Commodity Money with Divisible Goods

Stephen Eli Ahiabu*
University of Toronto†

February 2004

Abstract

I show that when goods are perfectly divisible, the fundamental and speculative equilibria of Kiyotaki and Wright (1989) can coexist. This validates welfare comparisons. The speculative equilibrium is always a better lubricated economy with a higher quantity of commodity money circulating. When goods with high storage costs start to circulate, they crowd out the circulation rate of goods with lower storage costs, resulting in a version of Gresham's law. A direct consequence is that the speculative equilibrium is not Pareto superior.

Keywords: Commodity Money, Gresham's Law
JEL Classification: C78, E59

*Department of Economics, 100 St. George Street, Toronto, Ontario, M5S 3G7, Canada. Email: sahiabu@chass.utoronto.ca, Tel/Fax: +1 (416) 963 4457

†This paper is a chapter in my doctoral dissertation at the University of Toronto. My gratitude goes to Professors Shouyong Shi (supervisor) and Miquel Faig for their helpful insights and comments. All remaining shortcomings are mine.

1 Introduction

The aim of this paper is to identify commodity money when goods are perfectly divisible. That is, what economic conditions can explain how certain goods became exalted to the status of mediums of exchange in the early economy?

An important progress in the literature that attempts to identify commodity money is Kiyotaki and Wright (1989) (KW). Their paper was the first to attempt an answer to this question using fully rational agents, strategic behaviour in a dynamic economy, with random pair-wise matching and without fiat money. In deciding on which goods to accept and which to reject in trade, agents compare storage costs and liquidity/marketability[1] properties of goods. The authors show that low storage cost goods always act as money. However, high marketability can lead agents to speculate and temporarily accept high storage cost goods, though not ultimately desired. This they term the speculative equilibrium. A third result is that there is no region in their parameter space for which any other equilibrium exists. Finally, these two equilibria cannot coexist. That is, there is no region of the parameter space that can simultaneously support both equilibria. In the current paper, we allow for fully divisible goods and test whether these two equilibria can be identified. Next, we ask if these two are the only equilibria and whether or not they can coexist.

Once we allow for divisibility, the analysis extends in two major dimensions, the first being bargaining. In Kiyotaki and Wright (1989), trade reduces to one-for-one exchange due to indivisibility. With divisibility however, agents can bargain over fractions of commodities and hence we employ a bargaining solution, which will be discussed later. Secondly, tractability may become a problem. Agents of a given type may hold a wide distribution of volumes of the different commodities; each agent's stocks depending on her own unique history of matches and trades. To prevent this, we employ the concept of households, each with a continuum of

[1] By marketability, they capture the probability of finding other agents willing to offer an agent her consumption good in return of her current stock.

members. By the law of large numbers, the state variable of a household becomes known, once the equilibrium policy decisions of that household are given. Thus, the model proposed in this paper is tractable in equilibrium. Specifically, if parameter values allow for a given equilibrium (say the Fundamental equilibrium), we can derive closed form solutions for the state variables and this enables us to make comments on welfare and velocity. Of course, off equilibrium, households can hold any mix of commodity money and production units they wish. Thus, given divisibility, off equilibrium distribution of state variables become non-tractable. Nevertheless, confirming an equilibrium requires that we evaluate and compare the payoffs to off equilibrium strategy alternatives, which have associated non-tractable state variables. Hence, to confirm our equilibria, we employ a numerical approach.

The advantage of our model (which comes straight forward from allowing divisibility and bargaining) is that it becomes possible to identify how the distribution of commodity money holdings in equilibrium depends on the matching function, the bargaining paradigm and model parameters. This is unlike in Kiyotaki and Wright (1989) and Renero (1999) where bargaining is simply one-for-one exchange and hence equilibrium distribution of commodity money depends only on the matching function. Since the only state variable for a household is money, the distribution of commodity money holdings has important implications for welfare. Thus, showing how this distribution depends on model parameters and not only the matching function is a non-trivial result in this paper.

In the next section, we review the literature on deep money models with particular attention paid to contributions at identifying commodity money. Section 3 presents the model studied in this paper. In section 4, we impose sufficient structure and assume parameter values which permit a stationary equilibrium. We characterize the equilibrium and show how they are verified in section 5, as well as comment on coexistence, welfare and velocity. Section 6 concludes.

2 Literature Review

Even in the early economy, the benefit of a medium of exchange has been widely recognized as it promotes trade and specialization. Starr (1972) accessed the role of money in lubricating an economy with decentralized exchange and stressed that the double coincidence of wants requirement generally precludes the fulfillment of excess demands through direct barter. The existence of a medium good that everyone is willing to temporarily acquire, though not ultimately desired enables a bilateral exchange pattern which fulfills all excess demands. Ostroy (1973) perceives money as a "record-keeping device" that permits the enforcement of budget constraints without requiring equal-value-exchange at each meeting. This way, money allows for eventual fulfillment of excess demands after a minimal number of meetings involving two given agents.

Rather than focusing on how money helped overcome the structural difficulties of a decentralized economy, another branch of the early literature concentrated on how certain commodities came to be "exalted into general mediums of exchange". Jones (1976) described an economy with a "monetary pattern of exchange" as one in which there is one good that always enters into every exchange. Any other good entering an exchange if purchased is not resold and if sold is not repurchased. His environment thus does not only forbid double coincidence of want but also long chains of indirect trade. Agents acquire their desired goods only through two-stage trades. The author then established how all agents come to choose the same intervening trade commodity, thus identifying commodity money. Though interesting, individuals are naïve concerning the equilibrium matching distribution and as a result, their strategies may not be optimal given the distribution. Iwai (1988) employs Jones' environment with fully rational agents but simple non-sequential trade strategies are optimal in his environment since matching is non-random.

In an economy without fiat money, agents may acquire intermediate goods either for their intrinsic value in economizing on storage costs or for their extrinsic value in finding desirable matches in subsequent periods. Kiyotaki and Wright (1989) considers a three-agent type economy with three types of indivisible goods. Each agent has a unit storage capacity. Agent type h consumes commodity h and produces $h+1$ modulo 3. Net utility from consumption and producing the next good is u'_h for agent type h. $\bar{c}_{h,j}$ is the cost to agent h for storing a unit of good j between periods. These costs are of order $\bar{c}_{h,1} < \bar{c}_{h,2} < \bar{c}_{h,3}, \forall\ h = 1, 2, 3$. Let $p_{h,j}$ be the density of agents of type h holding good j between periods. An agent consumes her consumption commodity instantaneously when she lays hands on it and hence $p_{h,h} = 0$. Production guarantees that $p_{h,h+1} > 0$. Agent h neither produces nor consumes good $h+2$. She thus has no reason to hold good $h+2$ except for use as intermediate units for future trade, thus, as commodity money. That is, good $h+2$ becomes commodity money only if $p_{h,h+2} > 0$.

From the point of view of agent h carrying good j, $p_{j,h}$ represents the marketability of good j. This is because a match with agent j represents a double coincidence of want in consumption. Again, for agent h carrying good j, $p_{i,h}$ can also represent marketability of good j. This occurs if agents of type i wish to acquire good j as commodity money.

Agent Type	1	2	3
Consumes	1	2	3
Produces	2	3	1
May use this good as money	3	1	2

The authors find that there exists an equilibrium in which all agent types play the “fundamental strategy”, which involves the acquisition of goods that they find cheapest to store. The Fundamental equilibrium is characterized by good 1 emerging as the sole medium of exchange. This happens when $p_{2,1} > 0$, while agent 1 declines to accept good 3 in exchange for her own produced good, 2. This rejection is worthwhile if good 3 does not hold overwhelming marketability potential in future exchange for consumption good 1, bearing in mind the storage costs. Thus, the condition for the fundamental equilibrium is $\bar{c}_{1,3} - \bar{c}_{1,2} > (p_{3,1} - p_{2,1}) \frac{\beta}{3} u_1'$, where β is the time discount factor.[2]

The concept of marketability adopted above follows long-held notions by Menger (1892) stressing the importance of the number of agents producing and consuming the different commodities. The significance of this concept is demonstrated when good 3 becomes so marketable that agent 1 begins to speculate and accept good 3 in exchange for cheaper-to-store good 2: $\bar{c}_{1,3} - \bar{c}_{1,2} < (p_{3,1} - p_{2,1}) \frac{\beta}{3} u_1'$. In the "speculative equilibrium" therefore, $p_{1,3} > 0$. The extrinsic properties of good 3 have helped it join in as a second medium of exchange. There is no other equilibrium and the above equilibria cannot coexist. In particular, it is possible to show that good 2 never emerges as money, partly because $\bar{c}_{3,1} < \bar{c}_{3,2}$.

Aiyagari and Wallace (1991) extends the model of Kiyotaki and Wright (1989) to $N \geq 3$ agent types and relaxes the storage capacity assumption. Wright (1995) extends to endogenize specialization in an evolutionary perspective. He finds among others that the fundamental equilibrium often affectionately described as the "natural equilibrium" [see Marimon, McGrattan, and Sargent (1990)] disappears when types are endogenous. Rather, there exists a unique speculative equilibrium and the only role played by storage costs is to influence the equilibrium distribution of agent types in the economy. In Cuadras-Morato and Wright (1997), agents are assumed to be specialists in production but generalists in consumption. Tastes are changing over time. Agents always accept goods that they consume currently but the interesting question is whether they will accept goods that are not currently desired. A high probability of desiring a good next period encourages its acceptance because the agent herself and trade partners are more likely to want that good in the next period. Goods in high demand thus seem more likely to end up serving as money. At the same time, if the number of producers of a good is large, holding demand constant, individuals may be less likely to accept that good now because it is easy to acquire when needed. Renero (1999) allows

[2] See Kiyotaki and Wright (1989).

agents to employ mixed strategies instead of pure trading strategies. Agents are indifferent between accepting and rejecting goods that are not their consumption good and he names the resulting position his "constructed equilibrium". He compares to the fundamental equilibrium in Kiyotaki and Wright (1989) and finds that the constructed equilibrium welfare dominates since it leads to more trades and production in equilibrium.

The unit storage capacity assumption adopted has unconventional implications for monetary policy [see Aiyagari and Wallace (1992)]. To go around this, Rupert, Schindler, Schevchenko and Wright (2000) introduce an environment with storage capacities for fiat money and for goods. They establish conditions under which fiat money remains the sole medium of exchange, even in matches in which there is the potential for barter.

The extension of Kiyotaki and Wright (1989) to determine endogenous commodity money while allowing for perfect divisibility of goods is yet to be addressed in the literature. It allows us for instance to consider an essential aspect of marketability in the early economy without fiat money, which is "terms of trade". This natural and appealing dimension poses at least two important hurdles, being bargaining and tractability. Shi (1999) assumes that agents belong to households with continuum of members. This caters for tractability as already discussed and is employed in the current paper. Other methods that help deal with tractability include market segmentation as in Lagos and Wright (2002) and a numerical approach as in Molico (1999). Popular varieties of bargaining frameworks have been long established and require no further discussion. In the next section, we present an environment with divisible goods, which we subsequently employ to identify commodity money given certain stationarity assumptions.

3 Economic Environment

The environment we consider is closely similar to Kiyotaki and Wright [1989] (KW) as described in section 2, but with minimal adjustments. Goods are durable and perfectly divisible. Our main decision-making unit is the household. Each household has a $[0,1]$ continuum of member agents and a unit storage capacity. These member agents make decisions, but only during successful matches in the market. Exactly a third of the households are of each type, $h = 1,2,3$. Good $h+2$ acts as money if it is held by household h, that is, if $p_{h,h+2} > 0$. In this case, $p_{h,j}$ is the proportion of storage space used by a representative type h household to store good j between periods. As explained earlier, $p_{h,h} = 0$ and $p_{h,h+1} > 0$. All other physical arrangements are as in KW.

Due to the unit storage capacity assumption, $p_{h,h+2}$ is also equivalent to the actual quantity of commodity money held by household h. Let $m_h \in [0,1]$ be household h's stock of commodity money and $1-m_h$ the stock of produced goods. The state variable m_h thus completely describes the household's portfolio at a given time.[3] The cost to household h of storing such a portfolio between periods is denoted $c_h(m_h)$. We choose the functional form:

$$c_h(m_h) = \overline{c}_{h,h+1}[2-m_h] + \overline{c}_{h,h+2}[I+m_h]$$

where I is an indicator function that takes the value one if $p_{h,h+2} > 0$ and zero otherwise. To be consistent with the ranking of storage costs in KW, we assume that $\overline{c}_{h,1} < \overline{c}_{h,2} < \overline{c}_{h,3}$, $h = 1,2,3$. Household members carry these goods individually along to the market, yet storage costs are borne collectively at the household level. A member cannot carry two types of goods at the same time. Therefore, $p_{h,h+2}$ also represents the number of agents to hold the m_h units of commodity money, each

[3] With divisible goods, we have $m_h \in [0,1]$ and not $m_h \in \{0,1\}$ as in KW. Notice that $m_h = 0$ does not mean that the household has no stocks, but rather has $1 - m_h = 1$ in produced goods. An implicit assumption is that total stocks are kept to unit capacity at the end of each period. This assumption becomes redundant if marginal utility is sufficiently large.

of whom enter the market with $\frac{m_h}{p_{h,h+2}}$ units. Thus, members holding the produced units each enter the market with the quantity $\frac{1-m_h}{p_{h,h+1}}$. Given unit storage capacity and $[0,1]$ continuum of agents, each agent goes to the market with a unit volume of goods: $\frac{m_h}{p_{h,h+2}} = \frac{1-m_h}{p_{h,h+1}} = 1$. We later use $p_{h,j}$ also to denote a type h agent holding good j.

A household has one choice variable, being the an accept or reject decision for units of commodity money in the current trade period. This is basically an instruction to member agents holding produced units whether or not to accept commodity money in exchange for those goods. This decision depends on the current state, and is denoted $D_h(m_h)$ for the representative type h household, or simply, D_h. D_h takes the value 1 if agents are instructed to accept commodity money and zero otherwise.

m_h			m'_h
$1-m_h$			$1-m'_h$
Choose	Markets Open	Markets Close	Storage Costs
$\longrightarrow$	$\longrightarrow$	$\longrightarrow$	$\longrightarrow$
$D_h(m_h)$	$\frac{1-m_h}{p_{h,h+1}}$	Pooling	$c(m'_h)$
	$\frac{m_h}{p_{h,h+2}}$	Consumption	
	Pairwise matching	Production	
	Trade qtys, Trade		

Next, markets open. Once in the market, types are fully observable. Agents match randomly and individually with other agents and may trade depending on the household's accept/reject instructions. Successfully matched agents then decide on s, being the quantity to sell and b, the quantity to request in return. Agents do not buy back from the market goods of type they produce. The sequence of activities in a typical household as summarized above.

3.1 Matching and Bargaining

Pairwise matching and bargaining yield trade quantities which eventually appear in the household's optimization. Hence, next, we outline the matching and bargaining environment as well as the terms of trade.

First, we consider matching. An agent from household h matches with other agents of types $1, 2, 3$, each at the probability $\frac{1}{3}$. When matched with fellow type h agents, there cannot be beneficial trade. Now, suppose an agent from household h goes to the market with produced goods $h + 1$. Denote this agent as $p_{h,h+1}$. Meetings between $p_{h,h+1}$ and $p_{h+1,h+2}$ can result in trade only if $D_h = 1$. The probability of such a match occurring is $\frac{1}{3} p_{h,h+1} p_{h,h+2}$. If $p_{h,h+1}$ meets $p_{h+1,h}$ there can be trade leading to consumption in both households. If $p_{h,h+1}$ meets $p_{h+2,h}$, there can only be trade if $D_{h+2} = 1$. This leads to consumption only at household h . And if $p_{h,h+1}$ meets $p_{h+2,h+1}$, there will not be trade since both agents hold the same commodity. Next, suppose an agent from household h enters the market with money units $h+2$. If matched with $p_{h+1,h+2}$, both agents hold like units and there is no trade. If $p_{h,h+2}$ meets $p_{h+1,h}$, there will not be trade because agent $h + 1$ does not buy back production units $h + 2$ from the market. If $p_{h,h+2}$ meets $p_{h+2,h}$, there will be trade and consumption in both households. Finally, if $p_{h,h+2}$ meets $p_{h+2,h+1}$, there will not be trade since h does not buy back production units $h + 1$ from the market. The probabilities of these other matches follow similarly. There are four possibilities for trade, three of which can lead to consumption at household h. Household h acquires commodity money during meetings between $p_{h,h+1}$ and $p_{h+1,h+2}$, and spends it during matches between $p_{h,h+2}$ and $p_{h+2,h}$.

Next, we consider bargaining and terms of trade. Let δy be the disutility from producing y units of output, $\delta \geq 0$. Also, let $u_h(c) = u_h' c$ be the utility that household h gains from consuming $c > 0$ units of good h and $u_h(c) = -\infty$ when $c = 0$, with $u_h' > \delta$. Trade involves the exchange of goods for goods and hence each trader is a buyer and at the same time, a seller. To simplify therefore, we assume the bargaining power $\theta \in \{0, 1\}$, taking each value with probability

$\frac{1}{2}$. In making or accepting these offers, agents are assumed to consider only the immediate implication of the exchange. These immediate effects are captured in stages 2 and 3 of the timing sequence only. Agents are naive about the impact of their trade decisions on storage costs and the continuation value.

For matches between $p_{h,j}$ and $p_{j,h}$, $j \neq h$, suppose agent $p_{h,j}$ offers to sell the quantity s and in the process, buy the quantity b. Household h receives net utility $u'_h b - \delta s$ whiles household j gets $u'_j s - \delta b$. On occasions when agent $p_{h,j}$ makes the take-it-or-leave-it offer, the optimal quantities are such that:

$$\begin{aligned} u'_j s - \delta b &= 0 \text{ and} \\ b &= 1\ . \end{aligned}$$

That is, since $u'_h > \delta$, agent $p_{h,j}$ exhausts the stocks of $p_{j,h}$, while leaving the latter with zero net benefit. The quantities exchanges are thus $s = \frac{\delta}{u'_j}$ and $b = 1$. The rest half of the time when the other agent, $p_{j,h}$, makes the offer, she chooses $S = \frac{\delta}{u'_h}$ and $B = 1$. For household h, the total net utility from both such offers is $\frac{1}{2}\left[u'_h - \delta \frac{\delta}{u'_j}\right] + \frac{1}{2}\left[u'_h \frac{\delta}{u'_h} - \delta\right]$, or simply $\frac{1}{2}\left[u'_h - \frac{\delta^2}{u'_j}\right]$. This analysis pertains to when $p_{h,h+1}$ matches with $p_{j,h}$, $j = h+1, h+2$ which are two of the four possible trade matches. There are two others to consider.

Suppose $p_{h,h+1}$ meets $p_{h+2,h}$ and $D_{h+2} = 1$. Then if the former makes the offer (s, b), household $h+2$ receives zero trade surplus. Hence, $b = s = 1$, since again, $p_{h,h+1}$ exhausts stocks of the trade partner. Hence, household h receives $\frac{1}{2}\left[u'_h - \delta\right]$. When $h+2$ makes the offers, then again $S = \frac{\delta}{u'_h}$ and $B = 1$ and household h gets zero net utility.

Finally, suppose $p_{h,h+1}$ meets $p_{h+1,h+2}$ and $D_h = 1$. If the former makes the offer, then we set $u'_{h+1} s - \delta b = 0$ and $b = 1$. Thus, $s = \frac{\delta}{u'_{h+1}}$. However, if the latter agent makes the offer, she sets $-\delta B + \delta S = 0$ and hence $B = S = 1$. Household h gets total net utility $\frac{1}{2}\delta\left[1 - \frac{\delta}{u'_{h+1}}\right]$ from both cases. We employ the above bargaining paradigm because using others such as the generalized Nash impose considerable challenges.[4]

The bargaining results above enable us ascertain the net utility gained from trade transactions in the market as follows:

$$\begin{aligned} U_h &= p_{h,h+1}p_{h+1,h+2}\frac{D_h}{6}\delta\left[1-\frac{\delta}{u'_{h+1}}\right]+p_{h,h+1}p_{h+1,h}\frac{1}{6}\left[u'_h-\frac{\delta^2}{u'_{h+1}}\right] \\ &\quad +p_{h,h+1}p_{h+2,h}\frac{D_{h+2}}{6}\left[u'_h-\delta\right]+p_{h,h+2}p_{h+2,h}\frac{1}{6}\left[u'_h-\frac{\delta^2}{u'_{h+2}}\right], \end{aligned} \quad (1)$$

$h=1,2,3$. The terms in this equation follow directly from the preceding discussion. Next, the commodity money accumulation equation is described as:

$$m'_h = m_h + p_{h,h+1}p_{h+1,h+2}\frac{D_h}{6}(2) - p_{h,h+2}p_{h+2,h}\frac{1}{6}\left\{\frac{\delta}{u'_{h+2}}+1\right\},\ h=1,2,3 \quad (2)$$

Household h acquires commodity money units when $p_{h,h+1}$ meets $p_{h+1,h+2}$. For both values of θ, $\theta\in\{0,1\}$, agent $p_{h,h+1}$ acquires a unit quantity. Commodity money is spent when $p_{h,h+2}$ meets $p_{h+2,h}$. Half of the time, agent $p_{h,h+2}$ spends $\frac{\delta}{u'_j}$ and the other half spends an entire unit. Choosing $D_h=0$ in the current period does not preclude trading away commodity money, since stocks of previously acquired units may exist. If no stocks exist, $p_{h,h+2}=0$ as the household cannot allocate members to hold units which do not exist. This cancels away the last terms in equations (1) and (2).

Household members are altruistic towards fellow members. Let $v(m_h,\mu(m))$ be the value function of a representative type h household, where $\mu(m)$ is the aggregate distribution of money across all types. Household h chooses $D_h(m_h,\mu(m))\in\{0,1\}$ so as to solve:

$$v(m_h,\mu(m)) = \max_{D_h\in\{0,1\}} R_h + \beta E v(m'_h,\mu(m'))$$

[4]For example, applying to trade match between p_{12} and $p_{2,1}$, we have: $\max_{s,b}\left[u_1(b)-\delta s\right]^{\theta}\times\left[u_2(s)-\delta b\right]^{1-\theta}+\lambda_1\left[\frac{1-m_1}{p_{1,2}}-s\right]+\lambda_2\left[\frac{m_2}{p_{2,1}}-b\right]$. This is only one trade match, yet with two Lagrange multipliers. Since we cannot impose symmetry among household types, we need to keep track of all multipliers separately and once we consider all possible trade encounters, the problem can become rather large.

subject to (1) and (2), where the return function R_h is given as:

$$R_h = U_h - c_h\left(m_h'\right)\ , \quad h = 1, 2, 3. \tag{3}$$

4 Equilibrium

Suppose the household's problem yields the policy function $D_h\left(m_h, \mu\left(m\right)\right)$, $h = 1, 2, 3$. A stationary equilibrium is defined as the value functions $v\left(m_h, \mu\left(m\right)\right)$ and a stationary distribution of commodity money $\mu\left(m_h^e\right)$, $h = 1, 2, 3$ such that given the random matching function, all household types solve their respective maximization problems and yield the time-consistent policy functions $D_h^e\left(m_h^e, \mu\left(m^e\right)\right)$, $h = 1, 2, 3$.

To find an equilibrium, our approach is to conjecture the equilibrium, characterize it and then show that it is in fact an equilibrium. Given a consistent choice, D_h, by all households $h = 1, 2, 3$, the state variables settle to a constant distribution. The commodity money stocks are given by a solution to a system of simultaneous equations. In a conjectured equilibrium e therefore, a household that accepts commodity money will hold some constant quantity $m_h^e \in (0, 1)$ while a household that rejects will hold $m_h^e = 0$.

4.1 The Fundamental and Speculative Equilibria

If good 1 emerges as the sole medium of exchange, this is characterized by $m_2 = p_{2,1} \in (0, 1)$, $m_1 = p_{1,3} = m_3 = p_{3,2} = 0$ and $p_{1,2} = p_{3,1} = 1$. Finally, $D_1 = D_3 = 0$, while $D_2 = 1$. Imposing stationarity on (2), we have:

$$\mu\left(m^F\right) = \left\{m_1^F, m_2^F, m_3^F\right\} = \left\{0, \frac{2u_1'}{3u_1' + \delta}, 0\right\} \equiv \left\{p_{1,3}^F\ ,\ p_{2,1}^F\ ,\ p_{3,2}^F\right\} \tag{4}$$

Equation (4) is the distribution of commodity money stocks among households in the fundamental equilibrium. This contrasts with $\{0, 0.5, 0\}$ in KW, dependent only on the matching rate. The difference is accounted for by bargaining, which reduces to one-for-one exchange in environments with indivisible goods. Commodity money holding by a type 2 household is increasing in the marginal utility of household 1 types and decreasing in the production disutility of the same household types. In a match between $p_{2,1}$ and $p_{1,2}$ in which the former makes the offer, the higher the value u_1', the higher the surplus agent $p_{2,1}$ derives from the match. Agent $p_{2,1}$ exploits this surplus by giving fewer units of commodity money to agent $p_{1,2}$. Thus, agent $p_{2,1}$ holds on to more units, raising m_2 at the household level. The reverse is the case for δ.

From (3), instantaneous return for each household type is:

$$R_1^F = p_{2,1}^F \frac{1}{6}\left[u_1' - \frac{\delta^2}{u_2'}\right] - 2\overline{c}_{1,2}$$

$$R_2^F = p_{2,1}^F \frac{1}{6}\left[u_2' - \frac{\delta^2}{u_1'}\right] + p_{2,3}^F \frac{1}{6}\delta\left[1 - \frac{\delta}{u_3'}\right] - \overline{c}_{2,3}\left[2 - m_2^F\right] - \overline{c}_{2,1}\left[1 + m_2^F\right]$$

$$R_3^F = p_{2,3}^F \frac{1}{6}\left[u_3' - \delta\right] - 2\overline{c}_{3,1}$$

where $p_{2,3}^F$, $p_{2,1}^F$ and m_2^F are as in (4).

For goods 1 and 3 to emerge as mediums of exchange in the Speculative equilibrium, we require $D_1 = D_2 = 1$, while $D_3 = 0$. Again, imposing stationarity on (2) for $h = 1, 2$, we have a system of two equations for the values m_1 $(= p_{13})$ and m_2 $(= p_{21})$. This we can solve simultaneously or iteratively to deliver $\mu\left(m_h^S\right) = \left\{m_1^S, m_2^S, m_3^S\right\}$, where $m_3^S = 0$. The resulting return functions, R_h^S, $h = 1, 2, 3$, can also be derived likewise.

5 Simulation, Results and Discussion

We employ value function iteration to confirm these equilibria. To verify the fundamental equilibrium for instance, we start by quantifying the related equilibrium distribution $\mu\left(m^F\right)$, as in (4). Next, we form a grid for each type as follows:

$$\begin{array}{ll} \text{Type 1:} & \left\{m_1^F, ...m_1^{F,Dev}\right\} \\ \text{Type 2:} & \left\{m_2^{F,Dev}, ...m_2^F\right\} \\ \text{Type 3:} & \left\{m_3^F, ...m_3^{F,Dev}\right\} \end{array}$$

where $m_h^{F,Dev}$ is the steady state money stock of a type h household that consistently deviates from the fundamental strategy, given all other households (including other type h households) continually play fundamental. For example, if household 2 consistently deviates to reject good 1, stocks of m_2 eventually run out. Thus, we know from inspection that $m_2^{F,Dev} = 0$. Call these grids m_h. Next, we use (2) and p_{hj} values from (4) to get a new grid, m'_h. We approximate this new grid back onto type $h's$ original grid, m_h, for $D_h = \{0, 1\}$. This gives the next period's state variable, depending on current state and decision. Now, we can iterate the value function for the representative type h household until stationary.[5] The fundamental equilibrium is confirmed if $D_1(0) = 0$, $D_2\left(m_2^F\right) = 1$, and $D_3(0) = 0$. A similar approach is used for the Speculative equilibrium. The resulting distribution of money in each equilibrium is:[6]

Money Distribution

	$h = 1$	$h = 2$	$h = 3$
m_h^F	0	0.3355	0
m_h^S	0.3329	0.7480	0

Similar to KW, the speculative equilibrium arises here because the marketability benefits to household 1 from accepting good 3 outweighs the storage cost disadvantage, given the distribution in each equilibrium. This outcome indicates that the results in KW can hold more generally and does not depend on their restrictive environment with indivisible goods.

[5] In the value function iteration for the representative type h household, we hold all other household's (including other type h household's) state and decision variables constant and consistent with the fundamental equilibrium. Thus, we do not consider collective deviations. Details are available at http://www.chass.utoronto.ca/~sahiabu/computational_macro/Endo.m.

[6] See the appendix for parameter choices.

5.1 Coexistence, Velocity and Welfare

The decision of household 1, $D_1 \in \{0,1\}$, is pivotal in outlining which equilibrium emerges. Suppose that a type 1 household with the state space $\{m_1, \mu(m)\} = \{0, \mu(m^F)\}$ and chooses $D_1 = 0$. Again, suppose the same household, this time with $\{m_1, \mu(m)\} = \{m_1^S, \mu(m^S)\}$ chooses $D_1 = 1$. We conclude that economic fundamentals are right for coexistence of equilibria, since both are steady states for this economy. For the parameters chosen above, we find that this indeed is the case. The no-coexistence conclusion in KW is a consequence of indivisibility, which limits the possible distributions facing households.[7] With divisible goods however, economic fundamentals allow more flexibility for the steady state distribution space. We thus find regions of the parameter space for which the discounted value of deviation strategies is less than those consistent with both equilibria.

Let the velocity of good h in equilibrium e be defined as $V_h^e = \frac{qty}{m_{h=2}^e}$, modulo 3, where qty is the quantity of commodity h that exchanges hands each period. From (2) the velocity of money become:

$$V_1^e = \left[\frac{p_{2,3}^e p_{3,1}^e}{3} + \frac{p_{2,1}^e p_{1,2}^e}{6}\left(1 + \frac{\delta_1}{u_1'}\right)\right]\frac{1}{2m_2^e}\ ,\ e = F, S \tag{5}$$

$$V_3^S = \left[\frac{p_{1,2}^S p_{2,3}^S}{3} + \frac{p_{1,3}^S p_{3,1}^S}{6}\left(\frac{\delta_3}{u_3'} + 1\right)\right]\frac{1}{2m_1^S}\ . \tag{6}$$

Using the money distribution above in each equilibrium, we have:

Velocity of Money

	V_1^e	V_3^e
$e = F$	0.1683	$n.a.$
$e = S$	0.1123	0.1683

This implies that in the fundamental equilibrium, a unit of commodity money (good 1) takes on average 5.94 periods to trade away. When good 3 begins to

[7] In their model with indivisible goods, coexistence requires that: $\overline{c}_{1,3} - \overline{c}_{1,2} = \left(p_{3,1}^F - p_{2,1}^F\right)\frac{\beta}{3}u_1' = \left(p_{3,1}^S - p_{2,1}^S\right)\frac{\beta}{3}u_1',$. This is not feasible since $p_{3,1}^F = p_{3,1}^S = 1$ and $p_{2,1}^F \neq p_{2,1}^S$.

act as money however, the turnover period for good 1 jumps to 8.91. When bad money with higher storage costs begin to circulate, it crowds out the circulation rate of good money, leading to a version of the Gresham's law. In fact, the sharp increase in the stocks of good 1 held as money is a direct result of the decline in its circulation. In the fundamental equilibrium, $p_{2,1}$ agents have $p^F_{1,2} = 1$ agents to trade with. In the speculative equilibrium however, the number of these trade partners fall to $p^S_{1,2} < 1$. Thus household 2 disposes off money less often resulting in a rise in m_2 as well as a fall in the circulation rate. This result is different from Renero (1999). Renero showed in his constructed equilibrium with coexistence that when agents employ mixed strategies and are indifferent between holding their respective non-consumption goods, the circulation rate of good 1 increases. With mixed strategies agents can adopt an always-trade policy, leading to higher frequent of trade and higher welfare.

The table below shows the effect on welfare between the two equilibria. With the bargaining framework used, each household receives positive return on half of the trades and loses nothing on the other half. That is, for household h, higher turnover for production good $h+1$ translates monotonically into higher household consumption in equilibrium. Since $u'_h > \delta$, this is always beneficial to household h. Given the results on velocity, the welfare implications are therefore below:

Welfare

	$h = 1$	$h = 2$	$h = 3$
R^F_h	-0.4081	-3.9563	8.9635
R^S_h	2.2254	3.4907	7.7055

The speculative equilibrium sees an improved turnover for good 3 whiles that of good 1 declines. Household 3 therefore suffers due to the decline in circulation of its produced goods, while household 2 benefits. For household 1, although production good 2 does not act as money, they also trade more often in the speculative equilibrium, leading to higher welfare.

5.2 Discussion

Perhaps the single most significant limitation to our results is that we impose uniformity in the allocation of stocks among household agents. Once stocks are known - i.e. m_h, $1 - m_h$ known, - agents are allocated to match, with each agent holding an equal volume $[m_h/p_{h,h+2} = (1 - m_h)/p_{h,h+1} \equiv 1]$. It may be useful to relax this assumption. First, this will require a more general functional form for utility (and/or production disutility) and a more elaborate bargaining specification. As explained earlier, the inappropriateness of imposing symmetry among household types implies that one must keep track of several Lagrange multipliers. The second challenge in the broader environment is that of finding a definite stationary state in which a specific good(s) consistently emerges as medium(s) of exchange. One can expect that households will vary quantities in order to affect the bargaining outcomes and hence their stocks of commodity money. It is also possible that periods of over-accumulation of commodity money may lead households to reject goods that they may other times decide to accept. That is, goods may cyclically lose acceptability.

We implicitly assumed that prior to matching, each agent of type h can hold stocks of either good $h+1$ or $h+2$, never both. However, after the match, they can carry quantities of both units at the same time. This essentially is a simplifying assumption, without which an agent can be in a match in which she has the option to sell units of either $h+1$ or $h+2$, if she is allowed to carry both prior to the match. This choice is not a trivial one (see Rupert, Schindler, Schevchenko and Wright 2000) and rather than delving into that debate, we used the above assumption. If agents are allowed to hold both units into matches, our results on velocity are likely to fail. Trade, consumption and production will rise leading possibly to a Pareto improvement.

In the bargaining framework used above, we did not include storage costs when stating a household's surplus from each trade encounter. As a result, trade partners do not have to compensate for storage cost implications of trade with household

h. This is unlikely to affect the results in this paper. If a household does not hold commodity money in a given equilibrium, it will be impossible during a match to bribe a single member agent to accept commodity money for two reasons. First, the bribe will have to be high enough to at least compensate for the constant cost part of storing positive units of the good ($I\overline{c}_{h,h+2}$). No such compensation can be adequate for the second reason, being the fact that each trade match encounter is infinitesimal (compared to $\overline{c}_{h,h+2} > 0$). The implication of this argument is that no commodity can emerge as money for the sole reason that there was positive surplus in a trade encounter which facilitated a bribe.

6 Conclusion

The current paper has tested the robustness of results found by Kiyotaki and Wright [1989] that attempts to identify commodity money in a strategic sequential random matching framework. The environment discussed is by design similar to the above paper to allow for close comparison. The equilibria described by these authors readily make themselves evident and for the same reasons.

The contributions of this paper are as follows. First, we find that at least in the simply model studied here, the following result in KW is robust when we make goods divisible: the existence of the fundamental and speculative steady states. Their second result that these are the only equilibria is let unverified. Although we find no new equilibria, this is not conclusive evidence against other equilibria since only a limited section of the infinite parameter space is testable by any computational exercise.

Unlike in KW and Renero (1999), we demonstrate how the equilibrium distribution of commodity money depends, not only on the matching function, but also on model parameters. Through bargaining in matches, households can influence their mix of goods and they do not rely solely on the matching function for this outcome. This distribution happens to be important, since commodity money is the only state variable for a household in the economy described in these class of models. The distribution of stocks also has important implications for velocity as well as welfare. We arrive at a non-trivial distribution of state variables compared to earlier papers on commodity money.

The third contribution is that we showed that the no-coexistence result in KW is a direct result of indivisibility. With divisible goods, the two equilibria can coexist. In the case of coexistence with mixed strategies as in Renero (1999), velocity of the lower storage cost good increases due to reduced unwillingness to part with it. Under coexistence with divisible goods however, the velocity of the lower storage cost good declines as it's velocity is crowded out by the higher storage cost good. Velocity is closely tied with production and consumption. Since velocity of the lower storage cost good declines in the speculative equilibrium, it's producer engages in less production, less consumption and less welfare. Gresham's law, rules out a Pareto improvement as we move from the fundamental to the speculative equilibrium.

Reference

Aiyagari S. R., and Wallace N. (1991) "Existence of Steady States with positive consumption in the Kiyotaki and Wright Model," Review of Economic Studies, vol 58 no 5 pp 901 – 916

Aiyagari S. R., and Wallace N. (1992) "Fiat Money in the Kiyotaki and Wright Model," Economic Theory, vol 2 no 4 pp 447 – 464

Azariadis, C., Bullard, J., and Smith, B. (2001) "Public and Private Circulating Liabilities," Journal of Economic Theory 99, pp 59 – 116

Balasko Y. and K. Shell, (1981) "The Overlapping Generations Model II: The Case of Pure Exchange with Money", Journal of Economic Theory 24, 112 – 142

Burdett, K. A. Trejos and R. Wright, (2001) "Cigarette Money," Journal of Economic Theory, 99, 117 - 142

Cuadras-Morato X. and R. Wright (1997) "Money as a medium of exchange when goods vary by supply and demand" Macroeconomic Dynamics, 1, 680–700.

Faig, M. "A Search Theory of Money and Commerce with Neoclassical Production." September 2002.

Iwai, K., (1988) "The Evolution of Money: A Search Theoretic Foundation of Monetary Economics." Working Paper no. 88-03. Philadelphia: University of Pennsylvania, Center for Analytic Research in Economics and Social Sciences,.

Jones, R. A., (1976) "The Origin and Development of Media of Exchange" The Journal of Political Economy, Vol. 84, No. 4, Part 1, pp. 757-776.

Kiyotaki, N., Wright, R. (1993). "A search-theoretic approach to monetary economics." American Economic Review 83, 63 - 77.

Kiyotaki, N. and R. Wright, (1989), "On money as a medium of exchange", Journal of Political Economy 97, 927-954.

Lagos, R., Wright, R., (2005) "A Unified Framework for Monetary Theory and Policy Analysis." Journal of Political Economy **113**, 463-484

Marimon, R., E. McGrattan and Sargent T.J. (1990), "Money as a medium of exchange in an economy with artificially intelligent agents". Journal of Economic Dynamics and Control, 14:329-373.

Menger, Karl. "On the Origin of Money." Economic Journal 2 (June 1892): 239-255.

Molico, Miguel. (1999) "The Distribution of Money and Prices in Search Equilibrium." unpublished manuscript, 1999.

Ostroy J., (1973), "The Information Efficiency of Exchange", American Economic Review, 63 (4), 597-610.

Renero J., (1999) "Does and Should Commodity Medium of Exchange have relatively low storage costs?" International Economic Review, vol 40 no 2 pp 251 –

264

Rupert, P., Schindler, M., Shevchen, A., Wright, R., (2000) "The Search-Theoretic Approach to Monetary Economics: A Primer" Economic Review, Federal Reserve, Bank of Cleveland, Quarter 4, vol 36, no. 4, pp 10 - 28

Shi, S., (1995) "Money and Prices: A model of Search and Bargaining" Journal of Economic Theory, 67, pp 467 – 496.

Shi, S. (1997) "A Divisible Search Model of Fiat Money," Econometrica, 65, pp. 75-102.

Shi, S. (1999), Search, inflation and capital accumulation, Journal of Monetary Economics 44, 81-103

Starr, R.M., (1972), "The Structure of Exchange in Barter and Monetary Economies", Quarterly Journal of Economics, 86 (2), 290-302.

Wright R., (1995) "Search, Evolution and Money" Journal of Economic Dynamics and Control, vol 19 no 1-2 pp 181 – 206

Appendix

Parameters	$h = 1, 2, 3$
u'_h	100
$c_{h,1}$	1
$c_{h,2}$	3
$c_{h,3}$	5
δ	1
β	.98