EC2020-Fall 2011
Problem Set 6 solutions

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MWG 6.C.3

PROVE THAT

(I) $E(u(x)) \leq u(E(x)) \Rightarrow \pi(x, \varepsilon, u) \geq 0 \quad \forall x, \varepsilon > 0$

(II) $\pi(x, \varepsilon, u) \geq 0 \Rightarrow u(\tfrac{1}{2}x + \tfrac{1}{2}y) \geq \tfrac{1}{2}u(x) + \tfrac{1}{2}u(y)$

$$\Rightarrow u'' < 0.$$

(I) LET $L = \begin{cases} x + \varepsilon & \tfrac{1}{2} \\ x - \varepsilon & \tfrac{1}{2} \end{cases}$

THEN $E(u(L)) \leq u(E(L))$

$\Rightarrow E(u(L)) \leq u(x)$

NOW CONSIDER $L' = \begin{cases} x + \varepsilon & \tfrac{1}{2} + \pi \\ x - \varepsilon & \tfrac{1}{2} - \pi \end{cases}$

S.T. $E(u(L')) = u(x)$ i.e. $\pi = \pi(x, \varepsilon, u)$

THEN $E(u(L')) \geq E(u(L))$

BY EXPECTED UTILITY DEF

$$\Rightarrow \left(\tfrac{1}{2} + \pi\right)(x + \varepsilon) + \left(\tfrac{1}{2} - \pi\right)(x - \varepsilon) \geq \tfrac{1}{2}(x + \varepsilon) + \tfrac{1}{2}(x - \varepsilon)$$

$$\Rightarrow \pi(x + \varepsilon) - \pi(x - \varepsilon) \geq 0$$

$$\Rightarrow \pi x + \pi\varepsilon - \pi x + \pi\varepsilon \geq 0$$

$$\Rightarrow 2\pi\varepsilon \geq 0$$

SINCE $\varepsilon > 0 \Rightarrow \pi \geq 0$ $\square$

(II) LET $z = \frac{x+y}{2}$, $\varepsilon = |x - z|$. THEN WE HAVE

LET $L = (z-\varepsilon, z+\varepsilon; \frac{1}{2} - \pi, \frac{1}{2}\pi)$, WHERE π IS CHOSEN SO THAT

$$E(u(L)) = u(z)$$

$$\Rightarrow \quad \left(\tfrac{1}{2} - \pi\right) u(z-\varepsilon) + \left(\tfrac{1}{2} + \pi\right) u(z+\varepsilon) = u(z)$$

AND $\pi \geq 0$ BY ASSUMPTION.

SINCE $u' \geq 0$, $u(z-\varepsilon) \leq u(z+\varepsilon)$

$$\Rightarrow \quad \tfrac{1}{2} u(z-\varepsilon) + \tfrac{1}{2} u(z+\varepsilon) < u(z)$$

BUT FROM THE DEF OF z AND ε

$$\Rightarrow \quad \tfrac{1}{2} u(x) + \tfrac{1}{2} u(y) \leq u\left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) \quad \square$$

MWG 6.C.16

$L = (G, B \,; p, (1-p))$

INITIAL WEALTH ω.

$$\text{WEALTH} \quad \hat{\omega} = \begin{cases} \omega + G & p \\ \omega + B & 1-p \end{cases}$$

(I) HOW MUCH WOULD INDIVIDUAL ACCEPT FOR L?

LET g DENOTE PRICE, g MUST SATISFY

$$u(w + g) = E(u(\hat{\omega})) = p\,u(w + G) + (1-p)\,u(\omega + B)$$

(II) HOW MUCH WOULD INDIVIDUAL PAY FOR L?

LET $\hat{g}$ DENOTE PRICE. $\hat{g}$ MUST SATISFY

$$u(\omega) = p\,u(\omega - \hat{g} + G) + (1-p)\,u(\omega - \hat{g} + B)$$

(III) $g = \hat{g}$? PROBABLY NOT. TO CHECK, WRITE (I) + (II) AS LINEAR EXPANSIONS AROUND $u(\omega)$. LET $u \equiv u(\omega)$, $u' \equiv u'(\omega)$. THEN, FROM (I)

$$u + u'g \cong p[u + u' \cdot G] + (1-p)[u + u'B]$$

$$\Rightarrow \quad u'g \approx p\,u' \cdot G + (1-p)\,u'B$$

$$\Rightarrow \quad g \approx pG + (1-p)B$$

FROM (II) $\quad u \approx p[u + u'(-\hat{g} + G)] + (1-p)[u + u'(-\hat{g} + B)]$

$$\Rightarrow \quad 0 \approx p(-\hat{g} + G) + (1-p)(-\hat{g} + B)$$

$$\Rightarrow \quad \hat{g} \approx pG + (1-p)B$$

THUS WE HAVE $q = \hat{q}$! THIS MUST RESULT FROM THE FACT THAT WE IMPOSED $u'(\omega) = u'(\omega+G) = u'(\omega+B)$.

THIS SUGGESTS THAT $q = \hat{q}$ IF $u'' = 0$.

LET'S TRY AGAIN, WITH 2^{ND} ORDER EXPANSION.

$$(I) \Rightarrow \quad u + u' \cdot q + \tfrac{1}{2}u'' q^2 \approx p\left[u + u'G + \tfrac{1}{2}u''G^2\right] + (1-p)\left[u + u'B + \tfrac{1}{2}u''B^2\right]$$

$$\Rightarrow \quad u'q + \tfrac{1}{2}u''q^2 \approx pG\left[u' + \tfrac{1}{2}u''G\right] + (1-p)B\left[u' + \tfrac{1}{2}u''B\right] \qquad *$$

$$(II) \Rightarrow \quad u = p\left[u + u'(G-\hat{q}) + \tfrac{1}{2}u''(G-\hat{q})^2\right] + (1-p)\left[u + u'(B-\hat{q}) + \tfrac{1}{2}u''(B-\hat{q})^2\right]$$

$$\Rightarrow \quad 0 = p\left[u'(G-\hat{q}) + \tfrac{1}{2}u''(G^2 - 2G\hat{q} + \hat{q}^2)\right] + (1-p)\left[u'(B-\hat{q}) + \tfrac{1}{2}u''(B^2 - 2B\hat{q} + \hat{q}^2)\right]$$

$$\Rightarrow \quad u'\hat{q} - \tfrac{1}{2}u''\hat{q}^2 = p\left[u' \cdot G + \tfrac{1}{2}u''(G^2 - 2G\hat{q})\right] + (1-p)\left[u' \cdot B + \tfrac{1}{2}u''(B^2 - 2B\hat{q})\right]$$

$$\Rightarrow \quad u'\hat{q} + \tfrac{1}{2}u''\hat{q}^2 = p\left[u' \cdot G + \tfrac{1}{2}u''G^2\right] + (1-p)\left[u' \cdot B + \tfrac{1}{2}u''B^2\right] + p\tfrac{1}{2}u''(-2G\hat{q}) + (1-p)\tfrac{1}{2}u''(-2B\hat{q}) + u''\hat{q}$$

SUBSTITUTING FROM $*$ AND SIMPLIFYING

$$\Rightarrow \quad u'\hat{q} + \tfrac{1}{2}u''\hat{q}^2 = u'q + \tfrac{1}{2}u''q^2 + u''\left[-pG\hat{q} - (1-p)B\hat{q} + \hat{q}\right]$$

$$\Rightarrow \quad \hat{q} = q \iff u'' = 0 \ \underline{\text{OR}} \ B = G = 1 \quad \text{i.e. NO GAMBLE.}$$

MWG 6.D.1

(a) IF L' F.O.S.D L WE HAVE

$$p_1' u_1 + p_2' u_2 + (1-p_1'-p_2')u_3 > p_1 u_1 + p_2 u_2 + (1-p_1-p_2)u_3$$

FOR EVERY u WITH $u_1 < u_2 < u_3$. AND
$u_2 - u_1 > u_3 - u_2$

$$\Rightarrow (p_1'-p_1)u_1 + (p_2'-p_2)u_2 + [p_1+p_2-p_1'-p_2']u_3 > 0$$

$$\Rightarrow (p_1'-p_1)u_1 + (p_2'-p_2)u_2 + -(p_1'-p_1)u_3 - (p_2'-p_2)u_3 > 0$$

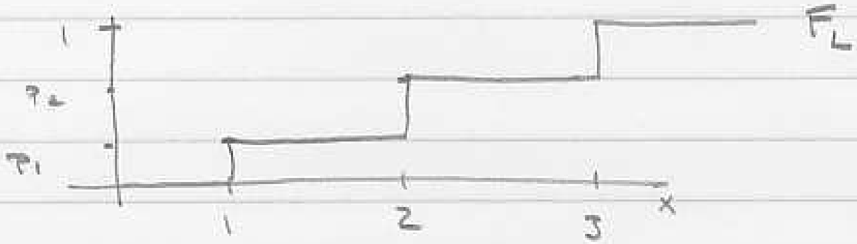
$$\Rightarrow (p_1'-p_1)(u_1-u_3) + (p_2'-p_2)(u_2-u_3) \geq 0$$

SINCE $u_1 \leq u_2 \leq u_3$ A SUFFICIENT CONDITION FOR THIS INEQUALITY IS

$$p_1' < p_1 \quad , \; p_2' < p_2$$

THIS IS THE SAME CONDITION WE GET IN (b)

(b) $L = (1, 2, 3 : p_1, p_2, (1-p_1-p_2))$ HAS DISTRIBUTION.

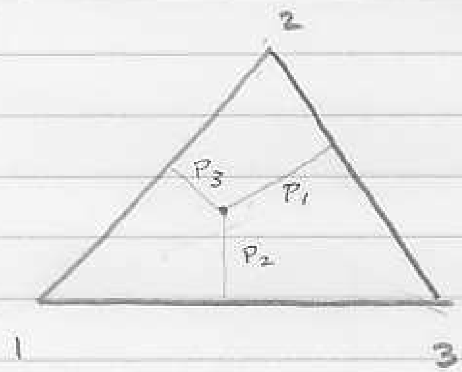


FOR L' TO F.O.S.D. L WE MUST HAVE

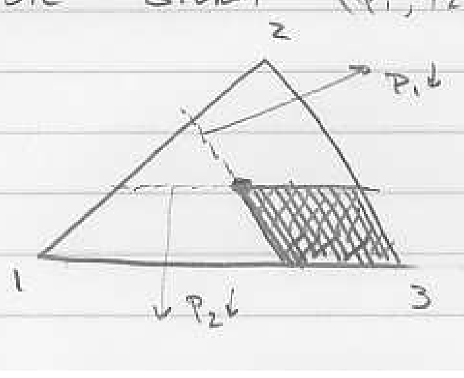
$F_{L'} \leq F_L$ EVERYWHERE.

$$\Rightarrow \quad p_1' \leq p_1$$
$$p_2' \leq p_2$$

IN THE SIMPLEX WE HAVE THAT THE DISTANCE FROM A POINT TO THE SIDE OPPOSITE THE i th VERTEX IS PROPORTIONAL TO p_i.
(SKETCH WITH PERSPECTIVE TO SEE THIS)



THUS, FOR GIVEN (p_1, p_2, p_3) REGION OF F.O.S.D IS:



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MWG 6.D.2

(1) IF F F.O.S.D. G THEN

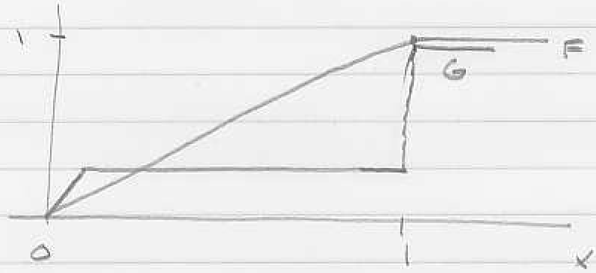
$$\int u(x)\,dF(x) \geq \int u(x)\,dG(x)$$

WHERE $u' \geq 0,\ u'' \leq 0$.

SINCE $u(x)=x$ SATISFIES $u' \geq 0,\ u'' \leq 0$ WE HAVE

$$\int x\,dF(x) \geq \int x\,dG(x)$$

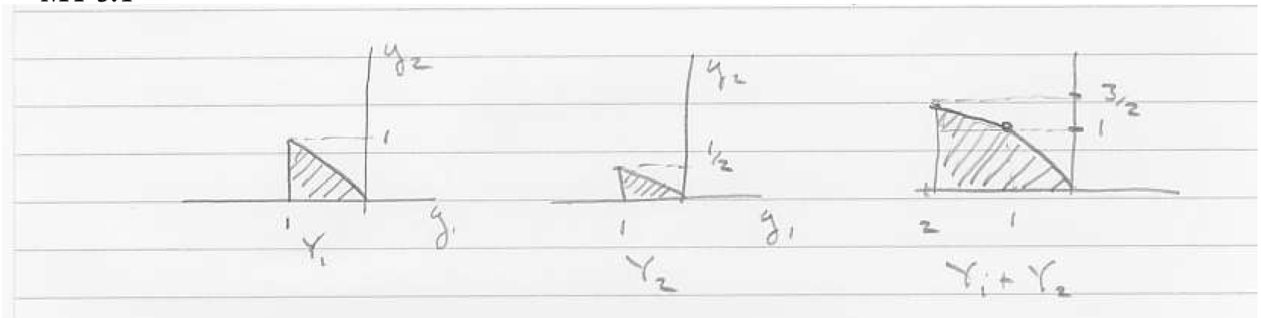
(2) CONSIDER



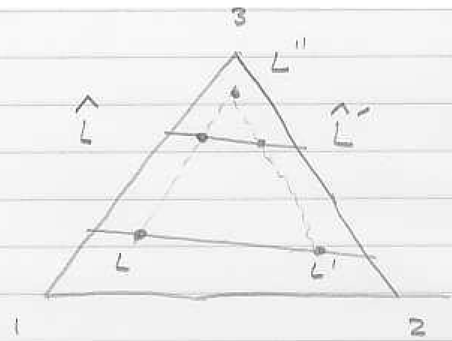
$E(F) \approx \frac{1}{2}$, $E(G) \approx 1$ BUT

$G \neg$ F.O.S.D. F.

MT 5.4



MT 6.1



$\hat{L} = \alpha L + (1-\alpha) L''$ $\qquad$ $\hat{L}' = \alpha L' + (1-\alpha) L''$

$\hat{L} \sim \hat{L}'$ FOLLOWS FROM $L' \sim L$ AND INDEPENDENCE

IF WE SHOW $\dfrac{\|\hat{L} - L''\|}{\|\hat{L}' - L''\|} = \dfrac{\|L - L''\|}{\|L' - L''\|}$ $\quad *$

THEN $\triangle L''\hat{L}\hat{L}'$ AND $\triangle L''L,L'$ SHARE AN ANGLE AND TWO SIDES ARE PROPORTIONAL

$\Rightarrow \quad \angle L''\hat{L}\hat{L}' = \angle L''L L'$

AND $\quad \angle L''\hat{L}'\hat{L} = \angle L''L'L$

$\Rightarrow \quad \overline{\hat{L}\hat{L}'} \parallel \overline{LL'}$

TO SEE $*$, NOTE THAT $\|\hat{L} - L''\| = \alpha\|L - L''\|$ ETC. $\Rightarrow$ $*$ HOLDS.

MT 6.2

$L_1 = (0, 1, 0)$

$L_2 = (\frac{11}{20}, 0 \ \frac{9}{20})$

$u' = \sqrt{u} \Rightarrow (u_1', u_2', u_3') = (1, 2, 3)$

$E(u(L)) = 4$
$E(u(L')) = \frac{11}{20} \cdot 1 + \frac{9}{20} \cdot 9 = \frac{92}{20} > 4 = E(u(L))$

IF WE DUPLICATE THIS EXERCISE USING u'
WE GET

$E(u'(L)) = 2$
$E(u'(L')) = \frac{11}{20} + \frac{9}{20} \cdot 3 = \frac{38}{20} < 2 = E(u'(L))$

THUS A MONOTONIC TRANSFORMATION YIELDS
PREFERENCE REVERSALS.