# EC2020-Fall 2011
# Problem Set 2 solutions

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**MWG 2.E.8**

SHOW THAT $\frac{d \ln x(p)}{d \ln p} = \frac{dx}{dp} \cdot \frac{p}{x}$ *

LET $z = \ln p \Rightarrow p = e^z$.

WE CAN REWRITE *

$$\frac{d \ln x(e^z)}{dz}$$

USING THE CHAIN RULE WE HAVE

$$= \frac{1}{x} \cdot \frac{dx}{de^z} \cdot \frac{de^z}{dz}$$

$$= \frac{1}{x} \cdot \frac{dx}{dp} \cdot p$$

IF WE ESTIMATE

$$\ln x = \alpha_0 + \alpha_1 \ln p_1 + \alpha_2 \ln p_2 + \alpha_3 \ln w$$

THEN (FOR EXAMPLE)

$$\frac{d \ln x}{d \ln p_1} = \alpha_1$$

THUS $\alpha_1$ IS PRICE 1 ELASTICITY OF DEMAND.

**MWG 3.B.1**

3B1

(1) $\succsim$ STRONGLY MONOTONE $\Rightarrow$ $\succsim$ MONOTONE

$\succsim$ STRONG MONOTONE

$\Leftrightarrow$ FOR $y, x \in X$

$y \geq x$ AND $y \neq x$ $\Rightarrow$ $y \succ x$

SINCE $\{y' \gg x\} \subset \{y \geq x\}$ AND $x \notin \{y' \gg x\}$

$\Rightarrow$ FOR $y, x \in X$

$y \gg x$ $\Rightarrow$ $y \succ x$

BUT $y \succ x \Leftrightarrow y \succsim x$ AND NOT $x \succsim y$

SO $y \succ x \Rightarrow y \succsim x$

$\Rightarrow$ FOR $y, x \in X$

$y \gg x \Rightarrow y \succsim x$ $\Rightarrow$ $\succsim$ MONOTONE $\square$

(2) $\succsim$ MONOTONE $\Rightarrow$ $\succsim$ LNS

IF $\succsim$ MONOTONE THEN $y \gg x \Rightarrow y \succ x$

LET $y' = \alpha x$ FOR $\alpha > 1$.

THEN $y' \gg x \Rightarrow y \succ x$.

LET $B_\varepsilon(x) \equiv \{x' \in X \mid \|x - x'\| \leq \varepsilon\}$

FOR ANY $\varepsilon > 0$ $\exists\, \alpha$ CLOSE ENOUGH TO 1

S.T. $y' \in B_\varepsilon(x)$

$\Rightarrow$ $\forall \varepsilon > 0$ $\exists\, y' \in B_\varepsilon(x)$ WITH $y \succ x$

$\Rightarrow$ $\succsim$ LNS $\square$

**MWG 3.C.3**

Say that for$\succeq$ on $X$, upper and lower contour sets are closed. Consider $(x_i,y_i)_{i=0}^{\infty}$ with $x_i \succeq y_i$ for all $i$, $x_i \to x$, $y_i \to y$, $x \neq y$. Show that $x \succeq y$

(By contradiction) Suppose $y \succ x$. If we can find $q \in X$ with $y \succ q \succ x$, then the problem is easy. If $X = R^L$ and $\succ$ is monotone we can always find such a $q$ (in fact, you should be able to construct such a $q$).

Let $U^q = \{z \in X | z \succ q\}$ and $L^q = \{z \in X | q \succ z\}$. By hypothesis both sets are open (why?). Since $y \succ q$, we have $y \in U^q$. Since $U^q$ open there exists $\epsilon > 0$ such that $B_\epsilon(y) \subset U^q$. Since $y_i \to y$ there exists $N$ such that for all $j > N$, $y_i \in B_\epsilon(y)$ which implies that, for all $j > N$, $y_i \succ q$. A similar argument establishes that for some $N'$, for all $j > N'$, $q \succ x_i$. Thus, for $j > \max\{N,N'\}$ we have $y_i \succ q \succ x_i$ and we contradict our hypothesis that $x_i \succeq y_i$ for all $i$.

Now suppose that there is no $q \in X$ such that $y \succ q \succ x$ (this is going to require that $X$ is discrete, or something similar). This is a little bit more complicated. We have two cases.

First, suppose that for some $N$, $y_i \succeq y$ and $x \succeq x_i$ for all $i > N$. Then, since $y \succ x$, we arrive at a contradiction immediately.

Second, if we are not in the first case, then we can construct a subsequence $(x_i')$ of $(x_i)$ which restricts attention to $x_i \succ x$, and, since the original sequence converges to $x$, so does the subsequence (there is a theorem here, you might try to prove it). Thus, we have constructed a sequence of consumption bundles converging to $x$ such that all elements of the sequence satisfy $x_i \succ x$. But, by hypothesis, there is no element $q$ of $X$ with $y \succ q \succ x$. This means that all elements in our sequence must satisfy $x_i \succeq y$, or that that they are in the upper contour set of $y$. By hypothesis, this set is closed. However, we have just demonstrated that $x$ is a limit point of this set and not in the set, a contradiction.

Comment: Obviously, the result that one definition of continuity implies the other is unlikely to change your life. What is important here is that you learn all of the definitions: closed, open, convergent, and how to work with them.

**MWG 3.C.6**

LET $u(x) = \left[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right]^{1/\rho}$ [FROM SOLUTION MANUAL]

(a) AS $\rho \to 1$ $u(x) \to \alpha x_1 + \alpha_2 x_2$ (EASY)

(b) FIRST, SINCE $\rho \ln$ Z MONOTONE, WE CAN WORK WITH

$$\tilde{u}(x) = \rho \ln[u(x)]$$

$$= \ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)$$

$$\widetilde{MRS}_{12} = \tilde{u}_1 / \tilde{u}_2$$

$$= \left[\frac{1}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho}\left[\rho \alpha_1 x_1^{\rho-1}\right]\right]\left[\frac{1}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho}\left[\rho \alpha_2 x_2^{\rho-1}\right]\right]^{-1}$$

$$= \frac{\rho \alpha_1 x_1^{\rho-1}}{\rho \alpha_2 x_2^{\rho-1}}$$

$$= \frac{\alpha_1 x_1^{\rho-1}}{\alpha_2 x_2^{\rho-1}}$$

AS $\rho \to 0$ $\Rightarrow$ $\widetilde{MRS}_{12} = \dfrac{\alpha_1 x_2}{\alpha_2 x_1}$

NOW CONSIDER $\hat{u} = x_1^{\alpha_1} x_2^{\alpha_2}$

$$\widehat{MRS}_{12} = \hat{u}_1 / \hat{u}_2 = \frac{\alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2}}{\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1}} = \frac{\alpha_1 x_2}{\alpha_2 x_1}$$

THUS, AS $\rho \to 1$ WE HAVE $\widehat{MRS}_{12} = \widetilde{MRS}_{12}$ FOR ALL $(x_1, x_2) \gg 0$. THUS, THE SLOPE OF INDIFFERENCE CURVES IS EVERYWHERE THE SAME.

(c) IT IS SUFFICIENT TO SHOW THAT FOR ANY $(x_1, x_2)$ WITH $x_1 \le x_2$, THEN

$$\tilde{u}(x_1, x_2) \to x_1 \text{ AS } \rho \to -\infty$$

FOR $\rho < 0$ AND $x_1 \ge 0, x_2 \ge 0$ WE HAVE

① $$\alpha_1 x_1^\rho \le \alpha_1 x_1^\rho + \alpha_2 x_2^\rho$$

$$\Rightarrow \quad \alpha_1^{1/\rho} x_1 \ge \left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho} \quad (\text{RECALL } \rho < 0)$$

② SINCE $x_1 < x_2$ AND $\rho < 0$ WE ALSO HAVE

$$\alpha_1 x_1^\rho + \alpha_2 x_2^\rho \le (\alpha_1 + \alpha_2) x_1^\rho$$

$$\Rightarrow \quad \left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho} \ge (\alpha_1 + \alpha_2)^{1/\rho} x_1$$

① + ② $\Rightarrow$

$$\underbrace{\lim_{\rho \to -\infty} \alpha_1^{1/\rho} x_1}_{\text{I}} \ge \lim_{\rho \to -\infty} \left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho\right)^{1/\rho} \ge \underbrace{\lim_{\rho \to -\infty} (\alpha_1 + \alpha_2)^{1/\rho} x_1}_{\text{II}}$$

BUT $\text{I} = \text{II} = x_1$ $\square$

**MWG 3.D.5**

(a) $u(x_1, x_2) = \left[x_1^{\rho} + x_2^{\rho}\right]^{1/\rho}$

UMP $\quad \max \left[x_1^{\rho} + x_2^{\rho}\right]^{1/\rho}$

$\quad$ s.t. $p_1 x_1 + p_2 x_2 = w$ $\quad$ ③

F.O.C. $\quad \nabla u = \lambda p$

$\Rightarrow$ ① $\quad \frac{1}{\rho}\left[x_1^{\rho} + x_2^{\rho}\right]^{\frac{1}{\rho}-1} \rho x_1^{\rho-1} = \lambda p_1$

② $\quad \frac{1}{\rho}\left[x_1^{\rho} + x_2^{\rho}\right]^{\frac{1}{\rho}-1} \rho x_2^{\rho-1} = \lambda p_2$

Solve ① + ② for $\lambda$ $\Rightarrow$

$$\frac{1}{p_1}\left[\frac{1}{\rho}\left[x_1^{\rho}+x_2^{\rho}\right]^{\frac{1}{\rho}-1}\rho x_1^{\rho-1}\right] = \frac{1}{p_2}\left[\frac{1}{\rho}\left[x_1^{\rho}+x_2^{\rho}\right]^{\frac{1}{\rho}-1}\rho x_2^{\rho-1}\right]$$

$$\Rightarrow \quad \frac{x_1^{\rho-1}}{p_1} = \frac{x_2^{\rho-1}}{p_2}$$

$$\Rightarrow \quad x_1 = \left[\frac{p_1}{p_2}\right]^{\frac{1}{\rho-1}} x_2 \quad ④$$

④ → ③ $\Rightarrow$

$$\left[p_1\left[\frac{p_1}{p_2}\right]^{\frac{1}{\rho-1}} + p_2\right] x_2 = w$$

$$\Rightarrow \quad x_2 = \left[p_1^{\frac{\rho-1}{\rho-1}} \cdot p_1^{\frac{1}{\rho-1}}\, p_2^{-\frac{1}{\rho-1}} + p_2^{\frac{\rho-1}{\rho-1}}\right]^{-1} w$$

$$= \left[\left[p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right] p_2^{-\frac{1}{\rho-1}}\right]^{-1} w$$

$$x_2 = \left(p_2^{\frac{1}{\rho-1}} w\right) \Big/ \left[p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right], \quad x_1 \text{ symmetric}$$

(d) $x_1(p, w) = \dfrac{p_1^{\frac{1}{\rho-1}}\, w}{[\qquad]}$ , $x_2 = \dfrac{p_2^{\frac{1}{\rho-1}}\, w}{[\qquad]}$

$$\frac{x_1}{x_2} = \left[\frac{p_1}{p_2}\right]^{1/\rho-1}$$

LET $z = p_1/p_2 \Rightarrow \dfrac{x_1}{x_2} = z^{1/\rho-1}$

$$\Rightarrow \frac{d}{dz}\left(x_1/x_2\right) = \frac{1}{1-\rho}\, z^{\frac{1}{\rho-1}-1}$$

$$\xi = \left[\frac{1}{1-\rho}\, z^{\frac{1}{1-\rho}-1}\right] \frac{z}{z^{\frac{1}{\rho-1}}}$$

$$= \frac{1}{1-\rho}\left(z^{\frac{1}{1-\rho}-1}\right)\left(z^{\frac{1}{1-\rho}-1}\right)^{-1}$$

$$= \frac{1}{1-\rho} \qquad \square$$

**MWG 3.D.6**

(a) SAY $\alpha+\beta+\gamma = k > 0$ SO THAT

$\frac{\alpha+\beta+\gamma}{k} = 1$. LET $f(z) = z^{1/k}$ SINCE $f' > 0 \ \forall k > 0$

$\tilde{u} = f(u)$ REPRESENTS THE SAME $\succsim$ AS $u$,

AND $\tilde{u} = (x_1 - b_1)^{\tilde{\alpha}} (x_2 - b_2)^{\tilde{\beta}} (x_3 - b_3)^{\tilde{\gamma}}$

WHERE $\tilde{\alpha} = \frac{\alpha}{k}$ ETC AND $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = 1$

(b) UMP MAX $(x_1 - b_1)^{\alpha} (x_2 - b_2)^{\beta} (x_3 - b_3)^{1-\alpha-\beta}$

S.T. $p_1 x_1 + p_2 x_2 + p_3 x_3 = w$ (1)

$\nabla u = \lambda p \implies \frac{\alpha}{x_1 - b_1} u = \lambda p_1$ (2)

$\frac{\beta}{x_2 - b_2} u = \lambda p_2$ (3)

$\frac{1-\alpha-\beta}{x_3 - b_3} u = \lambda p_3$ (4)

(2)+(3) $\implies \frac{\alpha}{(x_1 - b_1) p_1} = \frac{\beta}{(x_2 - b_2) p_2}$

$\implies (x_2 - b_2) = \frac{\beta p_1}{\alpha p_2} (x_1 - b_1) \implies$ (5) $x_2 = \frac{\beta p_1}{\alpha p_2} (x_1 - b_1) + b_2$

SIMILARLY

(2)+(4) $\implies$ $\implies$ (6) $x_3 = \frac{(1-\alpha-\beta) p_1}{\alpha p_3} (x_1 - b_1) + b_3$

(5) + (6) → (1) ⇒

$$p_1 x_1 + p_2\left[\frac{\beta p_1}{\alpha p_2}(x_1 - b_1) + b_2\right] + p_3\left[\frac{(1-\alpha-\beta)p_1}{\alpha p_3}(x_1 - b_1) + b_3\right] = \omega$$

$$\Rightarrow p_1 x_1 + \frac{\beta}{\alpha} p_1 (x_1 - b_1) + b_2 p_2 + \frac{\gamma}{\alpha} p_1 (x_1 - b_1) + b_3 p_3 = \omega$$

$$\Rightarrow p_1 x_1 + \frac{\beta}{\alpha} p_1 x_1 + \frac{\beta}{\alpha} p_1 b_1 + b_2 p_2 + \frac{\gamma}{\alpha} p_1 x_1 + \frac{\gamma}{\alpha} b_1 + b_3 p_3 = \omega$$

$$\Rightarrow p_1 x_1 \underbrace{\left[1 + \frac{\beta}{\alpha} + \frac{\gamma}{\alpha}\right]}_{k} = \omega - b_3 p_3 - b_2 p_2 - \underbrace{\left(\frac{\gamma}{\alpha} b_1 - \frac{\beta}{\alpha} b_1\right)}_{\tilde{k}} p_1$$

$$\Rightarrow x_1 = \frac{1}{k}\left(\frac{\omega}{p_1}\right) - \frac{b_3}{k}\frac{p_3}{p_1} - \frac{b_2}{k}\frac{p_2}{p_1} - \frac{\tilde{k}}{k}$$

$x_2$, $x_3$ SIMILAR.

SO, IF YOU ESTIMATE DEMAND EQUATIONS THAT ARE LINEAR IN RELATIVE PRICES, LIKE *, THE IMPLIED UTILITY FUNCTION LOOKS LIKE THE SAME UTILITY FUNCTION.

[IS THIS THE ONLY POSSIBLE UTILITY FUNCTION THAT COULD RESULT IN THESE DEMAND FUNCTIONS?]