EC2020-Fall 2011
Problem Set 3 solutions

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MT 3.1 (beware of typos – this is new)
We want to solve the following optimization problem:

$$\begin{aligned} \max \quad & (x_1x_2)^{1/2} \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 1 \\ & x_1 + 2x_2 \leq 1 \end{aligned}$$

Let λ be the multiplier for the first constraint, and let ϕ be the multiplier for the second constraint. Ignore the two non-negativity implicit constraints.

Our first order and K-T conditions are

$$\frac{(x_1x_2)^{1/2}}{2x_1} - 2\lambda - \gamma = 0 \tag{1}$$

$$\frac{(x_1x_2)^{1/2}}{2x_2} - \lambda - 2\gamma = 0 \tag{2}$$

$$(1 - 2x_1 - x_2)\lambda = 0,\ \lambda \geq 0,\ (1 - 2x_1 - x_2) \geq 0 \tag{3}$$

$$(1 - x_1 - 2x_2)\gamma = 0,\ \gamma \geq 0,\ (1 - x_1 - 2x_2) \geq 0 \tag{4}$$

There will generally be more than one way to proceed with this sort of problem. The principle is to check whether solutions exist to satisfy each/any possible combinations of constraints.

Case 1: $\lambda = 0$ **and** $\gamma > 0$ If we use 1 and 2 to solve for λ and γ, we get

$$\lambda = \frac{1}{3}\left(\frac{x_2}{x_1}\right)^{1/2} - \frac{1}{6}\left(\frac{x_1}{x_2}\right)^{1/2} \tag{5}$$

$$\gamma = \frac{1}{3}\left(\frac{x_1}{x_2}\right)^{1/2} - \frac{1}{6}\left(\frac{x_2}{x_1}\right)^{1/2} \tag{6}$$

$$\tag{7}$$

When $\lambda = 0$ 5 gives

$$\frac{1}{3}\left(\frac{x_2}{x_1}\right)^{1/2} = \frac{1}{6}\left(\frac{x_1}{x_2}\right)^{1/2} \tag{8}$$

Substituting 8 in 6 gives

$$\gamma = \frac{1}{2}\left(\frac{x_2}{x_1}\right)^{1/2} \tag{9}$$

Using 10 together with $\lambda = 0$ in 2 gives

$$x_1 = 2x_2 \tag{10}$$

If $\gamma > 0$ then from 4 we have $1 - 2x_1 - x_2 = 0$. Together with 10 this implies $(x_1,x_2) = (1/2,1/4)$. But from 3 we have $1 - 2x_1 - x_2 \geq 0$, a contradiction. Therefore, no solution exists when $\lambda = 0$ and $\gamma > 0$.

Case 2: $\lambda > 0$ **and** $\gamma = 0$ This is the same as case 1 by symmetry.

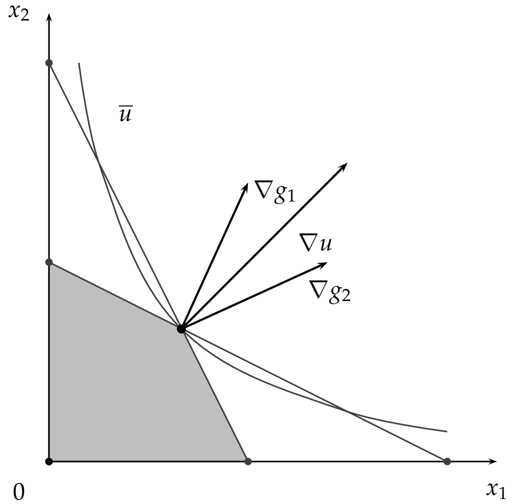
Case 3: $\lambda = 0$ **and** $\gamma = 0$ From 1 we have

$$\frac{(x_1x_2)^{1/2}}{2x_1} = 0.$$

This requires that $x_2 = 0$, which is not a maximum by inspection.
Case 3: $\lambda > 0$ **and** $\gamma > 0$ Thus far we have shown that there is no bundle satisfying the first order conditions if either or both of the constraints is slack. The only remaining possibility is that both constraints bind. In this case we must have

$$x_1 = x_2 = 1/3. \tag{11}$$

At this point, the first order conditions hold, the constraints hold, and, it is easy to check that $\gamma = \lambda = 1/6$, so this bundle satisfies the necessary conditions for a solution.



Here is the picture:

The constraint is shaded. Note that (1) the gradient of u lies in the convex cone formed by the constraint gradients and (2) constraint qualification holds.

Now we want to check second order conditions, and for practice, we'll do it a couple of different ways.

To show that $zD^2uz \leq 0$ for all $z \in R^2$, we first evaluate D^2u

$$u = (x_1x_2)^{1/2} \tag{12}$$

$$\nabla u = \begin{bmatrix} \frac{1}{2}x_1^{-1/2}x_2^{1/2} \\ \frac{1}{2}x_1^{1/2}x_2^{-1/2} \end{bmatrix} \tag{13}$$

$$D(\nabla u) = \begin{bmatrix} -\frac{1}{4}x_1^{-3/2}x_2^{1/2} & \frac{1}{4}x_1^{-1/2}x_2^{-1/2} \\ \frac{1}{4}x_1^{-1/2}x_2^{-1/2} & -\frac{1}{4}x_1^{1/2}x_2^{-3/2} \end{bmatrix} \tag{14}$$

Evaluating this last expression at $x_1 = x_2 = 1/3$ we have

$$D^2u = \begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{bmatrix} \tag{15}$$

If we pre- and post multiply by an arbitrary vector,

$$\begin{bmatrix} z_1z_2 \end{bmatrix} \begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{16}$$

we get $\frac{3}{4}(z_1 - z_2)^2$ which is non-positive for all $(z_1, z_2) \in R^2$, so D^2u is negative semi-definite, and the second order sufficient conditions hold. Thus, we have a solution to our problem.

We can also use the determinant based test of Theorem MD2 to check for concavity.

$$(-1)^1 \left| -\frac{3}{4} \right| \geq 0 \tag{17}$$

$$(-1)^2 \begin{vmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{vmatrix} = \left(\frac{9}{16} - \frac{9}{16} \right) \geq 0. \tag{18}$$

So D^2u is concave (again).

Finally, and also for practice, we want to check whether u is quasi-concave.

First note that $\nabla u|_{x^*} = (1/2, 1/2)$. We want to find the set of vectors z such that $z\nabla u = 0$. That is,

$$(z_1, z_2) \cdot (1/2, 1/2) = 0$$

By inspection, this requires that $z_1 = -z_2$. Thus, we want to check that for any $\alpha \in R$, if we pre and post multiply D^2u by $(\alpha, -\alpha)$, the result is non-positive. The mechanic of this exercise are very similar to one already completed above, and allows us to conclude that u is quasi-concave, a weaker sufficient condition for a maximum.

$$D^2u = \begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{bmatrix} \tag{19}$$

MT 3.2

SAY $u(x)$ CONTINUOUS AND $x(p,w)$ CONTINUOUS SHOW $u(x(p,w))$ CONTINUOUS.

$u(x)$ CONTINUOUS $\Leftrightarrow$ $\forall \delta > 0$, $\exists \varepsilon > 0$ S.T. IF $\|x - x'\| < \varepsilon$, THEN $|u(x) - u(x')| < \delta$

$x(p,w)$ CONTINUOUS $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists \gamma > 0$ S.T. IF $\|(p,w) - (p',w')\| < \gamma$, THEN $\|x - x'\| < \varepsilon$.

THUS, GIVEN $\delta > 0$, $\exists \varepsilon > 0$ S.T. IF $\|x(p,w) - x(p',w')\| < \varepsilon$ THEN $|V(p,w) - V(p',w')| < \delta$

AND $\exists$ $\gamma > 0$ S.T. IF $\|(p,w) - (p',w')\| < \gamma$ THEN $\|x - x'\| < \varepsilon$

IT FOLLOWS THAT FOR ALL $\delta > 0$, $\exists \gamma > 0$ S.T. IF $\|(p,w) - (p',w')\| < \gamma$ THEN $\|V(p,w) - V(p',w')\| < \delta$ $\square$

MT 3.3 (beware of typos – this is new)

Suppose that f is a discontinuous function from (0,1) to (0,1). Show that f is not concave.

To get the intuition, consider $f(x) = 0$ for $x \leq 1/2$ and $f(x) = 1$ for $x > 1/2$. Then $f(1/4) = 0$ and $f(3/4) = 1$ while

$$f(1/2) < \frac{1}{2}f(1/4) + \frac{1}{2}f(3/4),$$

so that f is not concave. We want to generalize intuition to functions that are worse behaved, e.g., $f(x) = 1$ at all rational numbers in (0,1) and 0 otherwise.

Without loss of generality, consider a discontinuity that occurs at $\alpha \in (0,1)$, and that for any $\epsilon > 0$ there exists δ such that for some $x \in (\alpha, \alpha + \epsilon)$, $f(x) > f(\alpha) + \delta$. That is, there is a discontinuous step up at α. Since f is bounded below at 0, for any $\gamma > 0$, $f(\alpha - \gamma) \geq 0$. The set of convex combinations of $(\alpha - \gamma, 0)$ and $(\alpha + \epsilon, f(\alpha + \epsilon))$ must lie below the set of convex combinations of $(\alpha - \gamma, f(\alpha - \gamma))$ and $(\alpha + \epsilon, f(\alpha + \epsilon))$, so if we can show that the first line passes above $f(\alpha)$ then we are done.

At α, the height of the line describing the first set of convex combinations is $\gamma \frac{f(\alpha)+\delta}{\gamma+\epsilon}$. By choosing ϵ sufficiently close to α we guarantee that this line passes above $f(\alpha)$ and we are done.