

# Reforming the direct–indirect tax mix<sup>1</sup>

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## Abstract

This paper provides a new framework for evaluating the welfare effects of tax reforms. It is shown that tax reforms are welfare improving if and only if they satisfy the following intuitive property: on average, consumer prices fall for commodities with high marginal excess burdens. The rule is then applied to analyze a shift from indirect to direct taxation. The welfare impact of such reforms can be decomposed into two effects: (i) the increase in welfare associated with substitution among taxed commodities, and (ii) the loss in welfare associated with substitution between commodities and leisure. On balance, direct tax reforms are desirable when inter-commodity substitution effects are large relative to commodity-leisure substitution effects. The analysis allows us to reconcile the apparently conflicting results of the tax reform and optimal taxation literatures.

*JEL Classification:* H21

*Keywords:* uniform commodity taxation, tax reform, excess burden

# 1 Introduction

Recently, proponents of flat-rate taxation of labour income have enjoyed renewed attention in policy debates in the U.S. and elsewhere. One argument common among policy analysts is that deductions and exemptions in the income tax system act like a system of differentiated commodity taxes. Flat tax reforms move toward taxing all transactions at a common rate, and consequently are held to reduce distortions in economic behaviour and the associated deadweight costs.<sup>1</sup> Optimal tax theory has long made clear, however, that this view may be misguided. A system of direct income taxation creates a tax wedge in the relative prices of taxed commodities and leisure, and this “one big distortion” can in general be less advantageous than the many, smaller distortions of a differentiated, indirect commodity tax system.

This paper analyzes the relationship between these arguments and develops a new heuristic for analyzing flat-rate income taxation. I provide a simple formula for calculating the impact of flat tax reforms on consumer welfare, which consists of the welfare gain from substitution among taxed commodities, and the welfare loss from substitution between commodities and leisure. Thus it is shown that flat tax reforms are more desirable the greater the degree of substitution possible among taxed commodities, and the less the degree of substitution between leisure and taxed commodities.<sup>2</sup> Informational requirements of our approach are parsimonious, moreover, requiring information only on initial tax rates and market demand elasticities.

This paper extends the work of Hatta (1986), who examined the welfare impact of a particular tax reform moving in the direction of uniform taxation, which is related to the flat tax reforms studied in this paper. Hatta argued such reforms improve welfare when, somewhat loosely, commodities that are pairwise net substitutes tend to be taxed at very different rates. In contrast, the welfare decomposition derived in this paper relies on inherent substitutability (i.e. Slutsky negativity) of expenditure-minimizing demands, rather than a restriction on initial tax rates and pairwise substitution effects. My paper also makes clear that it is the magnitude of such substitution effects relative to commodity–leisure substitution effects that is relevant to the analysis.

Moreover, the approach adopted here is quite general, and may be applied to analyze the efficiency impact of any small tax reform. Section 3 develops a number of useful “rules of thumb”

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<sup>1</sup>Another issue commonly addressed in flat tax proposals, but not addressed here, is tax rate progressivity. The analysis of this paper can however be applied to a number of related issues in the theory of taxation, including arguments for broad-based sales, value-added, and consumption taxes, and analysis of the effects of exemptions and credits within the income tax system.

<sup>2</sup>To avoid a misunderstanding common in the literature, note there is nothing special about leisure in the analysis, unless it is chosen arbitrarily to be the numeraire. The planner could choose to tax all commodities including labour at a uniform rate relative to, say, the tax on shoelaces. In this case, it is elasticities of shoelace demand that are relevant to the analysis.

of this kind. In particular, it is shown that the efficiency impact of any small tax reform can be identified with its impact on a compensated government revenue function (Proposition 1). This impact is shown to be positive if and only if the reform cuts tax rates on goods which, on average, have high marginal excess burdens (Proposition 2). These general results are then applied to flat tax reforms in Section 4, where the main “substitutability” decomposition (Proposition 3, described above) is established.

As well as providing intuitive rules for evaluating direct–indirect tax reforms, the analysis establishes a link between the earlier literatures on optimal taxation and on piecemeal tax reform, and helps to reconcile their sometimes disparate conclusions. Most research on the optimality of uniform taxation has examined the role of substitution between commodities and leisure in the analysis, to the exclusion of substitution effects among taxed commodities. Thus Sadka (1977) showed that uniform commodity taxation is optimal if and only if compensated wage elasticities of demand for all commodities are equal at the optimum.<sup>3</sup> In contrast, Hatta’s result stresses substitution effects among taxed commodities, but largely ignores commodity–leisure substitution. Proposition 4 helps reconcile these approaches. It is shown that Sadka’s preference restriction implies that a small tax reform always improves welfare, as gains from commodity substitution dominate the leisure distortion. Conversely, the welfare gains of flat tax reform are “robust,” holding for any initial tax rates, only if consumer preferences satisfy Sadka’s restriction.

## 2 The model

The model is standard. There is a single consumer and  $n+1$  commodities. Denoting the consumer’s net demands by  $X = (x^0, x^1, \dots, x^n)$  and consumer prices by  $Q = (q^0, q^1, \dots, q^n)$ , the budget constraint is  $Q'X \leq 0$ . Consumer preferences are represented by the utility function  $U(X)$ . The associated indirect utility and expenditure functions are

$$v(Q) = \max_X \{U(X) : Q'X \leq 0\} \tag{1}$$

$$e(Q, u) = \min_X \{Q'X : U(X) \geq u\}. \tag{2}$$

With the assumption that  $U$  is strictly concave, increasing and  $C^2$ , the compensated demands attaining the minimum in (2) are single-valued, continuously differentiable in  $Q$ , and satisfy  $X(Q, u) =$

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<sup>3</sup>This result builds on Corlett and Hague (1953), who established that, beginning from a uniform tax system, increasing a tax on a good more complementary with leisure would increase consumer welfare. More recently, Besley and Jewitt (1995) and others have extended this approach, establishing separability restrictions on preferences that are consistent with the Sadka condition and which are sufficient for optimality of uniform taxes.

$e_Q(Q, u)$  (Shephard's lemma).<sup>4</sup>

Production occurs under constant returns to scale, so that competitive profit-maximization by firms leads to fixed producer prices  $P = (1, p^1, \dots, p^m)$ , where commodity 0 serves as the numeraire for the economy. Government revenue is derived from linear taxes on transactions at rate  $t^i = q^i - p^i$ . There is no lump-sum taxation. Since consumer demands are degree-zero homogeneous in consumer prices, it is possible to normalize taxes so that  $t^0 = 0$ . Define the extended producer price and tax vectors  $P = (1, p)$  and  $T = Q - P$ , and let  $R(Q, u) = (Q - P)'X(Q, u)$  denote government revenue. (In what follows, it will also be convenient to refer to  $x, q$  and  $t$  as the demand, price and tax vectors with the 0th element deleted, e.g.  $x = (x^1, \dots, x^n)$ .) Government consumes an exogenously given  $G$  units of the numeraire, so that the aggregate production possibility constraint is  $P'X + G = 0$ . A tax policy  $(Q, u)$  is then feasible given revenue requirements if and only if

$$e(Q, u) = 0 \tag{3}$$

$$R(Q, u) = G. \tag{4}$$

Note that (3) and (4) imply that the production constraint holds identically.

### 3 A framework for evaluating tax reforms

A *tax reform* from initial prices  $Q$  is defined as a differentiable path  $Q(\theta)$  where  $\theta$  is a scalar parameter and  $Q(0) = Q$ . The primary subject of the paper is tax reforms moving toward flat-rate taxation of labour income, or equivalently from differentiated indirect taxation of commodity demands to direct taxation of income. The preliminary results in this section of the paper, however, apply to any feasible tax reform.

Consider a differential change in prices of taxed commodities  $dq$ , accompanied by a change in the price of the numeraire that balances the government budget. (Since the choice of numeraire is arbitrary, any small tax reform can be represented in this fashion.) To see how much the price of the numeraire must adjust, define  $q^0 = \phi(q, u)$  as the price which allows the consumer to attain utility  $u$  at prices  $q$ , so that

$$e(\phi(q, u), q, u) \equiv 0 \tag{5}$$

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<sup>4</sup>Throughout the paper, superscripts represent elements of a vector, whereas subscripts represent partial derivatives. Where the subscript is a vector, the result is a column vector of partial derivatives; hence  $e_Q = (\partial e / \partial q^0, \dots, \partial e / \partial q^n)'$ . Similarly,  $X_i$  denotes  $(\partial x^0 / \partial q^i, \dots, \partial x^n / \partial q^i)'$  and  $X_Q$  is the Slutsky matrix.

and define *compensated government revenue* as

$$R^c(q, u) = R(\phi(q, u), q, u). \quad (6)$$

A tax policy  $q$  is then feasible if and only if

$$R^c(q, u) = G. \quad (7)$$

Under weak conditions,  $R^c$  may be interpreted as a compensation function, measuring the consumer's willingness to pay for a tax reform. This simplifies the welfare evaluation of reforms considerably. To see this, note that when tax rates are changed income effects in commodity demands also lead to changes in government revenue. The total marginal effect of  $t^i$  on revenue is

$$\frac{d}{dq^i} R(Q, v(Q)) = R_i(Q, u) - \frac{R_u(Q, u)}{e_u(Q, u)} x^i(Q, u) \quad (8)$$

since (3) implies  $v_i = -x^i/e_u$ . When  $(dR/dq^i)/x^i > 0$ , the tax on commodity  $i$  is said to be *revenue increasing*.<sup>5</sup> With this definition, Proposition 1 establishes a condition under which the welfare effects of any feasible tax reform  $dq$  may be identified with its effects on compensated revenue.

**Proposition 1** *Suppose that  $t^0$  is revenue increasing. Then for any feasible tax reform  $dq$ ,  $du \geq 0$  if and only if*

$$dR^c = R_q^c dq \geq 0. \quad (9)$$

*Proof.* See appendix.

In seeking to characterize welfare-improving tax reforms, it is possible to ignore income effects and search for cases in which compensated revenue rises.

### 3.1 Tax reform and excess burden

Proposition 1 showed that welfare effects of tax reforms depend solely on their effects on an appropriately defined compensated tax revenue function. Revenue effects of a reform depend both on the tax rates initially levied and on the responsiveness of tax bases to rate changes. This section

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<sup>5</sup>This definition subsumes the cases of goods which are demanded and supplied by the consumer. For the latter group,  $x^i < 0$  and an increase in the tax rate represents an increase in the *subsidy* applied to the good, since  $q^i = 1 + t^i$  is the consumer price. For such goods, the tax is revenue increasing when  $dR/dq^i < 0$ .

provides a joint condition on initial tax rates and elasticities that is necessary and sufficient for a tax reform to be welfare increasing, and relates it to a common measure of tax distortions.

Define the *marginal excess burden* (MEB) of tax base  $i$  as the net marginal lump-sum compensation required to leave consumer utility unchanged, when one dollar of additional revenue is raised from the tax base, net of the additional revenue itself. The marginal compensating variation for an increase in  $t^i$  is  $e_i(Q, u)$  and, because the tax change is compensated, the tax rate change required to generate one dollar in revenue is  $1/R_i(Q, u)$ . The marginal excess burden is therefore

$$\text{MEB}^i = \frac{e_i(Q, u)}{R_i(Q, u)} - 1. \quad (10)$$

In what follows, however, it is more convenient to work with a normalization of  $\text{MEB}^i$ ,

$$\mu^i = \frac{\text{MEB}^i}{1 + \text{MEB}^i} = 1 - \frac{R_i}{e_i}. \quad (11)$$

For what follows, it is useful to observe that the normalized excess burdens satisfy an adding-up condition. Define  $L = \sum_{i>0} q^i x^i$  as expenditure on taxed commodities, and  $w^i = q^i x^i / L$  as the expenditure share of commodity  $i$ .

**Lemma 1**

$$\sum_{i>0} w^i \mu^i = \mu^0. \quad (12)$$

*Proof.* See appendix.

The lemma states that normalized excess burden of the numeraire commodity is just the average of normalized excess burdens for taxed commodities, weighted by initial expenditure shares. To derive the implications of Proposition 1 for excess burdens, observe that the marginal effect of increasing  $t^i$  on compensated revenue is, differentiating (5) and (6),

$$\begin{aligned} R_i^c &= R_i - \frac{e_i}{e_0} R_0 \\ &= x^i (\mu^0 - \mu^i), \end{aligned}$$

where (11) and Shephard's lemma has been applied in deriving the second equality. The total

differential of compensated revenue can then be expressed using (8) as

$$\begin{aligned}
 dR^c &= \sum_{i=1}^n x^i (\mu^0 - \mu^i) dq^i \\
 &= -L \sum_{i=1}^n w^i (\mu^i - \mu^0) d \log q^i.
 \end{aligned}
 \tag{13}$$

Consequently, Lemma 1 implies the summation in (13) equals minus the covariance of  $\mu^i$  and  $d \log q^i$ , weighted by initial expenditure shares, say  $\rho(\mu, d \log q)$ . Proposition 1 then yields the following corollary.

**Proposition 2** *Suppose that  $t^0$  is revenue increasing. Then a tax reform  $dq$  is welfare increasing if and only if  $\rho(\mu, d \log q) \leq 0$ .*

Proposition 2 indicates that a tax reform increases welfare when, on average, tax rates are reduced on bases with normalized excess burdens. This is a natural conclusion, and the condition is easily checked in applications.

## 4 Direct vs. indirect taxation

Consider a reform that moves from indirect to direct taxation of labour income in the following sense. Let the numeraire commodity be leisure, so that  $x^0 < 0$  measures labour supply. Beginning from any initial commodity tax system, consumer prices are “shrunk” proportionately toward producer prices, while the tax on labour income is increased to maintain revenue neutrality. This is a natural notion of a comprehensive reform towards direct taxation, inasmuch as all ad valorem tax rates are reduced proportionately and replaced by a tax on income at a common rate. For such a reform,

$$d \log q = -\tau d\theta. \tag{14}$$

where  $\tau^i = t^i/q^i$  is the initial ad valorem tax rate. Then Proposition 2 states  $du \geq 0$  if and only if  $\rho(\mu, \tau) \geq 0$ . The result gives a clear prescription for direct tax reform. When (and only when) high tax rates are levied on bases that are on average more highly distorted, welfare is improved by a movement toward direct taxation.

The next section elucidates what properties of consumer preferences are conducive to welfare gains from a movement toward direct taxation.

#### 4.1 Substitution effects and the desirability of reform

Define  $z(Q, u) = -x(Q, u)/x^0(Q, u)$  as the vector of compensated demands per unit labour supplied, and note that its Jacobian matrix is

$$z_q = -\frac{1}{x^0} \left( x_q - \frac{1}{x^0} x x'_0 \right). \quad (15)$$

Expanding (13) with (11) yields

$$\begin{aligned} dR^c &= - \sum_{i=1}^n \sum_{j=1}^n t^i t^j \left( x_j^i - \frac{x^i}{x^0} x_j^0 \right) d\theta \\ &= x^0 t' z_q t d\theta. \end{aligned} \quad (16)$$

Recall that  $x^0 < 0$ , so that Proposition 1 implies the following.

**Proposition 3** *Suppose that the labour tax is revenue increasing. Then a direct tax reform improves welfare if and only if*

$$t' z_q t \leq 0$$

where  $z_q(Q, u)$  is evaluated at  $u = v(Q)$ .

The result indicates that direct tax reform improves welfare as long as substitution possibilities among taxed commodities are large relative to substitution possibilities between leisure and commodities. When this is so, reducing distortions among taxed commodities results in efficiency gains which more than offset the effects of the increased distortion in the relative price of consumption to leisure. This is seen most clearly by expanding (16) to obtain

$$\begin{aligned} dR^c &= - \left( t' x_q t - \frac{1}{x^0} (t' x) (x'_0 t) \right) d\theta \\ &= -t' x_q t d\theta - \mu^0 G d\theta \end{aligned} \quad (17)$$

where (10) has been used to derive the second equality. The first term in the decomposition represents the inter-commodity substitution effects of the reform. Since  $x_q$ , a minor of the Slutsky matrix, is negative semi-definite, this effect on welfare is unambiguously positive, as the tax reform leads the consumer to substitute toward goods with high initial tax rates, raising compensated revenue. The second term in the decomposition represents the commodity–leisure substitution effects of the reform. Assuming  $\mu^0 > 0$ , so that the marginal excess burden of labour taxation

is positive, this effect offsets the gains from reducing commodity tax distortions. Thus when substitution possibilities among commodities are large relative to commodity–leisure substitution possibilities, the net effect is positive, and welfare rises as a result of a direct tax reform.

## 4.2 Optimal uniform taxation: A reconciliation

When  $z_q$  is negative semi-definite at initial prices  $Q$ , Proposition 3 implies that direct tax reform is desirable for any initial tax rates and producer prices. More generally, when substitution possibilities among taxed commodities are large relative to commodity–leisure substitution possibilities, a direct tax reform is a “robust” policy recommendation, in the sense that it improves welfare for a large set of initial tax systems.

This logic may be extended to derive a sufficient condition for the optimality of flat-rate income taxation. Suppose that  $z_q$  is globally negative semi-definite and  $t^0$  is globally revenue increasing. Then Proposition 3 indicates that direct tax reform is (weakly) welfare improving beginning from any initial tax system. It follows that the set of tax optima includes a flat-rate income tax system when our sufficient conditions for local welfare improvements hold globally.<sup>6</sup>

Sadka (1977) showed that flat-rate income taxation is optimal when wage elasticities of compensated demand for all commodities are globally equal, i.e. when there exists a scalar-valued function  $\alpha(Q)$  such that, for all  $Q$  and  $u = v(Q)$ ,

$$\frac{\partial x(Q, u)}{\partial q^0} \equiv x_0(Q, u) = \alpha(Q)x(Q, u). \quad (18)$$

Our analysis offers a further interpretation of the equal elasticities condition: it can be shown to be necessary and sufficient for  $z_q$  to be negative semi-definite, and hence for the global dominance of commodity substitution effects over commodity–leisure substitution effects.

**Proposition 4**  $z_q(Q, u)$  is negative semi-definite at  $(Q, u)$ , where  $u = v(Q)$ , if and only if the Sadka equal-elasticities condition (18) holds at  $(Q, u)$ .

*Proof.* See appendix.

When the equal-elasticities condition holds, an increase in labour taxation cannot distort relative commodity demands, so that the benefits of reducing distortions among taxed commodities must dominate. Conversely, for tax systems sufficiently close to uniformity, the positive commodity

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<sup>6</sup>Proof: Suppose any  $q^*$  is an optimal tax system. If the conditions of Proposition 3 hold globally, then there exists a path from  $q^*$  in  $\mathbf{R}_+^n$  over which utility is non-decreasing, and which converges to a uniform tax system  $\bar{q} = \gamma \mathbf{1}$ . Hence  $\bar{q}$  is optimal.

substitution effects of the reform are negligible, and distortions in leisure demand must dominate. This argument parallels Corlett and Hague (1953).<sup>7</sup>

### 4.3 Previous research

The objective of this paper—to relate welfare effects of direct tax reforms to substitution effects in consumption—is not new to the literature. Hatta (1986) examines reforms in which, beginning from any initial commodity tax vector, the highest tax rate ( $t^n$ , say) is decreased, while the lowest tax rate ( $t^1$ ) is increased to maintain revenue neutrality. For such a reform, (9) simplifies to

$$dR^c = (\mu^n - \mu^1) d\theta$$

and Proposition 1 implies welfare rises if and only if the excess burden of the high-tax base exceeds that of the low-tax base.

Hatta defines good  $i$  to be a substitute for the *compound* of other goods if

$$\sum_{j \neq i} p^j |t^j - t^i| x_j^i > 0,$$

i.e. if increasing prices of other goods in proportion to the distance between tax rates ( $t^i, t^j$ ) results in an increase in the compensated demand for good  $i$ . Furthermore, define  $\eta^i = \partial \log x^i / \partial \log q^0$  as the compensated wage elasticity of demand for  $i$ . Then Hatta shows (Theorem 2, p. 106) that decreasing  $t^n$  and increasing  $t^1$  to maintain revenue neutrality is welfare improving if: (i) good 1 is a substitute for the compound of other goods and so is good  $n$ ; and (ii)  $\tau^n \eta^n \geq \tau^1 \eta^1$ . Konishi (1989) establishes a parallel result for the notion of flat tax reform given in (14) and studied in this paper.

Hatta’s result leaves questions unanswered, however.<sup>8</sup> While Hatta stresses the role of substitution among commodities in condition (i), the result depends also on wage elasticities through condition (ii), and the relative importance of the two is unexplored. Similarly, little attempt has been made to relate the conditions to those suggested by the optimal tax literature.<sup>9</sup> Furthermore, compound elasticities are neither easily measured nor easily interpreted, so that verifying condition (i) appears to be difficult. (Note that goods are compound substitutes for all tax rates only if all

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<sup>7</sup>Note that Proposition 4 may also be interpreted as demonstrating that the Sadka condition holds if and only if the Jacobian of wage-compensated demand functions  $x(\phi(q, u), q, u)$  is negative semi-definite. Similarly, Besley and Jewitt (1995) have shown the Sadka condition holds if and only if wage-compensated labour supply  $x^0(\phi(q, u), q, u)$  is independent of prices. The equivalence of these two conditions can be shown directly.

<sup>8</sup>Gordon (1991) provides a more extensive discussion of the Hatta result along these lines.

<sup>9</sup>However, Konishi (1989) proves the “if” part of Proposition 4, although the technique of proof is quite different.

goods are pairwise net substitutes, which is an unreasonably strong condition.)

#### 4.4 Reform with many consumers

One of the most hotly debated aspects of flat tax reform is its distributional consequences, a consideration which has been absent from this analysis of a single-consumer economy.

The approach adopted here can be extended easily to analyze a many-consumer economy, to determine when *potential* Pareto improvements result from direct tax reforms, using the Scitovsky expenditure function

$$E(Q, u^1, \dots, u^H) = \sum_{h=1}^H e^h(Q, u^h)$$

in place of the individual expenditure function, and interpreting results in terms of elasticities of aggregate demand.<sup>10</sup>

When lump-sum compensation is possible for those who lose from the reform, these potential Pareto improvements can be made actual Pareto improvements. The amount of compensation required can in fact be computed directly from this analysis, since a taxpayer's marginal compensating variation for the reform can be shown to satisfy

$$dC^h = \frac{L^h}{H\bar{L}} \frac{dR^c}{1 + \mu^0} + L^{h0} \left( \frac{t'x^h}{L^h} - \frac{t'\bar{x}}{\bar{L}} \right) d\theta \quad (19)$$

where  $x^h$  is the vector of commodity demands and  $L^h$  is labour income for consumer  $h$ , and bars over variables indicate population averages. The first term in (19) is the share of consumer  $h$  in the aggregate efficiency change of the reform, where shares equal consumers' shares in aggregate labour income. The second term is the "transfer effect" of the reform. Since  $t'x^h/L^h$  is a consumer's average propensity to pay tax out of labour income, this term is proportional to the deviation of the average effective tax rate from the mean and to labour income. Flat tax reform results in transfers away from consumers with average effective tax rates below the mean in the economy. Thus lump-sum compensation, if feasible, takes a simple form for flat tax reforms.

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<sup>10</sup>One caveat is required, however. Guaranteeing that aggregate demand functions have a particular set of consistency properties may be difficult, because aggregate demands do not in general resemble those of a representative consumer. See Blackorby et al. (1990) for an analysis of this problem in the context of uniform optimal taxation.

## 5 Conclusion

This paper has developed a simple and intuitive framework for assessing the welfare effects of small tax reforms. Under weak conditions, the a reform causes welfare to rise in a single-consumer economy when it leads to an increase in compensated government revenue. This approach yields a common-sense heuristic for evaluating reforms: welfare rises when, on average, tax rates fall on bases with high marginal excess burdens.

As an application of the approach, I examined the case for tax reforms moving from indirect commodity taxation to direct, flat-rate taxation of labour income. It was shown the welfare impact of such reforms can be decomposed into two effects, the increase in welfare associated with substitution among taxed commodities and the offsetting decrease in welfare associated with substitution between commodities and leisure. On balance, such direct tax reforms are therefore welfare improving when substitution effects in the demand for taxed commodities are sufficiently large, and substitution effects between leisure and taxed commodities sufficiently small. Our sufficient conditions for flat tax reform to be desirable are consistent with, but more intuitive than, those derived from the optimal tax literature.

## Appendix

*Proof of Proposition 1.* First it is shown that  $R_u^c < 0$  if and only if  $t^0$  is revenue-increasing. To see this, differentiate the identity

$$R^c(q, v(Q)) \equiv R(Q, v(Q))$$

with respect to  $q^0$  to obtain

$$R_u^c(q, v(Q))v_0(Q) = \frac{dR(Q, v(Q))}{dq^0}.$$

Since  $v_0 = -x^0/e_u$  and  $(dR/dq^0)/x^0 > 0$  for a revenue-increasing tax, the result follows.

To prove the Proposition, a tax reform is feasible when it satisfies (7). Thus

$$du = -\frac{1}{R_u^c} (R_q^c dq).$$

Hence  $du \geq 0$  if and only if  $t^0$  is revenue increasing.  $\square$

*Proof of Lemma 1.* To verify (12), expand (11) to obtain  $\mu^i = -t'x_i/x^i$ , so that

$$\begin{aligned}\sum_{i>0} w^i \mu^i &= -\frac{1}{L} \sum_{i>0} q^i \sum_j t^j x_i^j \\ &= \frac{1}{L} \sum_j t^j q^0 x_0^j \\ &= \frac{q^0 x^0}{L} \frac{\sum_j t^j x_0^j}{x^0} \\ &= \mu^0\end{aligned}$$

where zero-degree homogeneity of demands ( $q^0 x_0^j + q^j x_q^j = 0$ ) has been applied in deriving the second equality, and the budget constraint in deriving the third.  $\square$

*Proof of Proposition 4.*

(i) Sufficiency: For any square matrix  $A$ , let  $A^{(m)}$  denote the  $m \times m$  leading principal minor of  $A$ . Suppose that (18) holds at  $Q$ . Homogeneity of demands then implies  $x_0^0 = \alpha x^0$ , and (15) reduces to

$$z_q = -\frac{1}{x^0} \left( x_q - \frac{1}{x_0^0} x_0 x_0' \right)$$

so that its minor determinants are

$$\det z_q^{(m)} = \left( -\frac{1}{x^0} \right)^m \det \left( x_q^{(m)} - \frac{1}{x_0^0} (x_0 x_0')^{(m)} \right)$$

The Slutsky matrix is

$$X_Q = \begin{bmatrix} x_0^0 & x_0' \\ x_0 & x_q \end{bmatrix}.$$

Applying a standard result on determinants of partitioned matrices (e.g., Murata, 1977),

$$\det X_Q^{(m+1)} = x_0^0 \det \left( x_q^{(m)} - \frac{1}{x_0^0} (x_0 x_0')^{(m)} \right) = x_0^0 (-x^0)^m \det z_q^{(m)}.$$

Since  $x_0^0 < 0$ ,

$$(-1)^{m+1} \det X_Q^{(m+1)} \geq 0 \implies (-1)^m \det z_q^{(m)} \geq 0.$$

Hence  $z_q$  is negative semi-definite because  $X_Q$  is.

(ii) Necessity: Let  $\eta^i = \partial \log x^i / \partial \log q^0$  denote the wage elasticity of demand for  $i$ . Suppose (18) does not hold, so that there exists a commodity  $i$  with  $\eta^0 > \eta^i$ . It suffices to find a single vector  $\hat{t} \in \mathbf{R}^n$  such that  $\hat{t}' z_q \hat{t} > 0$ . Consider

$$\hat{t} = \beta q + \delta e_i$$

where  $e_i$  is the unit vector with one as its  $i$ th element, and  $\beta > 0$  and  $\delta$  are arbitrary scalars. Observe that the homogeneity condition  $q^0 x_0^i + q' x_q^i = 0$  implies

$$\begin{aligned} q' z_q &= q' x_q - \frac{1}{x_0} (q' x) x'_0 \\ &= -q^0 x'_0 + q^0 x'_0 = 0'_n \end{aligned}$$

when the budget constraint  $Q'X = 0$  is satisfied. Thus

$$\hat{t}' z_q \hat{t} = \beta \delta (e'_i z_q q) + \delta^2 z_i^i. \quad (20)$$

Applying the homogeneity condition once again,

$$\begin{aligned} e'_i z_q q &= -\frac{1}{x_0} \left( x_q^{i'} q - \frac{x^i}{x_0} x_q^{0'} q \right) \\ &= -\frac{1}{x_0} \left( -q^0 x_0^i + \frac{x^i}{x_0} q^0 x_0^0 \right) \\ &= z^i (\eta^0 - \eta^i). \end{aligned}$$

Choose  $\delta$  small so that the second term in (20) is negligible, and

$$\hat{t}' z_q \hat{t} \approx \beta \delta z^i (\eta^0 - \eta^i) > 0 \quad \text{for } \delta > 0,$$

a contradiction. Since the choice of good  $i$  was arbitrary, it follows that  $z_q$  is negative semi-definite only if (18) holds.  $\square$

## References

- Besley, Timothy, and Ian Jewitt (1995) ‘Uniform taxation and consumer preferences.’ *Journal of Public Economics* 58, 73–84
- Blackorby, Charles, Russell Davidson, and William Schworm (1990) ‘Use and misuse of single-consumer results in a multi-consumer economy: The optimality of proportional commodity taxation.’ In *Measurement and modelling in economics*, ed. Gareth D. Myles (North-Holland)
- Corlett, W.J., and D.C. Hague (1953) ‘Complementarity and the excess burden of taxation.’ *Review of Economic Studies* 21, 21–30
- Gordon, James P.F. (1991) ‘Tax reform and uniformity: Explaining the hatta result.’ *Journal of Public Economics* 45, 257–261
- Hatta, Tatsu (1986) ‘Welfare effects of changing commodity tax rates toward uniformity.’ *Journal of Public Economics* 29, 99–112
- Konishi, Hideo (1989) ‘The welfare effects of shifting from an indirect tax to a direct tax.’ *Economic Studies Quarterly* 40, 212–219
- Murata, Yasuo (1977) *Mathematics for stability and optimization in economic systems* (Academic Press)
- Sadka, E. (1977) ‘A theorem on uniform taxation.’ *Journal of Public Economics* 7, 387–391