EC2020-Fall 2011
Problem Set 5 solutions

(Updated: 17 August 2011)

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MWG 4.B.1

(a) IF $\quad V_i(p, w_i) = a_i(p) + b(p)\, w_i$

THEN $\quad x_{i\ell}(p, w_i) = -\dfrac{\partial a_i/\partial p_\ell + \partial b/\partial p_\ell \; w_i}{b(p)}$

$$= -\left[\frac{\partial a_i/\partial p_\ell}{b(p)} + \frac{\partial b/\partial p_\ell}{b(p)}\, w_i\right]$$

$$\Rightarrow \quad \frac{\partial x_{i\ell}}{\partial w} = \frac{-\partial b/\partial p_\ell}{b(p)} \quad \text{i.e. A CONSTANT.}$$

THUS INCOME EXPANSION PATHS ARE LINEAR FOR ALL i.

(b) $\quad u = a_i(p) + b(p)\, w_i$

$$\Rightarrow \quad w_i = \frac{u}{b(p)} - \frac{a_i(p)}{b(p)}$$

$$\Rightarrow \quad e(p, u) = c(p)\, u + d_i(p)$$

FOR $\quad c(p) = \dfrac{1}{b(p)} \quad d_i(p) = \dfrac{a_i(p)}{b(p)}$

MWG 4.B.2

I CONSUMERS, L COMMODITIES

CONSUMERS DIFFER ONLY IN WEALTH LEVELS w_i

AND TASTE PARAMETER s_i

WE HAVE $v(p, w_i, s_i)$ AND $x(p, w_i, s_i)$

(a) FIX $(s_1, \ldots, s_I)$. SHOW THAT IF FOR ANY $(w_1, \ldots, w_I)$ AGG DEMAND CAN BE WRITTEN AS $x(p, \Sigma w_i)$ AND $\succsim_i$ HOMOTHETIC, THEN ALL $\succsim_i$ ARE IDENTICAL.

CONSIDER CONSUMERS i AND j. THEN

$$x_i(p, \Sigma w_k, s_i) = x_j(p, \Sigma w_k, s_j) = x(p, \Sigma w_i)$$

$$\Rightarrow \quad x_i(p, \Sigma w_k, s_i) = x_j(p, \Sigma w_k, s_j).$$

BY CHOOSING $(w_1, \ldots, w_I)$ S.T. $\Sigma w_i = w$ WE HAVE

$$x_i(p, w, s_i) = x_j(p, w, s_j) \qquad \forall w \gg 0$$

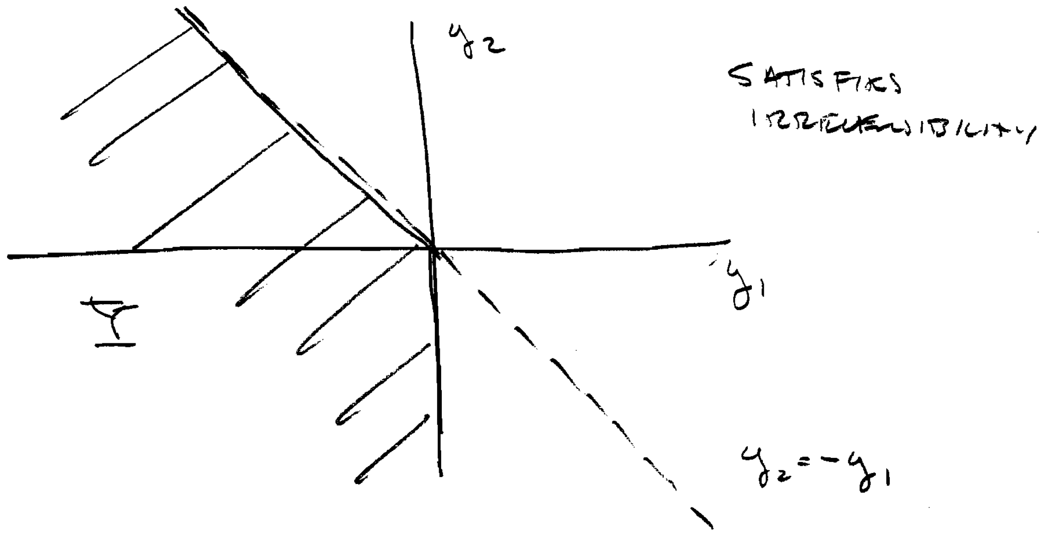
$\Rightarrow$ i AND j HAVE SAME $\succsim$ AND THE s_i DON'T MATTER.

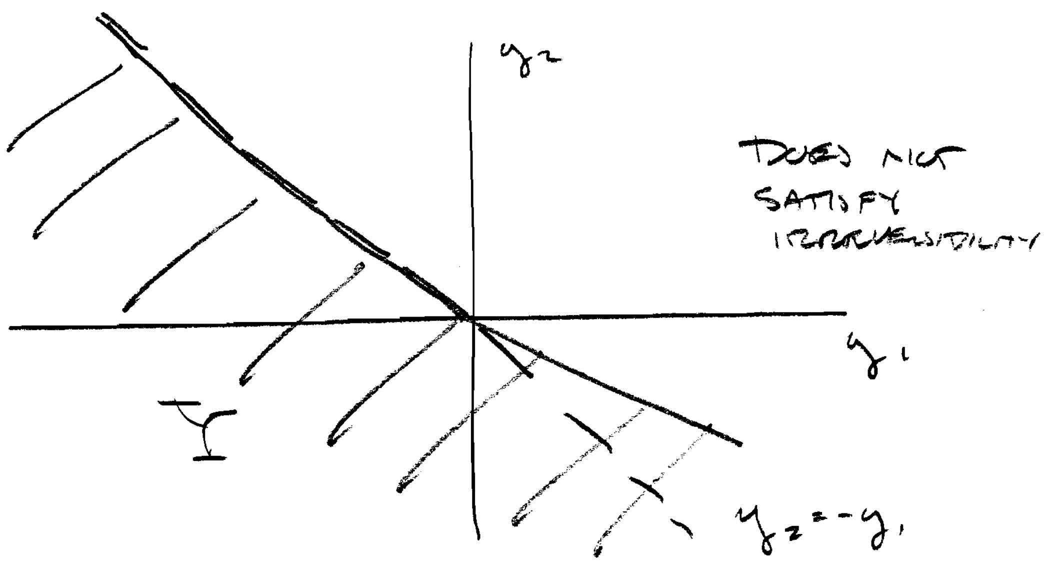
(b) THIS IS A BIT SILLY. ONE SILLY ANSWER IS IF

$$u_i = \min(x_1, \ldots, x_L) \qquad \forall i.$$

THEN THE s_i DON'T MATTER AND AGG DEMAND ALLOCATED ALL INCOME TO LOWEST PRICED GOOD.

MWG 5.B.1


y_2
SATISFIES IRREVERSIBILITY
y_1
Y
$y_2 = -y_1$


y_2
DOES NOT SATISFY IRREVERSIBILITY
y_1
Y
$y_2 = -y_1$

MWG 5.B.2

C.R.S. $\iff$ Y IS CRS IF AND ONLY IF

$y \in Y \Rightarrow \alpha y \in Y \quad \forall \alpha \geq 0.$

$f(z)$ IS H1 $\iff$ $\alpha f(z) = f(\alpha z) \quad \forall \alpha \geq 0$

SHOW CRS $\iff$ $f(z)$ IS H1.

SAY $f(z)$ IS H1 THEN $y = (f(z), z) \in Y$ BY DEFINITION OF f AND $(\alpha f(z), \alpha z) \quad \forall \alpha \geq 0$ IS FEASIBLE BY H1, THUS $\alpha y \in Y \quad \forall \alpha \geq 0.$

SAY Y IS CRS AND $y \in Y$ WITH $y = (f(z), z)$. BY CRS $\alpha y = (\alpha f(z), \alpha z) \in Y$ AND, SINCE y IS ON THE BOUNDARY OF Y, SO IS αy. THUS $\alpha y = (f(\alpha z), \alpha z)$, BY THE DEFINITION OF "PRODUCTION FUNCTION". IT FOLLOWS THAT

$$f(\alpha z) = \alpha f(z) \quad \forall \alpha \geq 0$$

$$\iff$$

$$f \text{ H1}$$

$\square$

MWG 5.B.3

① Y CONVEX $\Leftrightarrow$ GIVEN $y, y' \in Y$ THEN
$\alpha y + (1-\alpha) y' \in Y \quad \forall \alpha \in [0,1]$.

② $f(z)$ CONCAVE $\Leftrightarrow$ GIVEN z, z'
$f(\alpha z + (1-\alpha) z') \geq \alpha f(z) + (1-\alpha) f(z')$.

SHOW ① $\Leftrightarrow$ ②

① $\Rightarrow$ ② CONSIDER y, y' ON BOUNDARY OF Y.

THEN $y = (f(z), z)$, $y' = (f(z'), z)$

BY CONVEXITY $\alpha y + (1-\alpha) y' \in Y$

$\Rightarrow \quad \alpha f(z) + (1-\alpha) f(z') \leq f(\alpha z + (1-\alpha) z')$

$\Rightarrow \quad f$ CONCAVE.

② $\Rightarrow$ ① CONSIDER $y = (f(z), z)$ AND $y' = (f(z'), z')$

BY CONCAVITY $\alpha f(z) + (1-\alpha) f(z') \leq f(\alpha z + (1-\alpha) z')$

$\Rightarrow \quad (\alpha f(z) + (1-\alpha) f(z'), \alpha z + (1-\alpha) z') \in Y$

$\Rightarrow \quad Y$ CONVEX $\quad \square$

MWG 5.C.1

LET $\pi^* = \max py$

S.T. $y \in Y$

SAY $\pi^* \le 0$ AT y^* THEN $\pi(y=0) = 0$

SO $\pi^* = 0$

SAY $\pi(y) > 0$ SOME $y \in Y$

BY NON-DECREASING RETURNS TO SCALE,

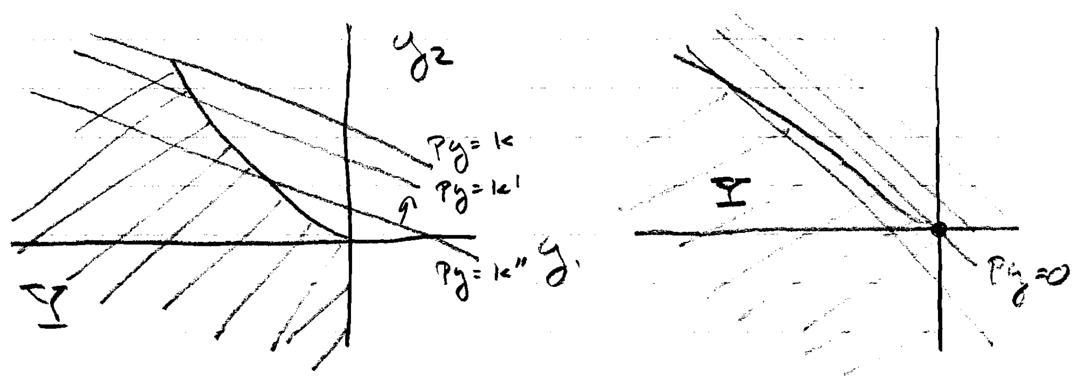
$\alpha y \in Y \quad \forall \alpha \ge 1$

$\Rightarrow \quad \pi(\alpha y) = p \alpha y = \alpha \pi(y)$

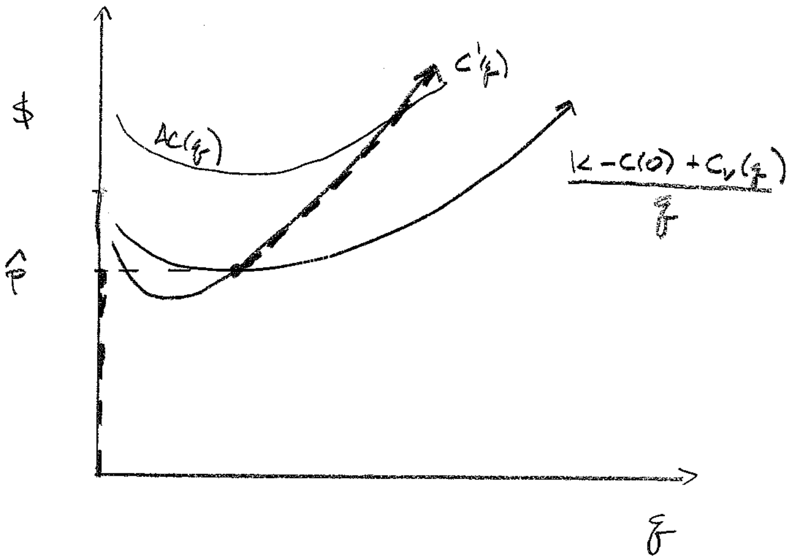
BUT $\alpha y \in Y$ FOR ARBITRARILY LARGE α

$\Rightarrow \pi^* = \infty \quad \square$

THIS PICTURE IS A BETTER WAY TO SEE THIS:



MWG 5.D.2



$$y^*(p) = \begin{cases} 0 & p < \hat{p} \\ MC(q) & p \geq \hat{p} \end{cases}$$

FOR $p < \hat{p}$ $\quad \pi = -C(0)$

FOR $p \geq \hat{p}$ $\quad \pi = \hat{p}\, q - (K + C_v(q))$

$$> K - C(0) + C_v(q) - (\qquad)$$

$$= -C(0)$$

WITH PARTIALLY SUNK COSTS, FIRM IGNORES SUNK PORTION OF COSTS AND ACTS AS IF IT FACES FIXED COST $K - C(0)$.

[a] $\succsim$ Homothetic $\iff \left[x \succ y \iff \beta x \succ \beta y \quad \beta \in \mathbb{R}_{++} \right]$

Say $u(x_1, x_2) \geq u(y_1, y_2)$

$\iff x_1^{\alpha} x_2^{1-\alpha} \geq y_1^{\alpha} y_2^{1-\alpha}$

$\iff \beta x_1^{\alpha} x_2^{1-\alpha} \geq \beta y_1^{\alpha} y_2^{1-\alpha}$

$\iff$

$(x_1\beta)^{\alpha}(x_2\beta)^{1-\alpha} \geq (y_1\beta)^{\alpha}(y_2\beta)^{1-\alpha}$

$\iff u(\beta x_1, \beta x_2) \geq u(\beta y_1, \beta y_2)$

[b] $\max \; x_1^{\alpha} x_2^{1-\alpha}$

S.T. $p_1 x_1 + p_2 x_2 = \omega_i$ (2)

F.O.C. $\frac{\alpha}{x_1} x_1^{\alpha} x_2^{1-\alpha} = \lambda p_1$

$\frac{1-\alpha}{x_2} x_1^{\alpha} x_2^{1-\alpha} = \lambda p_2$

$\Rightarrow \frac{\alpha}{x_1 p_1} = \frac{1-\alpha}{x_2 p_2}$

$\Rightarrow \alpha p_2 x_2 = (1-\alpha) x_1 p_1$

$\Rightarrow x_2 = \frac{(1-\alpha) p_1}{\alpha \; p_2} x_1$ (1)

(1) → (2)

$\Rightarrow p_1 x_1 + \frac{1-\alpha}{\alpha} p_1 x_1 = \omega_i$

$\Rightarrow x_1 \left(\frac{\alpha + (1-\alpha)}{\alpha} \right) = \frac{\omega_i}{p_1}$

$\Rightarrow x_1(p, \omega_i) = \frac{\alpha \omega_i}{p_1}$ (3) $\longrightarrow$

(3) + (1) $\Rightarrow$ $x_2(p, w_i) = \dfrac{(1-\alpha) w_i}{p_2}$

THUS

$$X_1(p, w_1, \ldots, w_I) = \sum \frac{\alpha}{p_1} w_i$$

$$= \frac{\alpha}{p_1} w \qquad \text{FOR} \quad w = \sum w_i$$

SIMILARLY

$$X_2(p, w_1, \ldots, w_I) = \frac{1-\alpha}{p_2} w$$

THUS AGGREGATE DEMAND DEPENDS ONLY ON AGGREGATE INCOME AND NOT ON HOW IT IS DISTRIBUTED.

[c] IN LIGHT OF [b] CONSUMER 1'S DEMAND WILL COINCIDE WITH AGGREGATE DEMAND.

[d] IN LIGHT OF [b] WE CAN WRITE

$$V_i(p, w_i) = x_1(p, w_i)^{\alpha} x_2(p, w_i)^{1-\alpha}$$

$$= \left[\frac{\alpha}{p_1} w_i\right]^{\alpha} \left[\frac{1-\alpha}{p_2} w_i\right]^{1-\alpha}$$

$$= \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} w_i$$

LET $k(p_1, p_2) = \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}$

SO $V_i(p, w_i) = k(p_1, p_2) w_i$ #

THE PLANNER WANTS TO SOLVE

$$V(p,\omega) = \max_{\omega_1,\dots,\omega_I} \sum V_i(p,\omega_i)$$

$$\text{S.T.} \quad \sum \omega_i = \omega$$

$$\Rightarrow V(p,\omega) = \max_{\omega_1,\dots,\omega_I} k(p) \sum \omega_i$$

$$\text{S.T.} \quad \sum \omega_i = \omega$$

$$\Rightarrow V(p,\omega) = \max k(p)\,\omega$$

$$\text{S.T.} \quad \sum \omega_i = \omega$$

$$\Rightarrow \quad V(p,\omega) = k(p)\,\omega \qquad \# \#$$

THAT IS, V IS INVARIANT TO HOW WEALTH IS DISTRIBUTED.

[e] FOR INDIVIDUALS WE HAVE, FROM # THAT

$$u_i = k(p)\,\omega_i$$

$$\Rightarrow \quad \omega_i = k^{-1}(p)\,u_i$$

SO $$e_i(p,u^0) = \left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha} u_i^0$$

FOR A CHANGE IN p_1 FROM p_1^0 AND u_i^0 TO p_1' AND u_i'

$$CV_i(p,\omega_i) = \omega_i - e(p_1', \bar{p}_2, u_i^0)$$

$$= \omega_i - \left[\frac{p_1'}{\alpha}\right]^{\alpha}\left[\frac{\bar{p}_2}{1-\alpha}\right]^{1-\alpha} u_i^0$$

$\rightarrow$

THUS, $\sum CV_i = w - \left[\frac{p_1}{\alpha}\right]^{\alpha}\left[\frac{p_2}{1-\alpha}\right]^{1-\alpha}\sum u_i^0$

FOR PLANNER WE HAVE, FROM ##

$$V(p,w) = k(p)w$$

$$\Rightarrow \quad u = k(p)\,w$$

$$\Rightarrow \quad e(p_1,p_2,u) = k(p)^{-1}u$$

$$\Rightarrow \quad CV(p,w) = w - \left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}u^0$$

BUT $u^0 = \sum_i u_i^0$ FOR ALL POSSIBLE WEALTH DISTRIBUTIONS (FROM (d)).

$$\Rightarrow \quad CV(p,w) = \sum_I CV_i(p,w_i)$$

MT 5.1

$\pi(\cdot)$ AND $y(\cdot)$ PROFIT AND SUPPLY FUNCTIONS FOR Y.
Y CLOSED AND SATISFIES FREE DISPOSAL.

(IV) $y(\cdot)$ IS H0

IF y^* SOLVES $\max py$ s.t. $y \in Y$

THEN y^* SOLVES $\max \alpha py$ s.t. $y \in Y$

$\Rightarrow$ $y(\cdot)$ IS H0.

(I) $\pi(\cdot)$ IS H1 FOLLOWS IMMEDIATELY FROM (IV).

(II) $\pi(\cdot)$ CONVEX: CONSIDER p, p' AND $p'' = \alpha p + (1-\alpha)p'$
$\alpha \in (0,1)$

$$\pi(p'') = \pi(\alpha p + (1-\alpha)p')$$
$$= [\alpha p + (1-\alpha)p'] \; y(\alpha p + (1-\alpha)p')$$
$$= \alpha p \, y(p'') + (1-\alpha)p' \, y(p'')$$

FROM DEF OF y $\le \alpha p \, y(p) + (1-\alpha) p' y(p')$

$$= \alpha \pi(p) + (1-\alpha)\pi(p')$$

$\square$

(iii) Y CONVEX THEN $Y = \{ y \in \mathbb{R}^L \mid py \leq \pi(p) \ \forall p \gg 0 \}$

FIRST, FLIP THIS AROUND

$$Y = \{ y \in \mathbb{R}^L \mid -py \geq -\pi(p) \ \forall p \gg 0 \}$$

SO THAT $\tilde{\pi}(p) = \min \ -py$
$\quad$ S.T. $y \in Y$

SINCE $Y \neq \phi$ AND Y CLOSED, $\tilde{\pi}(p)$ EXISTS.

$\tilde{\pi}(p)$ IS A SUPPORT FUNCTION FOR Y (SEE DEF 3.F.1)

WE THEN CONCLUDE THAT

$$Y = \{ y \in \mathbb{R}^L \mid -p \cdot y \geq \tilde{\pi}(p) \ \forall p \gg 0 \}$$

BY THE SAME REASONING AS WE USED IN THE 2ND ¶ OF P 65. $\square$

NOTE THAT THE THIRD ¶ OF P65 PROVIDES AN ALTERNATIVE PROOF OF CONCAVITY OF π.

(v) Y CONVEX THEN $y(p)$ CONVEX $\forall p$.
Y STRICT CONVEX $y(p)$ SINGLE VALUED.

PROOF: SAY Y CONVEX WITH $\hat{y}, \tilde{y} \in y(p)$

$$\Rightarrow \quad p\hat{y} = p\tilde{y} = \pi(p)$$

$$\Rightarrow \quad \alpha p\hat{y} + (1-\alpha) p\tilde{y} = \pi(p)$$

$$\Rightarrow \quad p(\alpha\hat{y} + (1-\alpha)\tilde{y}) = \pi(p)$$

BY CONVEXITY $\alpha\hat{y} + (1-\alpha)\tilde{y} \in Y$ SO $y(p)$ CONVEX.

IF Y STRICT CONVEX, WE HAVE

$$y'' = \alpha\hat{y} + (1-\alpha)\tilde{y} \in Y^{\circ} \Rightarrow \exists\ \varepsilon > 0 \text{ S.T.}$$

$$y'' + \varepsilon \in Y^{\circ}$$

BUT $p(y''+\varepsilon) > py'' = p\hat{y} = p\tilde{y} = \pi(p)$

A CONTRADICTION

$\therefore$ $y(p)$ MUST BE SINGLE VALUED.

$\square$

(VII) • FROM (IV) $y(p) = \nabla \pi(p)$.

THUS $Dy(p) = D^2\pi(p)$ IS IMMEDIATE.

- THAT $Dy(p)$ IS NEGATIVE SEMI-DEFINITE THEN FOLLOWS FROM CONCAVITY OF π.

- SYMMETRY FOLLOWS FROM THE SYMMETRY OF CROSS-PARTIAL DERIVATIVES.

- SINCE $y(p)$ IS HD

$$D_\alpha\, y(\alpha p) = D_p\, y(\alpha p)\cdot p = 0$$

EVALUATING AT $\alpha = 0$

GIVES $D_p\, y(p)\cdot p = 0$

(VI) THREE PROOFS OF $\nabla \pi(p) = y(p)$.

(A) SINCE $\pi(p)$ IS A SUPPORT FUNCTION FOR $\bar{Y}$ THE RESULT FOLLOWS IMMEDIATELY FROM THE DUALITY THM. $\square$

(B) $\nabla \pi(p) = \nabla \left(p\, y(p) \right)$

$$= p \nabla y(p) + y(p)$$

BUT $p \cdot \nabla y(p) = 0$ FOR ANY INTERIOR SOLUTION TO P.M.P.

$\Rightarrow \nabla \pi(p) = y(p)$. $\square$

(C) PROFIT MAX PROBLEM IS

$$\pi^*(p) \quad \max \; py \quad \text{s.t.} \quad F(y) = 0$$

$$\mathcal{L} = py + \lambda F(y)$$

BY ENVELOPE THM $\nabla \pi^*(p) = \nabla_p \mathcal{L}$

$$= y$$

THUS $\nabla \pi^*(p) = y(p)$ $\square$

MT 5.2

(a) $$\max \quad y + x^{1/2}$$

$$\text{S.T.} \quad p_y y + p_x x = w$$

$$y = \frac{1}{p_y}\left[w - p_x x\right] \qquad \#$$

$$\Rightarrow \max_x \quad \frac{w}{p_y} - \frac{p_x}{p_y} x + x^{1/2}$$

$$\text{F.O.C.} \quad -\frac{p_x}{p_y} + \frac{1}{2} x^{-1/2} = 0$$

$$\Rightarrow x^{-1} = \left[\frac{2p_x}{p_y}\right]^2$$

$$\Rightarrow x(p, w) = \left(\frac{p_y}{2p_x}\right)^2$$

From # $$y(p,w) = \frac{w}{p_y} - \frac{p_x}{p_y}\left(\frac{p_y}{2p_x}\right)^2$$

$$\Rightarrow y(p,w) = \frac{w}{p_y} - \frac{p_y}{4p_x}$$

Thus, $\frac{\partial x}{\partial w} = 0$ and $\frac{\partial y}{\partial w} = \frac{1}{p_y}$

(b) $V(p,w) = y(p,w) + [x(p,w)]^{1/2}$

$$\Rightarrow V(p,w) = \left[\frac{w}{p_y} - \frac{p_y}{4p_x}\right] + \frac{p_y}{2p_x}$$

$$\Rightarrow V(p,w) = \frac{w}{p_y} + \frac{p_y}{4p_x} = u^0 \qquad \#$$

SOLVING FOR $w \Rightarrow e(p_x, p_y, u^0) = p_y\left[u^0 - \frac{p_y}{4p_x}\right]$

FIX p_y AT $\bar{p}_y$ AND CONSIDER A CHANGE

FROM p_x^0 AND u^0 TO p_x^1 AND u^1.

THEN $CV(p,w) = w - e(p_x^1, \bar{p}_y, u^0)$

$$= w - \bar{p}_y\left[u^0 - \frac{\bar{p}_y}{4p_x^1}\right]$$

BUT $u^0 = \frac{w}{\bar{p}_y} + \frac{\bar{p}_y}{4p_x^0}$ (FROM #)

SO $CV(p,w) = \frac{-\bar{p}_y^2}{4}\left[\frac{1}{p_x^0} - \frac{1}{p_x^1}\right]$

SIMILARLY, $EV(p,w) = e(p_x^0, \bar{p}_y, u^1) - w$

$$= \bar{p}_y\left[u^1 - \frac{\bar{p}_y}{4p_x^0}\right] - w$$

BUT $u^1 = \frac{w}{\bar{p}_y} + \frac{\bar{p}_y}{4p_x^1}$ FROM #

$\rightarrow$

$$\Rightarrow \quad EV(p,w) = \overline{p_y}\left[\frac{w}{\overline{p_y}} + \frac{\overline{p_y}}{4p_x^1} - \frac{\overline{p_y}}{4p_x^0}\right] - w$$

$$= \frac{\overline{p_y}^2}{4}\left[\frac{1}{p_x^1} - \frac{1}{p_x^0}\right]$$

$$= \frac{-\overline{p_y}^2}{4}\left[\frac{1}{p_x^1} - \frac{1}{p_x^0}\right] = CV(p,w)$$

(C) FIX p_x AT $\overline{p_x}$ AND CONSIDER A CHANGE FROM p_y^0 AND u^0 TO p_y^1 AND u^1

$$CV(\overline{p_x}, p_y, w) = w - e(\overline{p_x}, p_y^1, u^0)$$

$$= w - p_y^1\left[u^0 - \frac{p_y^1}{4\overline{p_x}}\right]$$

BUT $u^0 = \frac{w}{p_y^0} + \frac{p_y^0}{4\overline{p_x}}$ FROM $*$, SO

$$CV(\overline{p_x}, p_y, w) = w - p_y^1\left[\frac{w}{p_y^0} + \frac{p_y^0}{4\overline{p_x}} - \frac{p_y^1}{4\overline{p_x}}\right]$$

$$= w\left[1 - \frac{p_y^1}{p_y^0}\right] + \frac{-p_y^1}{4\overline{p_x}}\left[p_y^1 - p_y^0\right]$$

$$EV(\bar{P}_x, P_y, w) = e(\bar{P}_x, P_y^0, u^1) - w$$

$$= P_y^0 \left[u^1 - \frac{P_y^1}{4\bar{P}_x} \right] - w$$

BUT $u^1 = \dfrac{w}{P_y^1} + \dfrac{P_y^1}{4\bar{P}_x}$

$$\Rightarrow \quad EV = P_y^0 \left[\left[\frac{w}{P_y^1} + \frac{P_y^1}{4\bar{P}_x} \right] - \frac{P_y^1}{4\bar{P}_x} \right] - w$$

$$= w \left[\frac{P_y^0}{P_y^1} - 1 \right] \neq CV(\bar{P}_x, P_y, w)$$

MT 5.3

FIRM COST MINIMIZATION PROBLEM IS

$$C(\omega, q) = \min_x \ \omega \cdot x \quad \text{S.T. } f(x) = q$$

$$\Rightarrow \quad \mathcal{L} = \omega \cdot x + \lambda(q - f(x)) \quad \Rightarrow \quad \omega = \lambda \nabla f \qquad \#$$

SHOW THAT $\dfrac{\partial}{\partial q} C(\omega, q) = \lambda$

$$C(\omega, q) = \omega \cdot x(\omega, q)$$

THUS
$$\frac{\partial}{\partial q} C(\omega, q) = \omega \nabla_q x(\omega, q)$$
$$= \lambda \nabla f \nabla_q x(\omega, q) \qquad \text{(FROM \#)}$$
$$= \lambda$$

SINCE $\nabla f \nabla_q x = \dfrac{dq}{dq} = 1$