Properties of the expectation operator $E(.)$

Let X and Y be random variables — they can be discrete or continuous. The following properties can be verified:

$$E(\alpha) = \alpha \quad \text{, where } \alpha \text{ is a constant.}$$

$$E(\alpha X) = \alpha E(X)$$

$$E(g(X) + h(X)) = E(g(X)) + E(h(X)),$$ where $g(.)$ and $h(.)$ are functions of X.

$$E(X+Y) = E(X) + E(Y).$$

Properties of the variance operator $Var(.)$

$$Var(\alpha) = 0$$

$$Var(\alpha X) = \alpha^2 \, Var\, X$$

What about $Var(X+Y)$?

It turns out that a bit of care is needed. In the case where X and Y are so-called 'independent' random variables (which implies that $Cov(X, Y) = 0$), then we can write

$$Var(X+Y) = Var(X) + Var(Y).$$

But this is not the general case. Let us examine that.

$$\mathrm{Var}(X+Y) = E\big(X+Y-E(X+Y)\big)^2$$
$$= E\big(X+Y-E(X)-E(Y)\big)^2$$

Let us expand this, to give 16 terms:

$$\begin{aligned}\mathrm{Var}(X+Y) = E\big(&X^2 + XY - XE(X) - XE(Y)\\ &+ Y^2 + XY - YE(X) - YE(Y)\\ &- E(X)X - E(X)Y + (E(X))^2 + E(X)E(Y)\\ &- E(Y)X - E(Y)Y + E(Y)E(X) + E(Y)^2\big)\end{aligned}$$

$$\begin{aligned}= E\big(&X^2 + Y^2 + (E(X))^2 + (E(Y))^2\\ &+ 2XY - 2E(X)X - 2E(Y)Y - 2E(X)Y\\ &- 2E(Y)X + 2E(X)E(Y)\big)\end{aligned}$$

(Sorry, that was a bit tedious...)
Now, given that this involves an expectation of a sum, and noting that $E(X)$ and $E(Y)$ are constants, which we can write μ_x and μ_y, then we have:

$$\begin{aligned}\mathrm{Var}(X+Y) &= E(X^2) + E(Y^2) + \mu_x^2 + \mu_y^2\\ &\quad + 2E(XY) - 2\mu_x E(X) - 2\mu_y E(Y) - 2\mu_x E(Y\\ &\quad - 2\mu_y E(X) + 2\mu_x\mu_y.\\ &= E(X^2) + E(Y^2) + \mu_x^2 + \mu_y^2\\ &\quad + 2E(XY) - 2\mu_x^2 - 2\mu_y^2 - 2\mu_x\mu_y\\ &\quad - 2\mu_y\mu_x + 2\mu_x\mu_y\\ &= E(X^2) - \mu_x^2 + E(Y^2) - \mu_y^2\\ &\quad + 2E(XY) - 2\mu_x\mu_y\end{aligned}$$

But we know we can write the variance of X as:

$$\begin{aligned}
Var(X) &= E(X - E(X))^2 \\
&= E(X - \mu_X)^2 \\
&= E(X^2 - 2\mu_X X + \mu_X^2) \\
&= E(X^2) - 2\mu_X E(X) + \mu_X^2 \\
&= E(X^2) - \mu_X^2
\end{aligned}$$

and similarly

$$Var(Y) = E(Y^2) - \mu_Y^2$$

then we have

$$\begin{aligned}
Var(X+Y) = {} & Var(X) + Var(Y) \\
& + 2\left(E(XY) - \mu_X \mu_Y\right)
\end{aligned}$$

What about $E(XY) - \mu_X \mu_Y$?

Consider

$$\begin{aligned}
Cov(X,Y) &\equiv E((X-\mu_X)(Y-\mu_Y)) \\
&= E(XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y) \\
&= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X \mu_Y \\
&= E(XY) - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y \\
&= E(XY) - \mu_X \mu_Y
\end{aligned}$$

Therefore, in general, we have

$$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y).$$