University of Toronto

Department of Economics
ECO2020 Final exam - Turner - October 20, 2011
Start time: 9:10 am - Duration: 90 minutes

Examination Aids: No notes or books are allowed, but you may use a calculator. Please turn your cell-phones off.

The exam is worth 100 **points in all, and will count for 90% of your grade.**

When you write up your answers, your goal should be to (1) be correct, and (2) convince your reader that your answer is correct. Answers which do not achieve these goals will not be awarded full credit. To accomplish the second objective, it is helpful if your work is legible and if all steps are presented, possibly with a line of explanation.

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1. (50) Consider an individual who values consumption (c) and leisure l according to the utility function $u(c,l) = \frac{c^{1-\eta}}{1-\eta} + \frac{1}{2}\frac{l^{1-\eta}}{1-\eta}$ for $\eta \in (0,1)$. The consumer has an endowment of one unit of leisure and can exchnage leisure for consumption at the wage rate w. In addition, the consumer gets consumption $a > 0$ from non-work income (e.g. social assistance).

   Thus the consumers optimization problem is

$$\max_{c,l} u(c,l)$$
$$\text{s.t. } c + wl \leq w + a$$
$$c \geq a$$

   (a) Describe all solution(s) which satisfy the first order necessary conditions and where $l > 0$ (that is, you can ignore solutions where $l = 0$).
   (b) Verify that the second order conditions hold.
   (c) Find the largest wage, $\overline{w}$ at which the consumer chooses $l = 1$.
   (d) Graph your solution(s) to this problem. Your graph should illustrate the first order condition for the solution, and you should provide a one or two line explanation of why your graph illustrates the first order conditions. If you find more than one solution, put each on its own graph.

2. (20) Consider a consumer with continuous utility function $u(x,y)$, with u increasing and strictly quasi-concave. The consumer faces a budget $p_0 x + y = w$. Let λ denote the lagrange multiplier from the consumer's utility maximization problem. Let v denote the consumer's indirect utility function. Prove that $\frac{\partial v(p,w)}{\partial w} = \lambda$.

3. (30) Let $c(w,q)$ be the cost function of a single output technology Y with production function $f(\cdot)$ and that $z(w,q)$ is the associated conditional factor demand. Assume that Y is closed and satisfies free disposal. Prove that:

   (a) (5) $c(w,q)$ is homogenous of degree one in w.
   (b) (5) $c(w,q)$ is non-decreasing in q.
   (c) (5) c is a concave function of w.
   (d) (5) If the set $\{z \geq 0 : f(z) \geq q\}$ is convex then $z(w,q)$ is a convex set.
   (e) (5) If the set $\{z \geq 0 : f(z) \geq q\}$ is NOT convex, draw a graph illustrating that $z(w,q)$ need not be a convex set.
   (f) (5) If $z(w,q)$ is a single point and $c(w,q)$ is differentiable with respect to w at $\overline{w}$, then $\nabla_w c(\overline{w},q) = z(\overline{w},q)$. Your proof should be based on first order conditions, and not make use of the duality theorem or the envelope theorem.

$$\text{MAX} \quad \frac{c^{1-\eta}}{1-\eta} + \frac{1}{2}\frac{\ell^{1-\eta}}{1-\eta}$$

$$\text{S.T.} \quad c + w\ell \le w + a$$
$$-c \le -a$$

(F.O.C.) $\quad c^{-\eta} = \lambda - \phi \qquad (1)$

$\frac{1}{2}\ell^{-\eta} = \lambda w \qquad (2)$

(C.S.) $\quad c + w\ell - w - a \le 0, \quad \lambda \ge 0, \quad \lambda(c + w\ell - w - a) = 0 \qquad (3)$

$c - a \ge 0, \quad \phi \ge 0 \qquad \lambda(c-a) = 0 \qquad (4)$

CASE 1, $\lambda = 0$ | IF $\lambda = 0$, $(1) \Rightarrow c^{-\eta} = \phi$.
BUT $\phi \ge 0$, SO THIS REQUIRES $c < 0$,
CONTRADICTING (4)

CASE 2, $\lambda > 0, \phi = 0$ | WITH $\phi = 0$ $\quad (1) + (2) \Rightarrow$

$c^{-\eta} = \lambda \qquad (1')$

$\frac{1}{2}\ell^{-\eta} = \lambda w \qquad (2')$

$(1') + (2') \Rightarrow \quad \frac{1}{2}\ell^{-\eta} = c^{-\eta} w$

SO $\quad \ell^{-\eta} = 2c^{-\eta} w$

$\Rightarrow \quad \ell = c\,(2w)^{-1/\eta} \qquad (5')$

$(3) \Rightarrow \quad c + w\ell = w + a \qquad$ (SINCE $\lambda > 0$)

USING $(5') \Rightarrow \quad c + w\,(2w)^{-1/\eta} c = w + a$

$$\Rightarrow \quad c^* = \frac{w + a}{1 + w\,(2w)^{-1/\eta}} \qquad (6')$$

TO SATISFY (4) FOR $\phi = 0$, WE REQUIRE

$$\frac{w+a}{1+w(2w)^{-1/n}} \leq a \qquad (7)$$

SO WE CAN SATISFY OUR F.O.C.'S FOR $\lambda > 0$, $\phi = 0$ ONLY IF (6) HOLDS.

$$(6)+(5') \Rightarrow \quad l^* = (2w)^{-1/n} \frac{w+a}{1+w(2w)^{-1/n}} \qquad (8)$$

$$(8)+(2) \Rightarrow \quad \lambda^* = \frac{1}{2w} (2w)^{-1/n} \frac{w+a}{1+w(2w)^{-1/n}}$$

CASE 3: $\lambda > 0, \phi > 0$ | $\phi > 0 \Rightarrow c^* = a$, FROM (4)

THUS, $\lambda > 0 \Rightarrow a + wl - w - a = 0$ FROM (3)

$\Rightarrow l^* = 1.$

THUS $l^* = 1$, $c^* = a$. USING THIS IN (1), (2) WE HAVE

$$a^{-n} = \lambda - \phi \qquad (1'')$$

$$\frac{1}{2} = \lambda w \qquad (2'')$$

SO $(2'') \Rightarrow \lambda^* = \frac{1}{2w}$

$$\Rightarrow$$

$$\phi^* = \frac{1}{2w} - a^{-n}$$

$\lambda > 0$ SINCE $w > 0$ AND FINITE.

$\phi > 0 \Rightarrow \frac{1}{2w} > a^{-n}$ OR $a^n > 2w$

(b)

$$\nabla u = \begin{bmatrix} c^{-n} \\ \frac{1}{2} l^{-n} \end{bmatrix}$$

$$D^2 u = D(\nabla u) = \begin{bmatrix} -n c^{-n-1} & 0 \\ 0 & -\frac{n}{2} l^{-n-1} \end{bmatrix}$$

THUS, $z D^2 u z =$

$$\begin{bmatrix} z_1 & z_2 \end{bmatrix} D^2 u \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= -nc^{-n-1} z_1^2 - \frac{-n}{2} l z_2^2$$

SINCE $n, c, l, z_1^2, z_2^2 > 0$ THIS IS NEGATIVE FOR ALL z.

$\Rightarrow$ SECOND ORDER CONDITIONS HOLD AT BOTH VALUES OF (c^*, l^*)

(c) IF $l^* = 1$ THEN $c^* = a$. THAT IS, THIS IS THE HIGHEST WAGE AT WHICH OUR PERSON DOESN'T WORK.

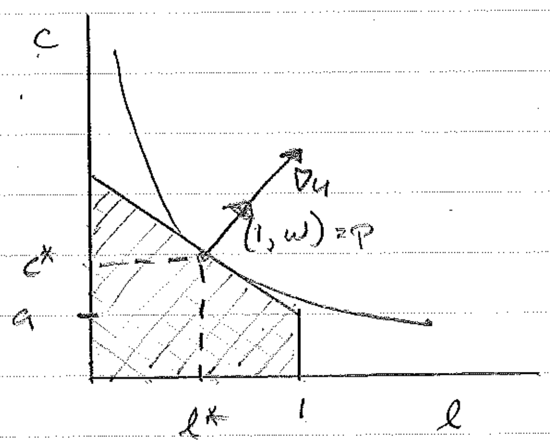
FROM (a) THIS HAPPENS WHEN

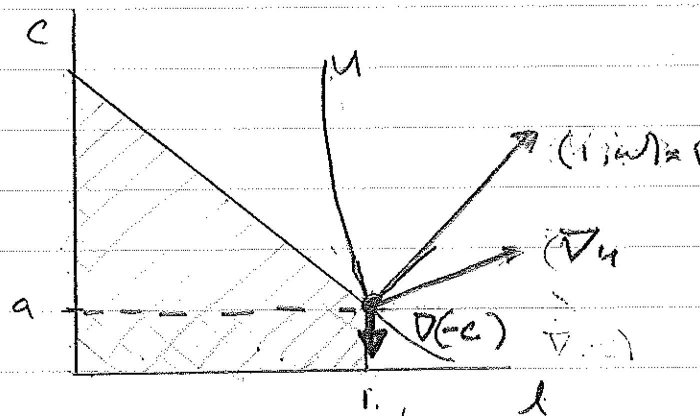
$\lambda > 0, \phi > 0 \Rightarrow a^\eta > 2w$

$$\Rightarrow w < \frac{a^\eta}{2}$$

SO OUR CRITICAL VALUE OF w IS $w = \frac{a^\eta}{2}$.

(d)





$\lambda > 0, \phi > 0$

$\lambda > 0, \phi = 0$

$\Rightarrow \nabla u = \lambda (1, w)^T$

$\nabla u = \lambda (1, w)^T + \phi \nabla(-c)$

3.

$$\frac{\partial v}{\partial w} = \frac{\partial u(x(p,w))}{\partial w}$$

USING THE CHAIN RULE,

$$= D_x u \cdot D_w x$$

BUT $D_x u = \lambda p$ FROM F.O.C

$$\Rightarrow \quad = \lambda p \cdot D_w x$$

FROM WALRAS LAW, $p \cdot x = w$

$$\Rightarrow \quad p \cdot D_w x = 1$$

$$\Rightarrow \quad \frac{\partial v}{\partial w} = \lambda$$

1.

a) $c(w,q) = \min \; w \cdot z$
$\quad$ s.t. $f(z) \geq q$

$c(\alpha w, q) = \min \; \alpha w \cdot z$
$\quad$ s.t. $f(z) \geq q$.

$= \alpha \left[\begin{array}{l} \min \; w \cdot z \\ \quad \text{s.t. } f(z) \geq q \end{array} \right] = \alpha c(w,q)$

b) SUPPOSE $\exists \; q, q'$ WITH $q' > q$

AND $c(w,q') < c(w,q)$. SINCE Y

SATISFIES FREE DISPOSAL WE CAN ALSO USE

INPUTS $z(w,q')$ TO PRODUCE q

$\Rightarrow \; c(w,q) = c(w,q')$, A CONTRADICTION.

THUS, $c(w,q)$ IS NON-DECREASING IN q.

$\rightarrow$

4. (c)

GIVEN ω, ω', LET $\omega'' = \alpha\omega + (1-\alpha)\omega'$ FOR $\alpha \in (0,1)$.

$$c(\omega'', q) = \omega'' z(\omega'', q)$$

$$= \alpha\omega z(\omega'', q) + (1-\alpha)\omega' z(\omega'', q)$$

$$\geq \alpha\omega z(\omega, q) + (1-\alpha)\omega' z(\omega', q)$$

$$= \alpha c(\omega, q) + (1-\alpha) c(\omega', q)$$

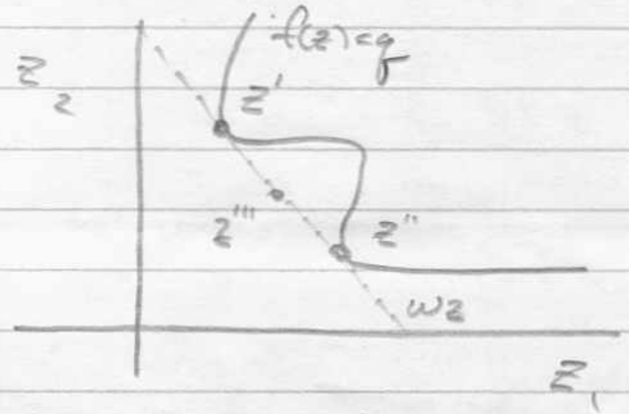
THUS $c(\alpha\omega + (1-\alpha)\omega', q) \geq \alpha c(\omega, q) + (1-\alpha) c(\omega', q)$

(d) CONSIDER $z', z'' \in z(\omega, q)$.
SINCE $\{z \geq 0 \mid f(z) \geq q\}$ IT FOLLOWS THAT

$$f(\alpha z' + (1-\alpha) z'') \geq q \quad \forall \alpha \in [0,1].$$

THUS $\alpha z + (1-\alpha) z' \in z(\omega, q)$.

(e) IF $\{z \geq 0 \mid f(z) \geq q\}$ NOT CONVEX THEN:



$z', z' \in z(\omega, q)$
BUT
$z''' \notin z(\omega, q)$.

THUS $z(\omega, q)$ NEED NOT BE CONVEX IF $\{z \geq 0 \mid f(z) \geq q\}$ NOT CONVEX.

4. f.

$$c(\omega, q) = \omega \cdot z(\omega, q)$$

$$\Rightarrow D_\omega c = z(\omega, q) + \omega \cdot \nabla_\omega z(\omega, q) \quad (1)$$

$z(\omega, q)$ SOLVES $\min \ \omega \cdot z$ s.t. $f(z) = q$

$$\Rightarrow \mathcal{L} = \omega \cdot z + \lambda(f(z) - q)$$

$$\nabla_z \mathcal{L} = \omega + \lambda \nabla f = 0$$

$$\Rightarrow \omega = -\lambda \nabla f \quad (2)$$

$(1) + (2) \Rightarrow D_\omega c = z(\omega, q) - \lambda \nabla f \, \nabla_\omega z(\omega, q)$

BUT $\nabla f \cdot \nabla_\omega z = 0$ AT AN OPTIMUM SINCE $q = f(z)$ IS CONSTANT

$$\Rightarrow D_\omega c = z(\omega, q)$$