the household's and firm's problem has a constant labour supply and a constant savings rate.

Given the values of $N_0, K_0,$ and $A_0,$ we can determine the values of consumption,, investment and output. i.e.,

$$l_0 = l,\ L_0 = lN_0, Y_0 = K_0^{\alpha}(L_0A_0)^{1-\alpha}$$
$$K_1 = sY_0, C_0 = (1-s)Y_0, w_0 = (1-\alpha)\frac{Y_0}{L_0}$$
$$r_0 = \alpha\frac{Y_0}{K_0} - 1$$

then using $K_1, A_1 = A_0e^g, N_1 = N_0e^n,$ and $l_1 = l,$ we can determine the values of Y_1, L_1, C_1, w_1, r_1 and $K_2.$ repeating this process we can determine the values for all dates.

Here the economy's output movements are driven by real shocks, not market failure

Here observed aggregate output movements represent the time varying Pareto optimum

The Model's Responses

$$\begin{aligned}\ln Y_t &= (1-\alpha)\ln A_t + \alpha\ln K_t + (1-\alpha)\ln L_t \\ &= (1-\alpha)\ln A_t + \alpha\ln sY_{t-1} + (1-\alpha)\ln(lN_t) \\ &= (1-\alpha)(\overline{A} + gt + \widetilde{A_t}) + \alpha\ln s + \alpha\ln Y_{t-1} \\ &\quad + (1-\alpha)\ln l + (1-\alpha)(\overline{N} + nt)\end{aligned}$$

Let $\widehat{Y}_t = \ln Y_t - \ln Y$ where Y is the value of output along the balanced growth path. Here Y would be the value where there are no shocks i.e., $\ln A_t = \overline{A} + gt,$ then

$$\begin{aligned}\ln Y_t &= (1-\alpha)(\overline{A} + gt) + \alpha\ln s + \alpha\ln Y_{t-1} \\ &\quad + (1-\alpha)\ln l + (1-\alpha)(\overline{N} + nt)\end{aligned}$$

so

$$\widehat{Y_t} = (1-\alpha)\widetilde{A_t} + \alpha\widehat{Y}_{t-1}$$

now we will use our knowledge about the $\widetilde{A_t}$ to determine how the white noise disturbances affect $\widehat{Y_t}.$ Given that this equation holds for all t:

$$\frac{\widehat{Y_t} - \alpha\widehat{Y}_{t-1}}{(1-\alpha)} = \widetilde{A_t} \text{ and}$$
$$\frac{\widehat{Y_{t-1}} - \alpha\widehat{Y}_{t-2}}{(1-\alpha)} = \widetilde{A_{t-1}}$$

then substituting these equations into the process

$$\widetilde{A_t} = \rho_A\widetilde{A_{t-1}} + \varepsilon_{A,t} \text{ we get}$$
$$\widehat{Y_t} = (\rho_A + \alpha)\widehat{Y_{t-1}} - \rho_A\alpha\widehat{Y_{t-2}} + (1-\alpha)\varepsilon_{A,t}$$

so $\widehat{Y}_t$ can be written as a linear combination of its last two values and a white noise disturbance. (this is an AR(2) process) This implies the following response of output when $\rho = .9$ and $\alpha = 1/3$, and $\varepsilon_{A,t} = \frac{1}{1-\alpha}$, and $\varepsilon_{A,0} = \ldots = \varepsilon_{A,t-1} = \varepsilon_{A,t+1} = \ldots = 0$

then $\widehat{Y}_t = (1-\alpha)\varepsilon_{A,t} = 1, \widehat{Y_{t+1}} = (\rho_A + \alpha)\widehat{Y}_t = (\rho_A + \alpha) = 1.23,$

$\widehat{Y_{t+2}} = (\rho_A + \alpha)\widehat{Y_{t+1}} - \rho_A\alpha\widehat{Y}_t = (\rho_A + \alpha)^2 - \rho_A\alpha = 1.22,$ similarly $\widehat{Y_{t+3}} = 1.14, \widehat{Y_{t+4}} = 1.03, \widehat{Y_{t+5}} = 0.94, \widehat{Y_{t+6}} = 0.84$

when demonstrates that this simple model can generate a hump-shaped response to a iid shock.

Problems with the model

- savings rate is constant→ consumption and investment are equally volatile
- labour input does not vary
- the model predicts that real wages are highly procyclical

In general there are problems with using this type of model to explain business cycles:

- Productivity shocks must be large and persistent in order to produce realistic business cycles
- the basic model's performance requires an empirically unreasonable degree of intertemporal substitution in labour supply

- predictions about behaviour of prices and wages can be counterfactual. i.e., real wages are highly procyclical in the model, but this is not generally seen in the data.
- Standard preferences are incompatible with the equity premium, i.e., the difference between the risk free rate of return and the average rate of return on equities.
- The use of the Solow residual leads to excessively volatile productivity shocks.

*******Much of the RBC Model's performance is linked to highly persistent technology shocks that are sufficiently volatile, a sufficiently elastic labor supply and empirically reasonable levels for shares of investment and consumption in output.

- Many of the extensions have focused on altering the labour market assumptions, e.g.,altering the timing of firm's decisions (factor hoarding), or altering the type of contracts offered workers this will be discussed after we have examined solution methods.
- Another strategy for making the basic RBC Model more consistent with the observed wage and employment variation, is to introduce an additional source of shocks. Christiano and Eichenbaum explore adding government and shocks to government expenditures to an RBC framework.

Why Add Government Shocks?

- Papers such as Edleberg, Eichenbaum and Fisher, and Ramey and Shapiro have found that in response to an unexpected increase in government expenditures:
  - wages fall
  - employment increases
  - output increases
  - consumption falls mildly
- Therefore if the data is responding to both technology and government expenditure shocks, we may see mildly procyclical wages along with highly procyclical employment in the data.

The Social Planner's Problem

The social planner maximizes the households' utility subject to the aggregate resource constraint. For simplicity assume that there is no growth and that $\delta = 0$ and that the population is fixed at N

$$\max_{\{c_t, l_t, K_{t+1}\}_0^\infty} E_o \sum_{t=0}^{\infty} e^{-\rho t}[\ln c_t + b\ln(1-l_t)]N$$

$$\text{s.t.}0 \le K_t^\alpha (A_t N l_t)^{1-\alpha} - Nc_t - K_{t+1} + K_t$$

for all t and all states of the world

The First Order Necessary Conditions:

$$E_t\{e^{-\rho t}\frac{N}{c_t} - N\lambda_t^s\} = 0$$

$$E_t\{e^{-\rho t}\frac{-b}{1-l_t}N + \lambda_t^s(1-\alpha)\frac{Y_t}{l_t}\} = 0$$

$$E_t\{-\lambda_t^s + \lambda_{t+1}^s\left[\alpha\frac{Y_{t+1}}{K_{t+1}} + 1\right]\} = 0$$

$$Y_t - Nc_t - K_{t+1} + K_t = 0$$

where $Y_t = K_t^\alpha(A_t N l_t)^{1-\alpha}$. These equations reduce to:

$$\frac{b}{1-l_t} = \frac{1}{c_t}(1-\alpha)\frac{Y_t}{l_t N}$$

$$e^{-\rho}E_t\left\{\frac{1}{c_{t+1}}\left[\alpha\frac{Y_{t+1}}{K_{t+1}} + 1\right]\right\} = \frac{1}{c_t}$$

$$Y_t - Nc_t - K_{t+1} + K_t = 0$$

Notice that these equations are the same as those in the decentralized economy when w_t and r_t are substituted out of the household's FONCs using the firm's FONCs, and the fact that $L_t = l_t N$

- These equations imply that the allocations in the decentralized economy are the same as the allocations in the social planner's problem.
- Thus, the first welfare theorem holds in this economy.

Definition *The First Welfare Theorem says that if markets are competitive and complete, and there are no externalities (and if the number of agents is finite), the decentralized equilibrium is Pareto-efficient i.e., it is impossible to make anyone better off without making someone else worse off.*