EC2020-Fall 2011
Problem Set 1 solutions

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MWG 1.B.1

SUPPOSE $\succsim$ SATISFIES COMPLETENESS AND TRANSITIVITY.
SHOW THAT

$$x \succ y \succsim z \Rightarrow x \succ z$$

IF $x \not\succ z$ THEN $z \succsim x$.
IF $x \succ y \succsim z$ THEN WE HAVE

$$y \succsim z \succsim x \Rightarrow y \succsim x$$

BUT $x \succ y \Rightarrow y \not\succsim x$ A CONTRADICTION. $\square$

MWG 1.B.2

SUPPOSE $\succsim$ COMPLETE AND TRANSITIVE

(I) SHOW THAT $\succ$ IS IRREFLEXIVE AND TRANSITIVE.

- BY COMPLETENESS $x \succsim x \quad \forall x$.
  BUT $x \succsim x \Longleftrightarrow x \not\succ x$
  $\Longrightarrow \succ$ IRREFLEXIVE $\forall x$

- SAY $x \succ y \succ z$ AND $x \not\succ z$. $\Rightarrow z \succsim x$ (3)
  FROM THE DEF OF $\succ$
  THEN WE HAVE $x \succsim y$ (2) $y \succsim z$ (1).
  $x \not\precsim y \qquad y \not\precsim z$

  FROM (1) + (3) AND TRANSITIVITY OF $\succsim$ $y \succsim x$ (4)

  BUT $x \succ y \Rightarrow y \not\succsim x$, A CONTRADICTION $\square$

(II) SHOW THAT $\sim$ IS REFLEXIVE, TRANSITIVE, SYMMETRIC.

- SINCE $\succsim$ COMPLETE, $x \succsim x$ AND $x \precsim x \Rightarrow x \sim x$ $\square$
- IF $x \sim y \sim z$ THEN
  $x \succsim y \succsim z \Rightarrow x \succsim z$ (1)
  AND $z \succsim y \succsim x \Rightarrow z \succsim x$ (2)
  (1) + (2) $\Rightarrow x \sim z$ $\square$
- IF $x \sim y$ THE $x \succsim y$ AND $y \succsim x \Rightarrow y \sim x$. $\square$

MWG 1.B.3

LET $f: \mathbb{R} \to \mathbb{R}$ WITH $f(x) > f(y) \Leftrightarrow x > y \quad \forall x, y \in \mathbb{R}$.
LET $u: X \mapsto \mathbb{R}$ REPRESENT $\succsim$.
SHOW THAT $f(u(\cdot))$ ALSO REPRESENTS $\succsim$.

u REPRESENTS $\succsim$ MEANS

$$u(x) \geq u(y) \Leftrightarrow x \succsim y$$

SUPPOSE $x \succsim y$. THEN $u(x) > u(y)$ OR $u(x) = u(y)$.
IF $u(x) = u(y)$ THEN $f(u(x)) = f(u(y))$.
IF $u(x) > u(y)$ THEN $f(u(x)) > f(u(y))$.
THUS $x \succsim y \Rightarrow f(u(x)) \geq f(u(y))$.

SUPPOSE $f(u(x)) \geq f(u(y))$. THEN $f(u(x)) = f(u(y))$ OR $f(u(x)) > f(u(y))$.

IF $f(u(x)) = f(u(y))$, MONOTONICITY OF $f \Rightarrow u(x) = u(y)$.
SINCE u REPRESENTS $\succsim \Rightarrow x \succsim y$.

IF $f(u(x)) > f(u(y))$, MONOTONICITY $\Rightarrow u(x) > u(y)$
SINCE u REPRESENTS $\succsim \Rightarrow x \succsim y$. $\square$

MWG 1.B.4

SUPPOSE (1) $u(x) = u(y) \Rightarrow x \sim y$ AND

(2) $u(x) > u(y) \Rightarrow x \succ y$

SHOW THAT

$$u(x) \geq u(y) \Leftrightarrow x \succsim y.$$

N.B. IF $A \Rightarrow B$ THEN $\neg B \Rightarrow \neg A$ [THIS IS THE "CONTRAPOSITIVE"]

APPLYING TO (1) AND (2) WE HAVE

(1') $x \succ y$ or $x \prec y \Rightarrow u(x) \neq u(y)$

(2') $x \precsim y \Rightarrow u(x) \leq u(y)$

$\therefore$ (1) + (2) $\Rightarrow$

$$u(x) \geq u(y) \Rightarrow x \succsim y \quad (3)$$

REVERSING THE ROLES OF x, y IN (2') GIVES

$$x \succsim y \Rightarrow u(x) \geq u(y) \quad (4)$$

(3) AND (4) TOGETHER GIVE $x \succsim y \Leftrightarrow u(x) \geq u(y)$

$\square$

MWG 1.B.5

- SUPPOSE THAT X IS FINITE AND $\succsim$ IS A RATIONAL PREFERENCE RELATION ON X. SHOW THAT THERE IS A UTILITY FUNCTION $u: X \to \mathbb{R}$ REPRESENTING $\succsim$.

- TO START, SUPPOSE THAT NO TWO ELEMENTS OF X ARE INDIFFERENT.

- LET $X = \{x_1, x_2, \ldots, x_N\}$. SINCE $\succsim$ IS COMPLETE AND TRANSITIVE WE CAN ORDER ELEMENTS OF X FROM LEAST PREFERRED TO MOST PREFERRED.

- LET $u(x)$ = RANK OF x. THAT IS, THE UTILITY OF THE LEAST PREFERRED ELEMENT IS 1, OF THE NEXT ELEMENT 2, ETC.

  THEN $u(x) > u(y) \Leftrightarrow x \succ y$ SO u REPRESENTS $\succsim$ ON X.

- NOW SUPPOSE THAT SOME ELEMENTS OF X ARE INDIFFERENT. CONSTRUCT $X^* \subset X$ SUCH THAT ONLY ONE MEMBER OF INDIFFERENT SETS IS RETAINED.

- CONSTRUCT u ON X^* AS ABOVE.

- EXTEND u TO X BY ASSIGNING ALL MEMBERS OF ANY INDIFFERENT SET THE SAME u.

$\square$

MWG 1.C.1

$(\mathcal{B}, C(\cdot))$ $\mathcal{B} = \{\{x,y\}, \{x,y,z\}\}$ AND
$C(\{x,y\}) = \{x\}$

SAY $C(\cdot)$ SATISFIES W.A.R.P. THEN

IF $x \in C(\{x,y\})$ THEN FOR ANY B
WITH $\{x,y\} \subset B$ AND $y \in C(B)$,
WE MUST HAVE $x \in C(B)$.

THUS, $C(\{x,y,z\})$ MUST EITHER
(i) NOT CONTAIN y
OR
(ii) CONTAIN BOTH y AND x
IF IT SATISFIES W.A.R.P.

$$\Rightarrow C(\{x,y,z\}) \subset \{\{x\}, \{z\}, \{x,y\}, \{x,z\}, \{x,y,z\}\}$$

BUT IF $C(\{x,y,z\}) \subset \{\{x,y\}, \{x,y,z\}\}$

THEN W.A.R.P. REQUIRES
$C(\{x,y\}) \subset \{\{y\}, \{x,y\}\}$.

THUS, IF W.ARP IS SATISFIED

$$C(\{x,y,z\}) \subset \{\{x\}, \{z\}, \{x,z\}\}$$

$\blacksquare$

MWG 1.D.1

THIS IS EASY IF YOU KEEP YOUR CHOICE STRUCTURE VERY SIMPLE.

$$\mathcal{B} = \{\{x, y, z\}\}$$

$$C(\{x, y, z\}) = \{x\}$$

THIS IS CONSISTENT WITH $x \succsim y \succsim z$

AND $x \succsim z \succsim y$

MWG 1.D.3

$X = \{x, y, z\}$

$\{\mathcal{B}, C(\cdot)\}$ $\quad \mathcal{B} = \{\{x,y\}, \{y,z\}, \{x,z\}, \{x,y,z\}\}$

$C(\{x,y\}) = \{x\}, \; C(\{y,z\}) = \{y\}, \; C(\{x,z\}) = \{z\}$

- SUPPOSE $C(\{x,y,z\}) = \{x\}$
  THEN W.A.R.P. $\Rightarrow$ $x \in C(\{x,z\}) = \{z\}$
  SO W.A.R.P. IS VIOLATED.

  SIMILARLY FOR $C(\{x,y,z\}) = \{y\}$ OR $\{z\}$.

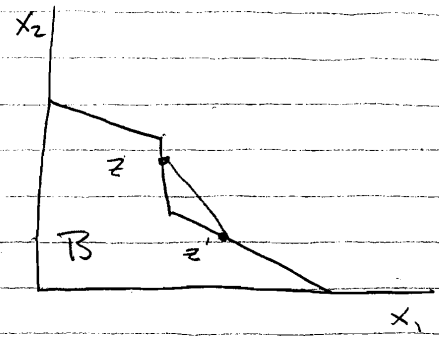
- SUPPOSE $C(\{x,y,z\}) = \{x,y\}$
  THEN W.A.R.P $\Rightarrow$ $y \in C(\{y,z\})$
  SO W.A.R.P. IS VIOLATED.

  SIMILARLY FOR $C(\{x,y,z\}) = \{x,z\}$ OR $\{y,z\}$.

- SUPPOSE $C(\{x,y,z\}) = \{x,y,z\}$
  THEN W.A.R.P. $\Rightarrow$ $z \in C(\{y,z\})$
  SO W.A.R.P. IS VIOLATED

$\square$

MWG 2.D.4



$z, z' \in B$, BUT $\frac{1}{2} z + \frac{1}{2} z' \notin B$

$\Rightarrow B$ NOT CONVEX

MWG 2.E.1

$$x_1(p,w) = \frac{p_2}{p_1+p_2+p_3} \frac{w}{p_1}$$

$$x_2(p,w) = \frac{p_3}{p_1+p_2+p_3} \frac{w}{p_2}$$

$$x_3(p,w) = \frac{\beta p_1}{p_1+p_2+p_3} \frac{w}{p_3}$$

$x(p,w)$ IS $H\phi$ $\Rightarrow$ $x(\alpha p, \alpha w) = x(p,w)$

$\forall \alpha, p, w, \quad \alpha > 0$

BUT $x_1(\alpha p, \alpha w) = \dfrac{\alpha p_2}{\alpha(p_1+p_2+p_3)} \cdot \dfrac{\alpha w}{\alpha p_1} = x_1(p,w)$

SIMILARLY x_2,

FOR $x_3(\alpha p, \alpha w) = \dfrac{\beta \alpha p_1}{\alpha(p_1+p_2+p_3)} \cdot \dfrac{\alpha w}{\alpha p_3} = x_3(p,w)$

SO x IS $H\phi$ $\square$

WALRAS LAW REQUIRES

$p \cdot x = w$

$\Rightarrow \quad \dfrac{p_2+p_3+\beta p_1}{p_1+p_2+p_3} \cdot w = w$

THIS IS TRUE IFF $\beta = 1$. $\square$

MWG 2.E.4

$$x_\ell(p, \alpha w) = \alpha x_\ell(p, w) \qquad \alpha > 0, \quad \forall \ell$$

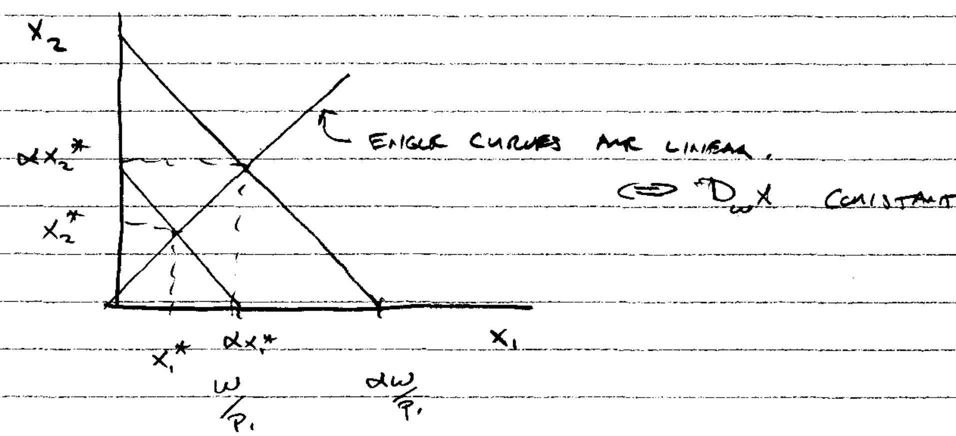
$$\Rightarrow \quad \frac{x_\ell(p, \alpha w)}{\alpha w} = \frac{x_\ell(p, w)}{w}$$

$$\Rightarrow \quad \frac{d}{dw} \frac{x_\ell(p,w)}{w} = 0 \qquad \text{i.e. RATIO } \frac{x_\ell}{w} \text{ CONSTANT}$$

$$\Rightarrow \quad \frac{w \frac{d}{dw} x_\ell - x_\ell}{w^2} = 0$$

$$\Rightarrow \quad w \cdot \frac{dx_\ell}{dw} = x_\ell$$

$$\Rightarrow \quad \frac{w}{x_\ell} \cdot \frac{dx_\ell}{dw} = 1 \quad \Leftrightarrow \quad \varepsilon_{\ell w} = 1 \qquad \square$$



MWG 2.F.1

I

DEF 2.F.1: W.A.R.P. IFF FOR ANY (p, w), (p', w'):

IF $p \cdot x(p', w') \leq w$ (A) THEN

AND $x \neq x'$ (B)

THEN $p' \cdot x(p, w) > w'$ (C)

II

DEF 1.C.1: W.A.R.P $\Leftrightarrow$ IF FOR SOME $B \in \beta$ WITH $x, y \in B$ WE HAVE $x \in C(B)$, THEN FOR ANY B' WITH $x, y \in B'$ WE MUST HAVE $x \in C(B')$.

SHOW THAT II $\Rightarrow$ I

LET $B(p, w) = \{x \mid p \cdot x \leq w\}$

SUPPOSE (0) $x(p, w) = C(B(p, w))$ AND

(1) $x(p', w') \in B(p, w)$

(2) $x(p', w') = C(B(p', w'))$

(3) $x \neq x'$

THEN II $\Rightarrow$ (4) $x(p, w) \notin B(p', w')$

(0) AND (2) HOLD BY DEF OF $x(p, w)$.

(1), (3), (4) CORRESPOND EXACTLY TO (A) (B) (C) (WITH SOME CHANGE OF NOTATION).

THUS II $\Rightarrow$ I

I $\not\Rightarrow$ II SINCE I CAN IMAGINE SITUATIONS WITH CONSTRAINT SETS THAT ARE NOT BUDGET SETS ~~WHERE II WOULD APPLY, BUT I WOULD NOT.~~

MWG 2.F.2

$$\omega = 8$$

$p_1 = (2, 1, 2)$ $\quad x_1 = (1, 2, 2)$

$p_2 = (2, 2, 1)$ $\quad x_2 = (2, 1, 2)$

$p_3 = (1, 2, 2)$ $\quad x_3 = (2, 2, 1)$

Show $\quad x_3 \succsim^* x_2 \succsim^* x_1 \succsim^* x_3$

$x_3 \succsim^* x_2 \iff p_3 \cdot x_2 \leq 8$

But $\quad p_3 \cdot x_2 = 1 \cdot 2 + 2 \cdot 1 + 2 \cdot 2 = 8 \quad \square$

$x_2 \succsim^* x_1 \iff p_2 \cdot x_1 \leq 8$

But $\quad p_2 \cdot x_1 = 2 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 = 8 \quad \square$

$x_1 \succsim^* x_3 \iff p_1 \cdot x_3 \leq 8$

But $\quad p_1 \cdot x_3 = 2 \cdot 1 + 1 \cdot 2 + 2 \cdot 2 = 8 \quad \square$

$\square$