Question 1

a) The expected nominal return, also equal to the actual nominal return, is given by

$$1 \times (1+i) = 1 + 0.05 = \underline{\$1.05}.$$

b) The expected real return is given by

$$1 \times (1+i) / (1+\pi^E) = 1.05/1.01 = \underline{\$1.039}$$

c) To compute the expected real rate of return (r^E) over this one-year horizon, we can write that

$$1 + r^E = \frac{1+i}{1+\pi^E} = \frac{1.05}{1.01}$$

$$\Rightarrow \quad r^E = \frac{1.05}{1.01} - 1 = 0.039 \text{ or } \underline{3.9 \text{ percent}}$$

d) The realized real return on the investment would be

$$1 \times (1+i) / (1+\pi^A) = \frac{1.05}{1.1} = \underline{\$0.954}.$$

e) The realized real rate of return (r^A) solves

$$1 + r^A = \frac{1+i}{1+\pi^A}$$

$$\Rightarrow \quad r^A = \frac{1.05}{1.1} - 1 = -0.0454 \text{ or } \underline{-4.5 \text{ percent}}.$$

Question 2

a) The formula for the expected real return over one year is given by

$$X(1+i)/(1+\pi^E).$$

The expected real rate of return satisfies

$$1+r^E = X(1+i)/(1+\pi^E)$$

$$\Rightarrow \quad r^E = X(1+i)/(1+\pi^E) - 1.$$

b) The expected real return would be given by

$$\frac{X(1+i_1)(1+i_2)}{(1+\pi_1^E)(1+\pi_2^E)}.$$

c) The expected real rate of return over the two-year horizon would satisfy:

$$X(1+r^E)^2 = \frac{X(1+i_1)(1+i_2)}{(1+\pi_1^E)(1+\pi_2^E)}$$

$$\Rightarrow \quad 1+r^E = \left[\frac{(1+i_1)(1+i_2)}{(1+\pi_1^E)(1+\pi_2^E)}\right]^{1/2}$$

$$\Rightarrow \quad r^E = \left[\frac{(1+i_1)(1+i_2)}{(1+\pi_1^E)(1+\pi_2^E)}\right]^{1/2} - 1.$$

d) The general formula for the expected real rate of return in this general case satisfies

$$X(1+r^E)^n = \frac{X(1+i_1)(1+i_2)\ldots(1+i_n)}{(1+\pi_1^E)(1+\pi_2^E)\ldots(1+\pi_n^E)} \equiv X\frac{\prod_{t=1}^{n}(1+i_t)}{\prod_{t=1}^{n}(1+\pi_t^E)}$$

$$\Rightarrow r^E = \left[\frac{\prod_{t=1}^{n}(1+i_t)}{\prod_{t=1}^{n}(1+\pi_t^E)}\right]^{1/n} - 1.$$

[Note that r^E is what is called a 'geometric average' fixed to be the same in each year.

The symbol " $\prod_{t=1}^{n}$ " is shorthand for $(1+x_1)(1+x_2)\ldots(1+x_n)$. It is the power series operator.]

Indexed debt question

a) Investing in the nominal instrument, the known nominal amount the investor would receive would be

$$1 + i = 1 + 0.1 = \$1.10$$

Investing in the real instrument, the investor would expect to receive a nominal payment given by

$$(1 + r)(1 + \pi^E) = 1.05 \times 1.02 = \$1.071$$

Note that this is an expected nominal return. The realized return could be higher or lower than this. If we assume that the investor is risk-neutral, then the expected nominal return is sufficient to guide choices.

In this case, given $\$1.10 > \$1.071$.

So the investor will choose to invest in the nominal instrument.

b) The break-even inflation rate (π^{BEIR}) satisfies

$$(1 + r)(1 + \pi^{BEIR}) = 1 + i$$

4.76%

$$\Rightarrow \quad \pi^{BEIR} = [(1+i)/(1+r)] - 1 = (1.1/1.05) - 1 =$$

So if $\pi^E < \pi^{BEIR}$ (as above), investors should invest in nominal bonds

Yield curve question

Given the desired two-year investment horizon, the investor can either invest today (time t) in a one-year bond at an interest rate of $i^1_t = 3$ percent, and then re-invest the proceeds in a year's time at rate i^1_{t+1}, or simply lock the money in a two-year bond at today's two-year rate, $i^2_t = 5$ percent.

In equilibrium, the one-year interest rate in one year's time, i^1_{t+1}, needs to be such that the expected nominal return from rolling over two one-year bonds (and note, that quantity is uncertain) should be equal to the nominal return on the two-year bond. Thus, in general:

$$(1) \quad (1 + i^1_t)(1 + i^1_{t+1}) = (1 + i^2_t)^2$$

[not 'squared' – rather, 'two-year']

[Implicitly, I am writing $i^{②}_{t+1} = i^{②}_t$.]

This implies that

$$1 + i^1_{t+1} = (1 + i^2_t)^2 / 1 + i^1_t$$

$$\Rightarrow \quad i^1_{t+1} = (1 + i^2_t)^2 / (1 + i^1_t) - 1$$

Thus in this case,

$$i^1_{t+1} = \frac{(1.05)^2}{1.03} - 1$$

$$= \underline{7.04 \text{ percent}}$$

Note the interpretation: if $i^1_t < i^2_t$, as in this example, then the so-called yield curve slopes up, and this indicates that the market expects that yields (or interest rates, virtually equivalently) will rise. The reverse is true if the yield curve slopes down: that signals the market's expectation that rates will fall.

If the 'arbitrage' condition given in (1) did not hold, then there would be clear profit opportunities. To illustrate, using (1), it is always possible to solve for i^1_{t+1}, as we did. Suppose you expect that the one-year rate in one year will be higher than that. This would mean that rolling over two one-year bonds would yield a higher payoff (in expectation) than the total return on the two-year investment. To show this, let your expected value on the one-year bond in one year be $i^{1,E}_{t+1} > i^{1}_{t+1}$. Then

$$\left(1+i^1_t\right)\left(1+i^1_{t+1}\right) = \left(1+i^2_t\right)^2 < \left(1+i^1_t\right)\left(1+i^{1,E}_t\right).$$

How could one exploit this profit opportunity? By selling two-year bonds (on the basis that their yield is too low) and buying one-year bonds today, in the expectation of being able to re-invest at an even higher rate in one-year bonds next year. The former selling behaviour would lead two-year yields to rise if enough people held the higher expectations $\left(i^{1,E}_{t+1} > i^1_{t+1}\right)$ – lower two-year bond prices would push the yields up; the latter buying behaviour would de-press the current one-year yields.

In equilibrium, yields would fully adjust, and there would be no potential arbitrage gains.