EC2020-Fall 2011
Problem Set 4 solutions

(Updated: 17 August 2011)

Matt Turner

MWG 3.G.1

WE KNOW THAT $w = e(p, v(p,w))$ *

DIFFERENTIATING * W.R.T. TO w GIVES

$$1 = \frac{d}{dw} e(p, v(p,w)) \qquad (1)$$

$$\Rightarrow \quad 1 = e_u \cdot \partial v / \partial w \quad \Rightarrow \quad e_u = \frac{1}{\partial v / \partial w}$$

DIFFERENTIATING * W.R.T. p GIVES

$$0 = \frac{\partial e}{\partial p_\ell} + \frac{\partial e}{\partial u} \cdot \partial v / \partial p_\ell \qquad (2)$$

(1) → (2) $\Rightarrow$
$$0 = \frac{\partial e}{\partial p_\ell} + \frac{\partial v / \partial p_\ell}{\partial v / \partial w}$$

USING ROY'S IDENTITY $\frac{\partial e}{\partial p_\ell} = x(p,w) = h(p,u)$

[I DON'T LIKE THIS TOO WELL, BUT IT'S THE BEST I COULD WORK OUT]

MWG 3.G.4

SAY $u(x) = \sum_\ell u_\ell(x_\ell)$. SHOW THAT ADDITIVE SEPARABILITY IS PRESERVED ONLY UNDER LINEAR TRANSFORMATIONS.

(a) $\Rightarrow$ LET $f(z) = \alpha z + \beta$.

THEN $f(u(x)) = f\left(\sum_\ell u_\ell(x_\ell)\right)$

$$= \alpha \sum_\ell u_\ell(x_\ell) + \beta$$

$$= \sum_\ell \alpha u_\ell(x_\ell) + \frac{\beta}{L} \quad \square$$

$\Leftarrow$ LET $f(z)$ S.T. $f' > 0$ AND $f'' \neq 0$

THEN $f(u(x)) \cong f(u^0) + f'(u^0)(u - u^0) + f''(u^0)\frac{1}{2}(u - u^0)^2$

LET $\alpha = f'(u_0)$, $\beta = f(u_0)$

$\Rightarrow f(u(x)) \approx \left[\sum_\ell \beta u_\ell(x_\ell) + \frac{-\alpha\beta + \beta}{L}\right]$ + NON-LINEAR TERM. $\square$

(b) PARTITION x INTO $y = (x_1, \ldots, x_k)$, $z = (x_{k+1}, \ldots, x_L)$

THEN $u(x) = \sum_1^k u_m(x_m) + \sum_{k+1}^L u_m(x_m)$

$$= \tilde{u}(y) + \hat{u}(z)$$

CONSIDER $x = (y, z)$ AND $x' = (y, z')$
$\hat{x} = (\hat{y}, z)$ AND $\hat{x}' = (\hat{y}, z')$

THEN $u(x) \geq u(x') \Leftrightarrow u(\hat{x}) \geq u(\hat{x}')$ $\square$

(c) SAY x SOLVES $\max \sum u_\ell(x_\ell)$

S.T. $px = \omega$

$$\Rightarrow \quad \frac{du_1}{dx_1} = \lambda p_1 \quad \text{AND} \quad \frac{du_2}{dx_2} = \lambda p_2$$

$$\Rightarrow \quad \frac{du_1}{dx_1}\frac{1}{p_1} = \frac{du_2}{dx_2}\frac{1}{p_2}$$

$$\Rightarrow \quad \frac{du_1/dx_1}{du_2/dx_2} \; * = \frac{p_1}{p_2}$$

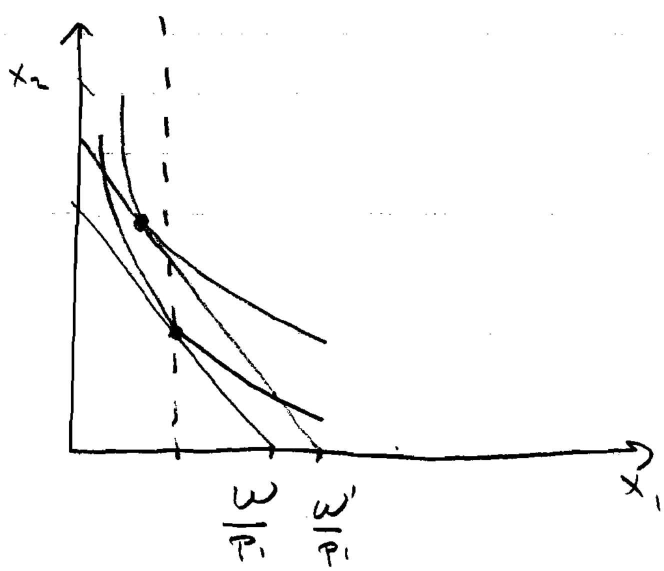
BY ADDITIVE SEPARABILITY, WE KNOW THAT $\frac{d}{dx_2}\left(\frac{du_1}{dx_1}\right) = 0$

BY CONCAVITY OF u_ℓ $\frac{d}{dx_2}\left(\frac{du_2}{dx_2}\right) < 0$.

THUS * MUST DECREASE AS x_2 INCREASES.

BUT A NECESSARY CONDITION FOR x_1 AN INFERIOR GOOD IS THAT * INCREASE AS x_2 INCREASES.

TO SEE THIS,



FOR x_1 TO BE INFERIOR WE MUST HAVE INDIF CURVES GET FLATTER AS x_2 INCREASES, HOLDING x_1 CONSTANT.

BUT SEPARABILITY RULES THIS OUT, GIVEN CONCAVITY.

MWG 3.G.9

COMPUTE THE SLUTSKY MATRIX FROM $V(p,w)$

USING ROY'S IDENTITY I CAN CALCULATE

$$x_\ell(p,w) = -\frac{\partial V/\partial p_\ell}{\partial V/\partial w}$$

DIFFERENTIATING AGAIN I CALCULATE $\frac{\partial x_\ell}{\partial w}$ AND $\frac{\partial x_\ell}{\partial p_\ell}$.

JUST WORK OUT DIAGONAL ELEMENTS TO LIGHTEN NOTATION.

BUT A DIAGONAL ELEMENT OF SLUTSKY TERM IS

THUS $$S_{kk} = \frac{\partial x_k}{\partial p_k} + \frac{\partial x_k}{\partial w} \cdot x_k$$

$$= \frac{\partial}{\partial p_k}\left[\frac{-\partial V/\partial p_k}{\partial V/\partial w}\right] + \frac{\partial}{\partial w}\left[\quad\right]\cdot\left[\quad\right]$$

OFF-DIAGONAL ELEMENTS SIMILAR $\square$

MWG 3.G.14

$p_1 = 1, \quad p_2 = 2, \quad p_3 = 3$

$$S = \begin{bmatrix} -10 & ? & ? \\ ? & -4 & ? \\ 3 & ? & ? \end{bmatrix}$$

SINCE S SYMMETRIC

$$S = \begin{bmatrix} -10 & ? & 3 \\ ? & -4 & ? \\ 3 & ? & ? \end{bmatrix}$$

SINCE $D_p h = S$ WE HAVE $\begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \frac{\partial h_1}{\partial p_2} & \frac{\partial h_1}{\partial p_3} \\ \frac{\partial h_2}{\partial p_1} & \frac{\partial h_2}{\partial p_2} & \frac{\partial h_2}{\partial p_3} \\ \frac{\partial h_3}{\partial p_1} & \frac{\partial h_3}{\partial p_2} & \frac{\partial h_3}{\partial p_3} \end{bmatrix} = S$

BUT $p \cdot D_p h = 0$

$$\Rightarrow p_1 \frac{\partial h_1}{\partial p_1} + p_2 \frac{\partial h_1}{\partial p_2} + p_3 \frac{\partial h_3}{\partial p_3} = 0$$

$$\Rightarrow -10 + 2 \frac{\partial h_1}{\partial p_2} + 9 = 0$$

$$\Rightarrow \frac{\partial h_1}{\partial p_2} = \frac{1}{2}$$

THIS LEADS TO

$$S = \begin{bmatrix} -10 & \frac{1}{2} & 3 \\ \frac{1}{2} & -4 & ? \\ 3 & ? & ? \end{bmatrix}$$

USING THE SAME TRICK ON THE SECOND ROW, WE HAVE

$$\frac{1}{2} - 8 + 3\,\frac{\partial h_2}{\partial p_3} = 0$$

$$\Rightarrow \quad \frac{\partial h_2}{\partial p_3} = \frac{5}{2}$$

SO

$$S = \begin{bmatrix} -10 & \frac{1}{2} & 3 \\ \frac{1}{2} & -4 & 5/2 \\ 3 & 5/2 & ? \end{bmatrix}$$

ONE MORE TIME

$$\Rightarrow \quad 3 + 5 + 3 \cdot \frac{\partial h_3}{\partial p_3} = 0 \Rightarrow \frac{\partial h_3}{\partial p_3} = -\frac{8}{3}$$

AND WE HAVE

$$S = \begin{bmatrix} -10 & 1/2 & 3 \\ 1/2 & -4 & 5/2 \\ 3 & 5/2 & -8/3 \end{bmatrix}$$

MWG 3.G.16

$$e(p,u) = \text{EXP}\left[\sum \alpha_\ell \ln p_\ell + \left(\prod p_\ell^{\beta}\right) u\right]$$

TO EASE NOTATION, LET $L=2$

$$\Rightarrow \quad e(p_1,p_2,u) = \text{EXP}\left[\alpha_1 \ln p_1 + \alpha_2 \ln p_2 + (p_1 p_2)^{\beta} u\right]$$

RECALLING THAT $e^{x+y} = e^x e^y$ AND $e^{\ln x} = x$

$$\Rightarrow \quad e(p_1,p_2,u) = p_1^{\alpha_1} p_2^{\alpha_2} e^{[(p_1 p_2)^{\beta}]u}$$

(a)

WE NEED RESTRICTIONS ON α_1, α_2 SO THAT e IS

(1) H1

(2) $D_2 e$ NEG. SEMI-DEF

(3) $e_p \geq 0$, $e_u > 0$

(4) $p D_{pp} e = 0$

(b) $v(p,w)$ MUST SOLVE

$$w = p_1^{\alpha_1} p_2^{\alpha_2} \text{EXP}\left((p_1 p_2)^{\beta} v(p,w)\right) \quad (1)$$

AND

$$u = v\left(p, p_1^{\alpha_1} p_2^{\alpha_2} \text{EXP}\left((p_1 p_2)^{\beta} u\right)\right) \quad (2)$$

(1) LOOKS MORE PROMISING, SO START WITH THAT. →

TRY $\quad V(p,w) = \dfrac{1}{(p_1p_2)^\beta} \ln\left[\dfrac{1}{p_1^{\alpha_1}p_2^{\alpha_2}} w\right]$

$$\Rightarrow \quad e\big(p, V(p,w)\big) = p_1^{\alpha_1} p_2^{\alpha_2} \,\mathrm{EXP}\left[(p_1p_2)^\beta \left[\frac{1}{(p_1p_2)^\beta} \ln\left(\frac{1}{p_1^{\alpha_1}p_2^{\alpha_2}}\right) w \right]\right]$$

$$= p_1^{\alpha_1} p_2^{\alpha_2} \,\mathrm{EXP}\left[\ln \frac{1}{p_1^{\alpha_1}p_2^{\alpha_2}} w \right]$$

$$= p_1^{\alpha_1} p_2^{\alpha_2} \left[\frac{w}{p_1^{\alpha_1}p_2^{\alpha_2}} \right]$$

$$= w$$

(c) I DON'T SEE HOW TO "VERIFY" ROY'S IDENTITY HERE SINCE IT JUST INVOLVES $V(p,w)$.

WE CAN CHECK SLUTSKY EQN BY

(1) CALCULATING $x(p,w)$ FROM V
CALCULATING $\dfrac{\partial x_\ell}{\partial p_k}$ AND $\dfrac{\partial x_\ell}{\partial w}$

(2) CALCULATING $h(p,u)$ FROM $e(p,u)$

(3) PLUGGING (1) + (2) INTO SLUTSKY AND CHECKING THAT THEY HOLD.

MWG 3.H.6

INTEGRABILITY PROBLEM IS

$$\frac{\partial e}{\partial p_\ell} = x_\ell(p, e(p))$$

HERE THAT GIVES

$$\frac{\partial e}{\partial p_\ell} = \frac{1}{p_\ell}\alpha_\ell\, e(p_\ell)$$

TRY $\quad e(p_\ell) = \text{EXP}(\alpha_\ell \ln p_\ell) + k_\ell(p_{-\ell})$

THEN $\quad \frac{\partial e}{\partial p_\ell} = \frac{\alpha}{p_\ell} e(p_\ell)$

REWRITING WE HAVE $e(p_\ell) = p_\ell^{\alpha_\ell} + k_\ell(p_{-\ell}) \quad \forall \ell.$

SO $\quad e(p, u) = p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots p_L^{\alpha_L} + k + f(u)$

SOLVES THIS SYSTEM,

AN UPPER CONTOUR SET $= \{x \in X \mid p \cdot x \geq e(p, u)\ \forall p \gg 0\}$

FOR $L=2$ AND $\bar{u}$ WE HAVE $V_u = \{x \in X \mid p_1 x_1 + p_2 x_2 \geq p_1^{\alpha_1} + p_2^{\alpha_2} + k\}$

SO INDIF CURVE IS $\quad p_1 x_1 + p_2 x_2 = p_1^{\alpha_1} + p_2^{\alpha_2} + k \qquad \forall p \gg 0$

SO

$$x_1 = p_1^{\alpha-1} + \frac{\tilde{k}}{p_1} \qquad x_2 = p_2^{\alpha_2 - 1} + \frac{\hat{k}}{p_2}$$

MWG 3.I.1

(WITH HELP FROM MANUAL)

WE ARE ASKED TO CALCULATE $F(x_0, y_0) - F(x_1, y_1)$ BY INTEGRATING $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ ALONG A PATH CONNECTING (x_0, y_0) TO (x_1, y_1). SINCE OUR ANSWER WON'T DEPEND ON THE PATH WE TAKE, WE ARE FREE TO CHOOSE ONE THAT IS CONVENIENT. e.g. $(x_0, y_0) \to (x_1, y_0) \to (x_1, y_1)$.

THUS FOR PRICE CHANGE FROM $(p_0, p_1) \to (q_0, q_1)$ WE HAVE

$$EV(p, q, w) = e(p, u^q) - e(q, u^q)$$

$$= e(p_0, p_1, u^q) - e(q_0, p_1, u^q) + e(q_0, p_1, u^q) - e(q_0, q_1, u^q)$$

$$= \int_{q_0}^{p_0} h_0(s, p_1, u^q)\,ds + \int_{q_1}^{p_1} h_1(q_0, s, u^q)\,ds \quad *$$

$$CV(p, q, w) = e(p, u^p) - e(q, u^p)$$

$$= e(p_0, p_1, u^p) - e(q_0, p_1, u^p) + e(q_0, p_1, u^p) - e(q_0, q_1, u^p)$$

$$= \int_{q_0}^{p_0} h_0(s, p_1, u^p)\,ds + \int_{q_1}^{p_1} h_1(q_0, s, u^p)\,ds \quad **$$

SINCE * AND ** DIFFER ONLY IN UTILITY LEVEL, IF THERE ARE NO WEALTH EFFECTS h_i INVARIANT TO u WE HAVE $CV = EV$.

MWG 3.I.2

TAX t ON GOOD 1, SO

$$(q_0, q_1) = (p_0, p_1 + t)$$

$$L = -T - EV(p, q, w) = e(q, u^q) - e(p, u^q) - T$$

$$= e(p_0, p_1 + t, u^q) - e(p_0, p_1) - T$$

$$L = \int_{p_1}^{p_1+t} h_1(p_0, s, u^q)\,ds - t\, h_1(p_0, p_1 + t, u^q)$$

RECALLING THAT $\frac{d}{dz}\int_A^z f(x)dx = f(z)$ WE HAVE

$$\frac{dL}{dt} = h_1(p_0, p_1 + t, u^q) - \left[h_1(p_0, p_1 + t, u^q) + t\frac{\partial h_1}{\partial p_1}\right]$$

$$= -t\frac{\partial h_1}{\partial p_1}$$

PROVIDED h_1 IS DIFFERENTIABLE AT p_0, p_1 AND $\frac{\partial h}{\partial p_1} > 0$ THEN $\frac{dL}{dt} > 0$ AND $\frac{dL}{dt} \to 0$ AS $t \to 0$

MWG 3.I.3

(WITH HELP FROM MANUAL)

$$p = (p_0, p_1) \qquad q = (q_0, p_1) \quad \text{WITH} \quad q_0 < p_1$$

LET $u^p = V(p, \omega)$ $\quad u^q = V(q, \omega)$. IT FOLLOWS THAT $u^p \le u^q$.

SINCE $\frac{\partial e}{\partial u} > 0$

$$\Rightarrow \quad e(p, p_1, u^p) \le e(p, p_1, u^q) \quad \forall p \gg 0$$

SINCE GOOD 0 INFERIOR WE HAVE

$$x_0(p, p_1, e(p, p_1, u^p)) \le x_0(p, p_1, e(p, p_1, u^q))$$

$$\forall p \gg 0$$

BUT $h(p, u) = x(p, e)$

$$\Rightarrow \quad h_0(p, p, u^p) \le h(p, p_1, u^q) \quad \forall p \gg 0$$

BUT $CV = \int h_0(p, p_1, u^q)\, dp$

$EV = \int h_0(p, p_1, u^p)\, dp$

SO $CV > EV$ IF x INFERIOR.