ENDOGENOUS BUYER-SELLER CHOICE AND DIVISIBLE MONEY IN SEARCH EQUILIBRIUM*

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Abstract

In the Lagos-Wright (2005) model, the quasi-linear preferences assumption is not necessary to generate simple distributions of money holdings if individuals choose endogenously to go to the search market as buyers or as sellers. The non-convex buyer-seller choice provides an incentive for gambling in lotteries, and, as a result, the value function has a linear interval. As long as this interval is the relevant one for evaluating their future utilities, individuals behave as if their preferences were quasi-linear. In the stationary equilibrium, individuals remain inside this linear interval if the money supply does not decline.

Keywords: monetary search, divisible money, lotteries.

JEL: E40.
1 Introduction

Monetary search models have provided rich insights into the foundations of money, and they have become the dominant paradigm in this field of economics. To facilitate tractability, early monetary search models made strong assumptions on the properties of money (indivisibility and limited storage capacity). These strong assumptions prevented the study of many interesting issues such as inflation. Thanks to the work of Shi (1997) and Lagos and Wright (2005), we now have two distinct frameworks that yield tractable monetary search models with divisible money. In Shi, individuals belong to large households where they rebalance their assets. In Lagos and Wright, individuals alternate between a centralized competitive market and a decentralized search market. The search market generates a demand for money while the competitive market allows individuals to rebalance their assets. This rebalancing leads to a degenerate distribution of money balances if utility is linear in a good traded in the competitive market (quasi-linear preferences). The present paper shows that, with some parametric restrictions, this strong assumption can be relaxed if individuals choose endogenously to go as buyers or as sellers to the search market.¹

The binary buyer-seller choice is non-convex. As a result, individuals generally benefit from randomization among alternative actions. In the present context, individuals can reduce their demand for money by gambling at the beginning of each period. If they win the gamble, they go to the search market as buyers. Otherwise, they go to the search market as penniless sellers. The effect of this randomization is that the value function has a linear segment. Therefore, as long as this segment is the relevant one for evaluating their future utilities, individuals behave as if their preferences were quasi-linear. The convenient effect of non-convexities has been extensively used in the literature following the Rogerson (1988) model with indivisible labor, where the non-convexity arises from the constraint that individuals cannot work part-time. In monetary economics, the Rogerson model has recently been used by Rocheteau, Rupert, Shell, and Wright (2007) to also relax the quasi-linear preferences assumed by Lagos and Wright (2005) in a related but distinct fashion to the one advanced here.

The rest of the paper is organized as follows. Section 2 describes the environment of the model. Section 3 studies the bargaining that takes place in a trade meeting in the search market. Section 4 formulates the maximization programs that characterize the optimal behavior of individuals. The conditions under which all individuals, conditional on their type, demand the same quantities of

¹This endogenous choice is present in Rocheteau and Wright (2007), but there preferences are still assumed to be quasi-linear.
money in a stationary equilibrium are established in Section 5. Section 6 discusses how similar results apply to a class of non-stationary equilibria. Section 7 concludes. An Appendix collects the proofs.

2 The Environment

The economy is composed of a continuum of measure one of individuals who live forever. Time is discrete and indexed by subscript $t$. Each period consists of two subperiods: day and night. During the day, all the individuals can produce and consume a nondurable good traded frictionlessly in a competitive spot market. Also, during the day, individuals are able to gamble their money holdings in atemporal fair lotteries. During the night, individuals are able to either consume or produce, but not both, another nondurable good, which is traded bilaterally in a search market where individuals can hide their true identity. Each individual must decide during the day either to go to the night market as a seller capable of producing the good or as a buyer ready to consume it. There are several reasons why this choice must be made ex-ante. For example, individuals may need to do some preparatory tasks during the day (acquire the proper technology) as in Rocheteau and Wright (2007), or they may have to search differently depending on the role they aim to play (e.g., buyers move around and sellers stay put).

At the night market, individuals are randomly matched in pairs, and trade can only take place in these matches. The matching technology yields constant returns to scale, so the probabilities that a buyer ($\pi^b_t$) and a seller ($\pi^s_t$) will be able to trade depend only on the relative abundance of the buyers and sellers searching. Therefore, these probabilities are functions of the fraction of individuals choosing to be buyers in period $t$ ($b_t$): $\pi^b_t = \pi^b (b_t)$ and $\pi^s_t = \pi^s (b_t)$. Both functions $\pi^b$ and $\pi^s$ are continuously differentiable. The function $\pi^b$ is decreasing, with terminal conditions: $\pi^b (0) = 1$ and $\pi^b (1) = 0$, whereas the function $\pi^s$ is increasing, with $\pi^s (0) = 0$ and $\pi^s (1) = 1$. Finally, the Law of Large Numbers implies: $b_t \pi^b (b_t) = (1 - b_t) \pi^s (b_t)$.

In this environment, money is essential because at night buyers can hide their identity and there is a lack of a double coincidence of wants. (See Kocherlakota, 1998, and Wallace, 2001.) This essentiality is not compromised by the existence of atemporal lotteries because those can be traded anonymously. Money is an intrinsically useless, perfectly divisible, and storable asset. The money supply grows at a constant gross rate $\gamma : M_{t+1} = \gamma M_t$, where $M_t$ is the aggregate quantity of money at the night of period $t$. New money is injected via a lump-sum transfer $\tau_t$ to all individuals at the
beginning of each day, so \( \tau_t = (1 - \gamma^{-1}) M_t \).

The objective of individuals is to maximize their expected lifetime utility. Preferences are additively separable, and the discount factor \( \beta \) belongs to the interval \((0, 1)\). The utility from daytime activities is given by a smooth concave function \( U(x_t, y_t) \), where \( x_t \) and \( y_t \) are respectively the quantities consumed and produced.\(^2\) The function \( U \) is increasing in \( x_t \) and decreasing in \( y_t \), and there is a quantity \( x^* \) such that \( U_1(x^*, x^*) = -U_2(x^*, x^*) \). Without loss of generality, the units used to measure utility are chosen so that \( U_1(0, x^*) = 1 \) and \( U(x^*, x^*) = 0 \). At night, the utility of a buyer is a concave function of the quantity consumed: \( u(q_t) \), and the disutility of a seller is a convex function of the quantity produced: \( c(q_t) \). Both \( u \) and \( c \) are smooth and increasing, \( u(0) = c(0) = 0 \), and there is a positive quantity \( q^* \) such that \( u'(q^*) = c'(q^*) \). Finally, the standard Inada conditions to ensure interior solutions apply: \( U_1(0, y) = u'(0) = \infty \), and \( U_2(x, 0) = c'(0) = 0 \).

3 The Terms of Trade in the Search Market

In a trade meeting at the night of period \( t \), the quantity supplied by the seller (male), \( q_t \), and the monetary payment made by the buyer (female), \( e_t \), are the outcome of a generalized Nash bargaining process. To characterize this process, let \( W_{t+1}(m_{t+1}) \) be the value function of both traders at the beginning of period \( t + 1 \), let \( m^b_t \) and \( m^s_t \) respectively be the money balances owned by the buyer and the seller, and let \( \alpha_t \) be the fraction of \( m^b_t \) that the buyer carries to the decentralized market. The money balances of the buyer play the dual role of being her wealth and her liquidity (means of payment). Allowing for the liquidity factor \( \alpha_t \) to be less than one means that the buyer has the freedom to leave some of her money balances at home, thus restricting her liquidity to be a fraction of her wealth. As shown below, wealthy buyers may take advantage of this self-imposed restriction to manipulate the terms of trade in their favor. In contrast, the money balances of the seller play the single role of being his wealth because he never needs to make payments in the search market. Therefore, there is no point in distinguishing what fraction of \( m^s_t \) is carried to the search market.

Define the trading surpluses of the buyer and the seller in a meeting where \( q_t \) is exchanged for \( e_t \) respectively as:

\[
S^b(q_t, e_t, m^b_t) = u(q_t) - z^b_t, \quad \text{where} \quad z^b_t = \beta \left[ W_{t+1}(m^b_t) - W_{t+1}(m^b_t - e_t) \right], \quad \text{and} \\
S^s(q_t, e_t, m^s_t) = z^s_t - c(q_t), \quad \text{where} \quad z^s_t = \beta \left[ W_{t+1}(m^s_t + e_t) - W_{t+1}(m^s_t) \right].
\]

\(^2\)For the logic of the model there is no need for the day good, so one could assume as in Faig and Jerez (2007) that the only good is the night good.
The variables $z^b_t$ and $z^s_t$ are the buyer’s utility cost and the seller’s utility reward of the payment $e_t$. In equilibrium, these changes in utility will simplify nicely because $W_{t+1}$ will be evaluated inside a linear interval. However, to prove that it is optimal for individuals to remain in this interval, one has to describe the out-of-equilibrium outcomes, and to do so one requires the general definitions in (1) and (2).

Assuming the buyer has bargaining power $\theta$, the outcome of the bargaining process is the solution to the following maximization program:

$$\max_{(q_t,e_t)} \left[ S^b(q_t,e_t,m^b_t) \right]^\theta \left[ S^s(q_t,e_t,m^s_t) \right]^{1-\theta},$$  \tag{3}$$

subject to the cash constraint $e_t \leq \alpha_t m^b_t$, and the individual rationality constraints $S^b(q_t,e_t,m^b_t) \geq 0$ and $S^s(q_t,e_t,m^s_t) \geq 0$. The first-order conditions that characterize the solution to this program can be conveniently stated as:

$$\frac{u(q_t) - z^b_t}{z^s_t - c(q_t)} = \frac{\theta}{1-\theta} \frac{u'(q_t)}{c'(q_t)}, \quad \text{and}$$  \tag{4}$$

$$\frac{u(q_t) - z^b_t}{z^s_t - c(q_t)} \geq \frac{\theta}{1-\theta} \frac{W_{t+1}'(m^b_t - e_t)}{W_{t+1}'(m^s_t + e_t)}, \quad \text{with equality if } e_t < \alpha_t m^b_t. \tag{5}$$

Because of the Inada conditions imposed on $u$ and $c$, as long as the first derivative of $W_{t+1}$ is finite at 0 (it will be), the choice of $q_t$ is interior, and the trade surpluses of buyers and sellers are strictly positive. So, the first-order condition for the optimal choice of $q_t$ is just equation (4). In contrast, the choice of $e_t$ may be constrained by the cash carried by the buyer to the search market in which case the Kuhn-Tucker condition (5) applies.

The first-order condition (4) implicitly determines a schedule mapping $z^s_t$ and $z^b_t$ onto $q_t$. Using (1) and (2), this schedule is summarized by a time independent function $f : q_t = f \left( z^s_t, z^b_t \right)$, which shape is determined by $\theta$ and the properties of $u$ and $c$. Using the Implicit Function Theorem, the derivatives of this function satisfy:

$$f_1 = \frac{\theta u'}{u' c' + (1-\theta) c'' S^b - \theta u'' S^s} > 0, \quad \text{and}$$  \tag{6}$$

$$f_2 = \frac{(1-\theta) c'}{u' c' + (1-\theta) c'' S^b - \theta u'' S^s} > 0. \tag{7}$$

With the assumptions already stated about $\theta$, $u$, and $c$, there is no guarantee for $f$ being concave, so the optimal choices made by individuals are difficult to characterize and may not be unique. Following Lagos and Wright (2005), these sidetracking complications are prevented by indirectly strengthening the assumptions about $u$ and $c$ with the following assumption about $f$:
A1: The second derivatives of the composite function \((u \circ f)\) satisfy: \((u \circ f)_{11} < 0\), \((u \circ f)_{12} \leq 0\), and \((u \circ f)_{22} u' + (u \circ f)_{12} c' \leq 0\).

In the special case \(z^b_t = z^b_t = z_t\), which holds if \(W_{t+1}\) is linear or if \(m^b_t = 0\) and \(e_t = m^b_t\), A1 implies that \(u \circ f\) is a concave function of \(z_t\), which is the analogous assumption made by Lagos and Wright (2005). As in that paper, \(\theta \approx 1\), and \(c\) linear together with \(u''u'' \geq u'u'''\) are sufficient conditions for A1 to be satisfied (see Lemma 1 in the Appendix).

4 Optimal Behavior

We are now ready to characterize the optimal plans for a typical individual in period \(t\) in an environment that will end up being consistent with an equilibrium. Let \(q_t(m^b_t, \alpha_t, m^s_t)\) and \(e_t(m^b_t, \alpha_t, m^s_t)\) be the functions that describe the outcome of the Nash bargaining process described in the previous section. Conditional on the individual being a buyer, the utility of having money balances \(m^b_t\) at the beginning of the night is given by a value function \(V^b_t(m^b_t)\) that satisfies:

\[
V^b_t(m^b_t) = \pi^b_t \max_{q_t(m^b_t, \alpha_t, m^s_t)} \left[ u \left( q_t \left( m^b_t, \alpha_t, m^s_t \right) \right) + \beta W_{t+1} \left( m^b_t - e_t \left( m^b_t, \alpha_t, m^s_t \right) \right) \right] dF_t(m^s_t) + \left( 1 - \pi^b_t \right) \beta W_{t+1} \left( m^b_t \right),
\]

(8)

where \(F_t(m^s_t)\) is the wealth distribution of the sellers present in the search market in period \(t\). Analogously, conditional on the individual being a seller, the value function \(V^s_t(m^s_t)\) satisfies:

\[
V^s_t(m^s_t) = \pi^s_t \max_{q_t(m^b_t, \alpha'_t, m^s_t)} \left[ -c \left( q_t \left( m^b_t, \alpha'_t, m^s_t \right) \right) + \beta W_{t+1} \left( m^b_t + e_t \left( m^b_t, \alpha'_t, m^s_t \right) \right) \right] dG_t(m^b_t, \alpha'_t) + \left( 1 - \pi^s_t \right) \beta W_{t+1} \left( m^s_t \right),
\]

(9)

where \(G_t(m^b_t, \alpha'_t)\) is the joint distribution of the wealth and liquidity factors of the prospective buyers.

During the day, if the individual decides to be a buyer at the approach of night, the utility attained with money balances \(\tilde{m}_t\) is given by:

\[
W^b_t(\tilde{m}_t) = \max_{\{x^b_t, y^b_t, m^b_t\}} \left[ U \left( x^b_t, y^b_t \right) + V^b_t(m^b_t) \right],
\]

subject to \(m^b_t\) being non-negative and the budget constraint:

\[
(\tilde{m}_t + \tau_t - m^b_t) \phi_t = x^b_t - y^b_t.
\]

(11)
Analogously, conditional on the individual deciding to be a seller at night, the value function $W^s_t(\tilde{m}_t)$ satisfies:

$$W^s_t(\tilde{m}_t) = \max_{\{x^s_t, y^s_t, m^s_t\}} \left[ I^d (x^s_t, y^s_t) + V^s_t (m^s_t) \right],$$

subject to $m^s_t \geq 0$ and

$$(\tilde{m}_t + \tau_t - m^s_t) \phi_t = x^s_t - y^s_t.$$  \hspace{1cm} (13)

At the beginning of the day, the individual chooses the trading role to be played at night by picking the best of the two possible roles:

$$W_t(\tilde{m}_t) = \max \left\{ W^b_t(\tilde{m}_t), W^s_t(\tilde{m}_t) \right\}. \hspace{1cm} (14)$$

This optimal choice depends on the money balances $\tilde{m}_t$. For example, in Figure 1, with low money balances, the individual prefers to be a seller, while with large money balances the individual prefers to be a buyer. Also, as seen in the figure, even if the functions $W^b_t$ and $W^s_t$ are concave, the function $W_t$ is not. This non-concavity implies that the expected utility of the individual may be increased by gambling on fair lotteries. For example, if $m_t$ is in the interior of the interval $[\tilde{m}^s_t, \tilde{m}^b_t]$, it is optimal for the individual to purchase a lottery ticket with payoff $\tilde{m}^b_t - \tilde{m}^s_t$ and a probability of winning $\varpi_t = (m_t - \tilde{m}^s_t) / (\tilde{m}^b_t - \tilde{m}^s_t)$. For this lottery to be fair, the price of the ticket must be $\varpi_t (\tilde{m}^b_t - \tilde{m}^s_t)$, so that the individual ends up with money balances $\tilde{m}^b_t$ with a lottery win and $\tilde{m}^s_t$ otherwise. The expected utility of these two outcomes is depicted by the straight line connecting $W_t(\tilde{m}^b_t)$ and $W_t(\tilde{m}^s_t)$, which is above $W_t(m_t)$ in the interval $(\tilde{m}^s_t, \tilde{m}^b_t)$. In general, the ability to play in fair lotteries implies that the value function $W_t(m_t)$ at the beginning of period $t$ must satisfy:

$$W_t(m_t) = \max_{\{\varpi_t, \tilde{m}^b_t, \tilde{m}^s_t\}} \left\{ \varpi_t W_t(\tilde{m}^b_t) + (1 - \varpi_t) W_t(\tilde{m}^s_t) \right\}, \hspace{1cm} (15)$$

subject to $\varpi_t = (m_t - \tilde{m}^s_t) / (\tilde{m}^b_t - \tilde{m}^s_t)$ and $\varpi_t \in [0, 1]$. Geometrically, this means that the function $W_t$ is the concave hull of $\tilde{W}_t$, so $W_t$ is concave although, typically, it has linear segments. The point of the paper is to show that these linear segments are the relevant portions of the value function in an important class of equilibria.

[Figure 1 should go around here]

Even if the value function at the beginning of each period is not linear, program (8), which characterizes how the money balances should be allocated when the individual goes to the search market as a buyer, is analytically tractable in an environment that will prove to be consistent with an equilibrium. The following proposition summarizes the key results of this analysis:
Proposition 1: Consider a search market where no sellers carry money balances and the terminal value function for all traders is the same increasing and concave function $W_{t+1}$. Suppose A1 holds. For all $m^b_t$, it is optimal for a buyer to restrict the liquidity carried to the search market, so that the Nash bargaining solution is cash constrained. Furthermore, this implies that the trade surplus attained by the buyer is a strictly concave function of $\alpha_t$, so that the optimal fraction of money balances carried to the search market is unique and characterized by $\alpha_t \leq 1$ and:

$$\frac{dS^b}{d\alpha_t} \geq 0, \text{ with equality if } \alpha_t < 1.$$  

Finally, the payment made by the buyer in a trade meeting is a non-decreasing function of $m^b_t$. (The proof is in the Appendix).

5 Stationary Monetary Equilibrium

In equilibrium, individuals behave optimally as defined by the Bellman equations (8) to (15); the terms of trade in the decentralized market are consistent with the Nash bargaining conditions (4) and (5); and the market of goods for money in the centralized market clears. The equilibrium is monetary if money is valued, and the equilibrium is stationary if the real value of the quantity of money is the same in all periods. To facilitate the exposition, this section characterizes a monetary stationary equilibrium, but it does so using a strategy of analysis that can be readily applied to the wide class of monetary non-stationary equilibria discussed in the next section.

Proposition 2: Assuming $\gamma \geq 1$, in a stationary monetary equilibrium the following statements hold:

1- In the night search market, the quantity traded, $\hat{q}$, and the fraction of individuals who choose to be buyers, $\hat{b}$, are given by the following system of equations:

$$\frac{u'(\hat{q})}{g'(\hat{q})} = 1 + \frac{\gamma \beta^{-1} - 1}{\pi^b(\hat{b})}, \text{ and}$$

$$\pi^s(\hat{b}) [g(\hat{q}) - c(\hat{q})] = \pi^b(\hat{b}) [u(\hat{q}) - g(\hat{q})] - g(\hat{q}) (\gamma \beta^{-1} - 1),$$

where the function $g$ is defined as:

$$g(q) = \frac{\theta u'(q) c(q) + (1 - \theta) c'(q) u(q)}{\theta u'(q) + (1 - \theta) c'(q)}.$$  

The quantity of money paid in a transaction satisfies: $\beta \phi_{t+1} \hat{e}_t = g(\hat{q})$. Sellers have no money balances, and buyers have and carry $\hat{m}^b_t = \hat{e}_t$. 

2- In the day centralized market, all individuals consume and produce the efficient quantity \( x^* \), and the value of money satisfies:

\[
\beta \phi_t M_t = \gamma \tilde{b} g (\tilde{q}) ,
\]

so \( \phi_t = \gamma \phi_{t+1} \).

3- For all \( t \), the value function \( W_t \) is increasing and concave, and has a constant slope \( \phi_t \) in a linear interval \([0, \bar{m}_t]\), where \( \bar{m}_t = \bar{m}_t^b - \tau_t \).

The key of proving Proposition 2 is to construct the value function \( W_t \) in statement 3. Unfortunately, standard recursive methods are not applicable because the required monotonicity for these methods does not necessarily hold in the presence of Nash bargaining. To avoid this difficulty and at the same time handle non-stationary equilibria, the proof, which is included the Appendix, constructs \( W_t \) as the limiting value function in a sequence of finite horizon economies with a carefully chosen terminal value function.

Let \( \Phi_t \) be the distribution function of the initial holdings of money balances in period \( t \). As shown in Proposition 2, the fraction \( b_t \) does not depend on the exact shape of \( \Phi_t \) as long as its support is in \([0, \bar{m}_t]\). To see how this is possible, note that the Law of Large Numbers implies that \( b_t \) is the integral across all individuals of the probability of being a buyer:

\[
b_t = \int_0^{\bar{m}_t} \frac{m - \bar{m}_t^s}{\bar{m}_t^b - \bar{m}_t^s} d\Phi_t (m) = \frac{M_t}{\bar{m}_t^b}.
\]

And, this integral has the same value for all distributions \( \Phi_t \) with support in \([0, \bar{m}_t]\). At the beginning of period \( t + 1 \), the distribution \( \Phi_{t+1} \) has all mass concentrated in two points: 0 and \( \bar{m}_t \). Sellers who succeed in trading and buyers who do not end up with \( \bar{m}_t \) balances, and the remaining individuals end up with zero balances. Since the measure of successful buyers must be the same as the measure of successful sellers, the fraction of individuals with \( \bar{m}_t \) balances is just the initial measure of buyers \( \tilde{b} \).

The condition \( \gamma \geq 1 \) plays a key role in ensuring that individuals stay in the linear interval of the value function \( W_t \). If \( \gamma < 1 \), the government would have to impose a lump-sum tax to finance the steady decline in the money supply. Consequently, individuals who started a period with zero money balances would not be able to afford any lottery tickets, and they would have to consume more than they would produce during the day to pay the tax. Changes in their money balances would have to be absorbed with changes in consumption and production during the day.
and because of decreasing marginal utility in these activities, the value function would not be locally linear.

6 Non-stationary Equilibria

As is typical in monetary economies, in addition to the stationary monetary equilibrium, there is a large number of non-stationary equilibria where rational expectations about future inflation determine alternative current values of money. This section shows that the strategy used to construct the stationary equilibrium can be used with minor modifications to construct a subset of the non-stationary equilibria. As long as \( t > t + 1 \) and \( m_b t \leq m_{t+1} \), the proof of Proposition 2 does not need stationarity to obtain that all individuals consume and produce \( x^* \) during the day, and that \( q_t \) and \( b_t \) are the solutions to (37) and (38). Combining these two equations and using \( b_t \pi^b (b_t) = (1 - b_t) \pi^a (b_t) \), we obtain

\[
\frac{b_t}{1 - b_t} = \frac{u(q_t) - g(q_t) \pi'(q_t)}{g(q_t) - c(q_t)}.
\]

For all \( q_t \in (0, q^*) \), \( g(q_t) - c(q_t) \) is strictly positive, and so (22) defines a function mapping \( q_t \) onto \( b_t : b(q_t) \). Substituting this function back into (37), and using \( b_t m^b_t = M_t \) (demand for money equals supply of money), and \( \beta \phi_{t+1} m^b_t = g(q_t) \) (terms of trade from Nash bargaining), results into the following difference equation:

\[
b(q_t) g(q_t) = \frac{\beta}{\gamma} \left[ \pi^b (b(q_{t+1})) \left[ \frac{\pi'(q_{t+1})}{g'(q_{t+1})} - 1 \right] + 1 \right] b(q_{t+1}) g(q_{t+1})
\]

(23)

Therefore, arguments analogous to those applied in Proposition 2, imply that any path \( \{q_t\} \) satisfying (23) that stays in \( (0, q^*) \) is a non-stationary monetary equilibrium if it can be guaranteed that for all \( t \) (i) \( \phi_t > \beta \phi_{t+1} \) and (ii) \( m^b_t \leq m_{t+1} \). Using \( b_t m^b_t = M_t \), \( \beta \phi_{t+1} m^b_t = g(q_t) \), the government budget constraint, and \( m_{t+1} = m^b_{t+1} - \tau_{t+1} \), these two conditions can be restated as: (i) \( \gamma b(q_t) g(q_t) \geq \beta b(q_{t+1}) g(q_{t+1}) \) and (ii) \( b(q_{t+1}) \leq \gamma b(q_t) / \left[ 1 + (\gamma - 1) b(q_t) \right] \). One can easily check that if the function \( b(q_t) \) is monotonically increasing, then all paths starting with \( q_0 < q \) satisfy these two conditions, so that they are non-stationary equilibria, and \( \{q_t\} \) converges monotonically to zero (the non-monetary equilibrium). Unfortunately, the function \( b(q_t) \) may not be monotonically increasing for some specifications of \( u \) and \( c \),\(^3\) in which case either the equilibria have more

\(^3\)For example, with random matching, a linear cost function, an isoelastic utility function, and \( \theta = 0.5 \), the function \( b(q_t) \) is monotonically increasing if and only if the coefficient of RRA is 0.4 or lower.
complicated dynamics or they cannot be characterized as solutions to (23) because individuals move out of the linear intervals of their value functions.

7 Conclusion

The binary buyer-seller choice introduces a natural non-convexity, which gives rise to individuals’ benefiting from playing in lotteries. As a result of these gambles, the value function is linear in an interval of money balances where individuals remain under fairly general conditions. Therefore, even without quasi-linear preferences, a model similar to Lagos and Wright (2005) has a simple distribution of money, and so it is analytically tractable.

Appendix

Lemma 1: Sufficient conditions for A1 to be satisfied are: (i) \( \theta \approx 1 \) and (ii) \( A \equiv u'c' (c'u'' - u'c'') + (2u'c'' + c'u'') (u'c'' - c'u'') \leq 0 \). (If \( c \) is linear, \( A \leq 0 \) is equivalent to \( u'u'' \leq u''u'' \).)

Proof: Applying the chain rule of differentiation on \( u(f(z^b_t, z^b_t)) \), and using (4) and the Implicit Function Theorem to determine the derivatives of \( f \) yields:

\[
(u \circ f)_{11} = \frac{f_1^2}{\Delta} \left\{ (2 - \theta) u' (c'u'' - u'c'') + \frac{(1 - \theta) S^b}{u'c'} \left[ A - 2 (u'c'' - c'u'') \right] \right\},
\]

\[
(u \circ f)_{12} = \frac{f_1 f_2}{\Delta} \left\{ (1 - \theta) u' (c'u'' - u'c'') + \frac{(1 - \theta) S^b}{u'c'} \left[ A - (u'c'' - c'u'') \right] \right\},
\]

and

\[
(u \circ f)_{22} + (u \circ f)_{12}c' = \frac{f_2^2}{\Delta^2} (1 - \theta) S^b \left[ A - \theta (u'c'' - c'u'') \right],
\]

where \( \Delta = u'c' + (1 - \theta) c''S^b - \theta u''S^s > 0 \). The first terms inside the brackets in the first two equations are always negative. The remaining terms vanish as \( \theta \) approaches one, and they are non-positive if \( A \leq 0 \). Finally, both \( f_1 \) and \( f_2 \) are positive, so the lemma follows.

Proof of Proposition 1: Suppose that at the night of period \( t \), a (female) buyer had \( m^b_t \) balances, and that the bargaining solution were interior. Let \( \tilde{\alpha}_t \) be the fraction of money balances she would have to carry to exactly pay for a trade: \( \tilde{\alpha}_t m^b_t = e_t \left( m^b_t, \tilde{\alpha}_t, 0 \right) \). All \( \alpha_t \in [\tilde{\alpha}_t, 1] \) would yield the same trade surplus, but reducing \( \alpha_t \) below \( \tilde{\alpha}_t \) would move the bargaining solution from being interior to being constrained. The following argument shows that such a move would improve her trade surplus.
Since $\tilde{\alpha}_t m^b_t = e_t$, the effect of a small reduction of $\alpha_t$ below $\tilde{\alpha}_t$ can be calculated applying the Implicit Function Theorem to (4) and using the definitions of $z^b_t$ and $z^s_t$ in (1) and (2):

$$\frac{dq_t}{d\alpha_t} = \frac{\theta u'(q_t) \beta W'_{t+1} (e_t) m^b_t + (1 - \theta) c'(q_t) \beta W'_{t+1} (m^b_t - e_t) m^b_t}{u'(q_t) c'(q_t) - \theta u''(q_t) S^s(q_t, e_t, 0) + (1 - \theta) c''(q_t) S^b(q_t, e_t, m^b_t)}$$

$$= \frac{c'(q_t) \beta W'_{t+1} (m^b_t - e_t) m^b_t}{u'(q_t) c'(q_t) - \theta u''(q_t) S^s(q_t, e_t, 0) + (1 - \theta) c''(q_t) S^b(q_t, e_t, m^b_t)},$$

where the second equality follows from (5) which holds with equality at $\alpha_t = \tilde{\alpha}_t$. Since $u'' < 0$, $c'' \geq 0$, $S^s$ and $S^b > 0$, and $\theta \in (0, 1)$, the following inequality follows:

$$\frac{dq_t}{d\alpha_t} < \frac{\beta W'_{t+1} (m^b_t - e_t) m^b_t}{u'(q_t)}.$$

Therefore,

$$\frac{dS^b_t}{d\alpha_t} = u'(q_t) \frac{dq_t}{d\alpha_t} - \beta W_{t+1}' (m^b_t - e_t) m^b_t < 0.$$

Consequently, the buyer could increase $S^b_t$ by reducing $\alpha_t$ below $\tilde{\alpha}_t$, which leads to a strictly cash-constrained bargaining solution, and $e_t = \alpha_t m^b_t$.

As long as the cash constraint is binding, differentiation of (1), using $q_t = f(z^s_t, z^b_t)$, yields:

$$\frac{dS^b_t}{d\alpha_t} \frac{1}{\beta m^b_t} = (u \circ f)_1 W'_{t+1} (e_t) + [(u \circ f)_2 - 1] W'_{t+1} (m^b_t - e_t).$$

(24)

Since $W_{t+1}'$ is a decreasing function and (6) and (7) imply that $(u \circ f)_1$ and $[1 - (u \circ f)_2]$ are positive, the derivative $dS^b_t/d\alpha_t$ is strictly decreasing with respect to $\alpha_t$ as long as the following condition holds:

$$(u \circ f)_{11} [W'_{t+1} (e_t)]^2 + 2 (u \circ f)_{12} W'_{t+1} (e_t) W'_{t+1} (m^b_t - e_t) + (u \circ f)_{22} [W'_{t+1} (m^b_t - e_t)]^2 < 0.$$  

(25)

Such inequality is guaranteed by A1 together with (4) and (5).

Consider an increase in the balances $m^b_t$ owned by the buyer. Depending on $m^b_t$, she may choose to carry all her money to the market ($\alpha_t = 1$) or to leave some at home ($\alpha_t < 1$). As long as $\alpha_t = 1$, $e_t = m^b_t$, so an increase in $m^b_t$ increases $e_t$ by the same amount. If $\alpha_t < 1$, the first-order condition that characterizes the money balances carried to the market is:

$$(u \circ f)_1 W'_{t+1} (e_t) - [1 - (u \circ f)_2] W'_{t+1} (m^b_t - e_t) = 0.$$  

(26)

The argument in the previous paragraph shows that the expression in the left-hand-side of (26) is strictly decreasing with respect to $e_t$. Furthermore, the third inequality in A1, together with (4)
and (5), implies that this expression is increasing with respect to \( m_t^b \). Therefore, an increase in \( m_t^b \) has to weakly raise \( e_t \).

**Proof of Proposition 2:** For each period \( t \), the value function \( W_t \) is going to be constructed as the limit of a sequence of value functions \( W_t^1, W_t^2, \ldots \) for economies similar to the one described above, except for having a finite horizon, which corresponds to the superscripts in the functions. The exogenously given terminal value functions in these finite economies are: \( W^0_{t+T}(m_{t+T}) = w + \phi_{t+T}m_{t+T} \), where \( T \) is the economy’s horizon, \( w = (1 - \beta)^{-1} \{ \pi^s(\hat{q}) [g(\hat{q}) - c(\hat{q})] + (\gamma - 1) \beta^{-1}\hat{b}g(\hat{q}) \} \), and \( \phi_{t+T} \) satisfies (20).

The finite economy with a one-period horizon (\( T = 1 \)) is characterized as follows. The buyer’s utility cost (1) and seller’s utility reward (2) of payment \( e_t \) are identical and linear in \( e_t : z_t^b = z_t^s = \beta\phi_{t+1}e_t \equiv z_t \). With this simplification, the Nash bargaining condition (4) becomes: \( z_t = g(q_t) \). Moreover, Proposition 1 implies that the cash constraint must be binding, so \( e_t = \alpha_t m_t^b \). Therefore, the value function \( V_t^b(m_t^b) \) of a buyer in a search market where fraction \( b_t \) of traders are buyers satisfies:

\[
V_t^b(m_t^b) = \pi^b(b_t) \max_{\alpha_t \in [0,1]} \left[ u \left( g^{-1}(\beta\phi_{t+1}\alpha_t m_t^b) \right) - \beta\phi_{t+1}\alpha_t m_t^b \right] + \beta \left( w + \phi_{t+1}m_t^b \right).
\]  

(27)

Assumption A1 implies that the function \( u(g^{-1}(\cdot)) \) is strictly concave, so \( V_t^b(m_t^b) \) is increasing and concave. Because of the Inada conditions, the slope of \( V_t^b \) at the origin is infinite, and it is \( \beta\phi_{t+1} \) for \( m_t^b \) above a certain threshold, which induces an interior value of \( \alpha_t \). Since \( z_t \) does not depend on \( m_t^s \), conditions (4) and (5) imply that neither the terms of trade nor \( S_t^s \) depend on \( m_t^s \). Consequently, \( V_t^b(m_t^s) \) is affine with slope \( \beta\phi_{t+1} \). In a search market where fraction \( b_t \) of traders are buyers who carry positive money balances (they will), a seller with \( m_t^s = 0 \) obtains a positive trading surplus, but a deviant buyer with \( m_t^b = 0 \) attains a zero trading surplus, so \( V_t^b(0) < V_t^s(0) \).

Moreover, (18) implies that the expected trade surplus of a buyer carrying \( \hat{m}_t^b \) exceeds the trade surplus of a seller carrying the same amount of money, so \( V_t^b(\hat{m}_t^b) > V_t^s(\hat{m}_t^b) \). Combining these two inequalities, \( V_t^b \) and \( V_t^s \) must cross once and only once. Given (10) and (12), the same must be true for \( W_t^b \) and \( W_t^s \) as displayed in Figure 1. Also, the strict concavity of \( U \) and \( V_t^b \) implies the strict concavity of \( W_t^b \).

Because of the Inada conditions, the optimal choice of \( x_t^b, y_t^b \), and \( m_t^b \) must be interior, so this choice must satisfy:

\[
U_1 \left( x_t^b, y_t^b \right) = -U_2 \left( x_t^b, y_t^b \right), \quad \text{and} \\
U_1 \left( x_t^b, y_t^b \right) \phi_t = \beta \left[ \pi^b(b_t)\alpha_t \left[ \frac{u'(q_t)}{g'(q_t)} - 1 \right] + 1 \right] \phi_{t+1},
\]  

(28)

(29)
where $q_t = g^{-1}(\beta \phi_{t+1}\alpha_t m^b_t)$. Analogously, the optimal choice of $x^s_t$, $y^s_t$, and $m^s_t$ must satisfy:

$$U_1(x^s_t, y^s_t) = -U_2(x^s_t, y^s_t), \quad \text{and}$$

$$U_1(x^s_t, y^s_t) \phi_t \geq \beta \phi_{t+1} \quad \text{with equality if } m^s_t > 0. \quad (31)$$

The ability to play fair lotteries implies that as long as $m_t \in [\bar{m}^s_t, \bar{m}^b_t]$, the individual plays a lottery to end up with either the money balances $\bar{m}^s_t$ or $\bar{m}^b_t$ characterized by (see Figure 1):

$$W^s_t(\bar{m}^s_t) = W^b_t(\bar{m}^b_t), \quad \text{and}$$

$$W^s_t(\bar{m}^s_t) = W^b_t(\bar{m}^b_t - \bar{m}^s_t) = W^b_t(\bar{m}^b_t) - W^s_t(\bar{m}^s_t). \quad (33)$$

If the individual wins the lottery and gets $\bar{m}^b_t$, the optimal choice is to be a buyer, otherwise it is best to be a seller. Therefore, all buyers end up with the same money balances $\bar{m}^b_t$. Using the Envelope Theorem, (32) and (33) simplify into

$$U_1(x^s_t, y^s_t) = U_1(x^b_t, y^b_t) \quad (34)$$

$$U_1(x^s_t, y^s_t) \left[ x^b_t - y^b_t + x^s_t - y^s_t + \phi_t (m^b_t - m^s_t) \right] = U(x^s_t, y^s_t) - U(x^s_t, y^s_t) + V^b_t(m^b_t) - V^s_t(m^s_t). \quad (35)$$

For a given pair $\phi_t$ and $b_t$, (11), (13), (16), (27) to (35) describe the optimal behavior of a representative individual. To characterize an equilibrium, these equations must be combined with the stationary condition $\phi_t = \gamma \phi_{t+1}$ and market clearing during the day. Equations (28), (30), (34), and market clearing imply:

$$x^b = y^b = x^s = y^s = x^s. \quad (36)$$

Hence, $W^s_t(\bar{m}^s_t) = W^b_t(\bar{m}^b_t) = U_1(x^s_t, y^s_t)$ and $\phi_t = \phi_t$ follows from the Envelope Theorem. So, the slope of the value function $W^1_t$ in the interval $[0, \bar{m}_t]$ is $\phi_t$. Moreover, since $W^b_t(\bar{m}^b_t) = \pi^b(b_t)dS^b_t/dm^b_t + \beta \phi_{t+1}$ and $\phi_t > \beta \phi_{t+1}$, the derivative $dS^b_t/dm^b_t$ must be positive. Given that $\alpha_t m^b_t = e_t$, $dS^b_t/dm^b_t$ must also be positive, so that condition (16) implies $\alpha_t = 1$. Also, condition (31) implies $m^s_t = 0$ because $\phi_t > \beta \phi_{t+1}$. Using these results, together with (8), (9), (36), and $\beta \phi_{t+1} m^b_t = g(q_t)$, conditions (29) and (35) simplify into:

$$\phi_t = \beta \left[ \pi^b(b_t) \left[ \frac{u'(q_t)}{g'(q_t)} - 1 \right] + 1 \right] \phi_{t+1} \quad (37)$$

$$\pi^s(b_t) \left[ g(q_t) - c(q_t) \right] = \pi^b(b_t) \left[ u(q_t) - g(q_t) \right] - g(q_t) \left( \frac{\phi_t}{\beta \phi_{t+1}} - 1 \right). \quad (38)$$
With the stationary condition $\phi_t = \gamma \phi_{t+1}$, these two equations are identical to (17) and (18), so $b_t = \hat{b}$, $q_t = \hat{q}$, and $m^b_t = \hat{m}^b_t$. Moreover, stationarity combined with the definition of $\phi_{t+1}$ implies (20). Finally, one can check that $W^1_t(0) = w$, so $W^1_t$ is identical to $W^0_t$ in the interval $[0, \bar{m}_t]$. For $m_t > \bar{m}_t$, $W^1_t$ is identical to $W^b_t$, so it is increasing, concave, and flatter than $\phi_t$.

We can now turn our attention to the two-period economy to construct $W^2_t$. The analysis of the one-period economy applies directly to period $t + 1$, so the value function $W^1_{t+1}$ has slope $\phi_{t+1}$ for $m_{t+1} \leq \bar{m}_{t+1}$ and is flatter than $\phi_{t+1}$ for $m_{t+1} > \bar{m}_{t+1}$. At the night of period $t$, the characterization of Nash bargaining is identical to the one-period economy as long as $m^b_t \in [0, \bar{m}_{t+1}]$. This interval includes $\hat{m}_t$ because $\bar{m}_{t+1} = \gamma \bar{m}_t = \gamma (\hat{m}^b_t - \hat{r}_t) = \hat{m}^b_t[\gamma(1-\hat{b}) + \hat{b}] > \hat{m}^b_t$. If $m^b_t > \bar{m}_{t+1}$, $W^1_{t+1}$ is not linear, which prevents an explicit solution to the Nash bargaining problem. However, despite this difficulty, Lemma 2 proves that in an environment consistent with the equilibrium described in Proposition 2, the value function $V^b_t$ is flatter than $\phi_t$ for $m^b_t > \bar{m}_{t+1}$. Consequently, the Envelope Theorem implies that $W^b_t$ must be flatter than $\phi_t$ for $m_t > \bar{m}_t$. Therefore, for $m_t \in [0, \bar{m}_t]$, the optimal choices in period $t$ are the same in the one- and in the two-period economies, so $W^2_t (m_t) = W^1_t (m_t)$. For $m_t > \bar{m}_t$, the optimal choices in the two economies differ, but $W^2_t (m_t)$ is still increasing, concave, and flatter than $\phi_t$.

The same argument can be applied recursively to all the finite economies, so that all the functions in the sequence $W^1_t, W^2_t, \ldots$ are increasing and concave, have slope $\phi_t$ in $[0, \bar{m}_t]$, and are flatter than $\phi_t$ for $m_t > \bar{m}_t$. In the limit, the value function $W_t$ for the infinite horizon economy must also be increasing, concave, have slope $\phi_t$ in $[0, \bar{m}_t]$, and not be steeper than $\phi_t$ elsewhere. Moreover, Lemma 2 implies that $W_t$ must be flatter $\phi_t$ for $m_t > \bar{m}_t$. In conclusion, as long as the initial money holdings are in $[0, \bar{m}_t]$, the infinite-horizon economy has the same equilibrium in period $t$ as all the finite horizon economies, which proves statements 1 to 3 in Proposition 2.

**Lemma 2:** Consider a decentralized market where, except for a possible deviator, no sellers carry money balances, all buyers carry $\hat{m}^b_t$ (as defined in Proposition 1), and all traders share the same concave terminal value function $W_{t+1}(m_{t+1})$, which first derivative is equal to $\phi_{t+1}$ for $m_{t+1} \leq \hat{m}^b_t$, and belongs to the interval $(0, \phi_{t+1}]$ for $m_{t+1} > \hat{m}^b_t$. Assume $\phi_t \geq \beta \phi_{t+1}$. A deviating individual who starts the night of period $t$ with money balances in excess of $\hat{m}^b_t$ will choose to be a buyer: $V^b_t (m) \geq V^b_t (m)$ for all $m > \hat{m}^b_t$. Furthermore, the value function of this individual is positive and flatter than $\phi_t$: $0 < V^b_t (m^{b}_t) < \phi_t$.

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4 A supplementary Appendix available on request from the author shows that $W_t$ is the limit of a sequence of value functions with these properties.
Consider first the case buyer may choose to carry all her money to the market and with a terminal value function with a constant slope equal to \( \phi_{t+1} \). Since \( V^b_t(m) - V^b_t(m) = \pi^t_S^b(m) - \pi^t_S^b(m) \), the argument in the construction of Figure 1 implies that \( \pi^t_S^b(m) > \pi^t_S^b(m) \). Therefore, to prove \( V^b_t(m) > V^b_t(m) \) for all \( m \geq \hat{m}^b_t \), it is sufficient to show that \( S^b_t(m) \geq \hat{S}^b_t(\hat{m}^b_t) \) and \( S^b_t(m) < \hat{S}^b_t(\hat{m}^b_t) \) for all \( m^b, m^s \geq \hat{m}^b_t \).

Proof: Let \( \hat{S}^b_t(m) \) and \( \hat{S}^s_t(m) \) be respectively the trading surpluses attained by an individual starting the night with \( m \) balances and with a terminal value function with a constant slope equal to \( \phi_{t+1} \). Since \( V^b_t(m) - V^b_t(m) = \pi^t_S^b(m) - \pi^t_S^b(m) \), the argument in the construction of Figure 1 implies that \( \pi^t_S^b(m) > \pi^t_S^b(m) \). Therefore, to prove \( V^b_t(m) > V^b_t(m) \) for all \( m \geq \hat{m}^b_t \), it is sufficient to show that \( S^b_t(m) \geq \hat{S}^b_t(\hat{m}^b_t) \) and \( S^b_t(m) < \hat{S}^b_t(\hat{m}^b_t) \) for all \( m^b, m^s \geq \hat{m}^b_t \).

Differentiating (1) with respect to \( m_t^b \) yields:

\[
\frac{dS_t^b}{dm_t^b} = [1 - (u \circ f)] \beta \left[ W_{t+1}^t (m_t^b - \varepsilon_t) - W_{t+1}^t (m_t^b) \right] + \frac{dS_t^b}{d\varepsilon_t} \frac{de_t}{dm_t^b} \geq [1 - (u \circ f)] \beta \left[ W_{t+1}^t (m_t^b - \varepsilon_t) - W_{t+1}^t (m_t^b) \right] \geq 0, \tag{39}
\]

where the first inequality follows from Proposition 1 and the Kuhn-Tucker condition (16), and the second inequality follows from \((u \circ f) < 1 \) - see (7) - and the concavity of \( W_{t+1} \). Since \( S_t^b \) is an increasing function of \( m_t^b \), \( S_t^b(m_t^b) \geq S_t^b(\hat{m}^b_t) \) for all \( m_t^b \geq \hat{m}^b_t \). Moreover, because the interval of \( W_{t+1} \) used to evaluate \( S_t^b(\hat{m}^b_t) \) has a constant slope \( \phi_{t+1} \), \( S_t^b(\hat{m}^b_t) = \hat{S}_t^b(\hat{m}^b_t) \), so that \( S_t^b(m_t^b) \geq \hat{S}_t^b(\hat{m}^b_t) \) for all \( m_t^b \geq \hat{m}^b_t \).

Differentiating (2) with respect to \( z_t^s \), (6) implies that \( S_t^s \) is an increasing function of \( z_t^s \):

\[
\frac{dS_t^s}{dz_t^s} = 1 - c'f_t \geq 0. \tag{40}
\]

Therefore, since \( z_t^s \leq \beta \phi_{t+1} \varepsilon_t \), we have \( S_t^s(m_t^s) \leq \beta \phi_{t+1} \varepsilon_t - c \left( f \left( \beta \phi_{t+1} \varepsilon_t, \beta \phi_{t+1} \varepsilon_t \right) \right) = \hat{S}_t^s (\beta \phi_{t+1} \varepsilon_t) \). Moreover, if \( W_{t+1} \) has a constant slope, then

\[
\frac{d}{d\varepsilon_t} \hat{S}_t^s (\beta \phi_{t+1} \varepsilon_t) = (1 - c'f_t - c'f_2) \beta \phi_{t+1} \geq (1 - c'f_t - u'f_2) \beta \phi_{t+1} > 0, \tag{41}
\]

where the first inequality follows from \( u' \geq c' \), as implied by (4) and (5), and the second inequality follows from \( c'f_t + u'f_2 < 1 \) - see (6) and (7). Therefore, since \( \varepsilon_t \) is at most \( \hat{m}^b_t \) (the money carried by the potential buyers), \( S_t^s(m_t^b) \leq \hat{S}_t^s (\hat{m}^b_t) \) for all \( m_t^b \), including all \( m_t^b \geq \hat{m}^b_t \).

Because of Proposition 1, a (female) buyer who has money balances \( m_t^b \) will never choose to carry the money balances that lead to an interior Nash bargaining solution. Depending on \( m_t^b \), the buyer may choose to carry all her money to the market \((\alpha_t = 1)\) or to leave some at home \((\alpha_t < 1)\). Consider first the case \( \alpha_t = 1 \). Since sellers carry no balances and the buyer is spending all of hers, \( z_t^b = z_t^s = \beta \left[ W_{t+1}^t (m_t^b) - W_{t+1}^t (0) \right] \equiv z_t^s \). Therefore,

\[
V_t^b(m_t^b) = \pi^b(u \circ f)(z_t^b, z_t^s) + \left( 1 - \pi^b \right) \beta W_{t+1}^t (m_t^b) + \pi^b \beta W_{t+1}^t (0). \tag{43}
\]
Assumption A1 implies that \( u \circ f \) is a strictly concave function of \( z_t \), and the concavity of \( W_{t+1} \) implies that \( z_t \) is a concave function of \( m_t^b \). Therefore, \((u \circ f)(z_t, z_t)\) is a strictly concave function of \( m_t^b \), so \( V_t^b \) is locally strictly concave for values of \( m_t^b \) that lead to \( \alpha_t = 1 \). Moreover, for these values of \( m_t^b \) the slope of \( V_t^b \) is positive because the following statements hold: The function \((u \circ f)\) is increasing in \( z_t \), \( z_t \) is increasing in \( m_t^b \), and \( W_{t+1}'(m_t^b) > 0 \). Consider now the case \( \alpha_t < 1 \). Since the choice of \( \alpha_t \) is interior, the Envelope Theorem implies:

\[
\frac{dV_t^b}{dm_t^b} = p^b [1 - (u \circ f)_2] \beta W_{t+1}' \left( m_t^b - e_t \right) + \{1 - \pi_t^b [1 - (u \circ f)_2]\} \beta W_{t+1}'(m_t^b) \tag{44}
\]

where the first inequality follows from the concavity of \( W_{t+1} \) and \((u \circ f)_2 \in (0, 1)\) -see (7)-, and the second and third inequalities follow directly from the assumptions of the Lemma. Also, (44) implies that the slope of \( V_t^b \) is positive because \( W_{t+1}' > 0 \). In summary, \( V_t^b \) is upward sloping, and either it is strictly concave \((\alpha_t = 1)\) or its slope is strictly less than \( \phi_t \) \((\alpha_t < 1)\). Moreover, \( V_t^b \) is strictly concave at a neighborhood of \( \hat{m}_t^b \) where \( \alpha_t = 1 \) and \( V_t^{b*}(\hat{m}_t^b) = \phi_t \) (Envelope Theorem applied to (27) using (37)). Therefore, \( 0 < V_t^{b*}(m_t^b) < \phi_t \) for \( m_t^b > \hat{m}_t^b \).
References


Figure 1

Value Function

Utility