Frictional Assignment

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This version: November, 1998

Abstract

This paper examines the time-consuming process of matching the two sides of a market each having diverse characteristics. This is cast in a labor market setting where workers of different skills need be matched with different machine qualities to produce output. I characterize the efficient allocation and then show that it can be decentralized by a competitive framework. A prominent feature of the frictional assignment is that each skill level is associated with a market tightness in addition to a machine quality. The differential market tightness as an additional allocative device implies that the assignment is not always positively assortative, i.e., high quality machines are not necessarily assigned to high skills even though machine qualities and skills are complementary in production. The market mechanism that decentralizes the efficient assignment has the feature that firms post wages to attract workers in addition to choosing machine qualities. A steady state is established and numerical exercises are used to show that the differential market tightness for different skills is also quantitatively important for the wage function and wage distribution.

JEL classification: D33, J31, L11.
Keywords: Frictional matching; Market tightness; Skills; Machines; Wage distribution.

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1. Introduction

This paper deals with the fundamental issue of resource allocation in a frictional economy, the *frictional assignment*, which refers to the following two-sided matching problem. The two sides of a market, each side having diverse characteristics, need be matched with each other in order to produce "output". The matching process is time-consuming and the level of output depends on the pair's characteristics. The market can be the labor market where workers of different skills need be matched with machines of different qualities, or the loan market where projects of different qualities need be financed by different loan provisions, or the marriage market where men and women with different attributes seek for marriages. What does the efficient assignment look like in this frictional world? How can a competitive market decentralize the efficient allocation? And, What division of the match surplus does the frictional assignment imply? These are the questions addressed here. To be specific, I will focus on the matching problem in the labor market between machine qualities and worker skills.

The emphasis on frictions can be easily motivated by the existence of persistent unemployment and under-utilized machines. Until recently such frictions have been ignored in the assignment literature. The classic pieces by Tinbergen (1951) and Becker (1973) deal exclusively with frictionless matching environments, i.e., economies where a match can be formed instantaneously if it is advantageous for the two sides to do so.[1] A central result there, that the market yields efficient assignments, fails in general in the frictional environment with random-matching even when agents on each side of the market are homogeneous (see Diamond (1982), Mortensen (1982), Pissarides (1990) and Hosios (1990)). This failure makes one wonder whether agents can design other mechanisms to capture the large unrealized gains in the frictional matching environment.

Besides efficiency, there are other issues that motivate the examination of a frictional assignment. The first is about the nature of the assignment and the second is about wage inequality between skilled and unskilled workers. In a frictionless matching world, the market assignment is

[1] See Sattinger (1993) for a survey and Jovanovic (1998) for a dynamic model.

positive in the sense that high machine qualities are allocated to high skills, provided that the two sides are complementary in the production function. The positive assignment implies that the wage differential between a high-skill worker and a low-skill worker is attributed to the difference in machine qualities assigned to them as well as to the skill difference itself. With frictions it is far from obvious whether the assignment is positive and so, to understand wage inequality, one must know what the frictional assignment looks like.

To address these issues I examine a large market where workers differ in skills and skills are complementary with machine qualities in production. To emphasize the difference between a frictional assignment and a frictionless one, skills are assumed to be perfectly observable so as to make it straightforward to characterize the frictionless assignment. There is free entry by firms which choose machine qualities to match with workers. The matching process is time-consuming, as each worker can choose at most one firm in a period to form a match. I characterize the efficient allocation, show that it can be decentralized by a competitive framework, and then explore the implications on wage inequality.

The main departure of the efficient frictional assignment from a frictionless one is that it is not always positive, even though machine qualities and worker skills are always complementary in production. This is because a frictional assignment must assign a "right" number of firms (i.e., tightness) as well as a "right" machine quality to each skill in order to ensure efficiency and, as a result, high-skill workers might be compensated by sufficiently less tight markets that make it unnecessary to assign to them high quality machines at the same time. To ensure a positive assignment, machine qualities and skills must be sufficiently complementary in production.

There is a realistic market mechanism that decentralizes the efficient assignment. The key feature of the decentralization is that firms post wages to affect the number of matches they get. The philosophical reason for this mechanism to be efficient is that firms, the side of the market that incurs the cost of making a match (the cost of vacancy), are given the "property rights" to post wages to divide the match surplus. This generates differential market tightness for different skills that rewards the firm's choice of machine qualities and so implements the efficient outcome. In contrast, the lack of such property rights, as characterized by an exogenous matching function and an exogenous division of the match surplus, contributes to the failure of the markets in delivering efficiency in the random matching model mentioned above.[2]

The differential market tightness for different skills is also quantitatively significant for the wage differential between skills. Calibration exercises show that the friction increases wage inequality more through the differential market tightness than through changes in the machine quality assignment. A general technological progress that increases all skills' productivity in the same proportion benefits low-skill workers more than high-skill workers.

This paper builds on wage-posting models analyzed previously by Peters (1991), Montgomery (1991), Burdett et al. (1996), Moen (1997) and Acemoglu and Shimer (1998). In particular, Moen (1997) shows that the efficient allocation in a frictional matching economy can be decentralized by a wage-posting framework. The main limitation of these previous wage-posting models is that one or two sides of the markets are assumed to be homogeneous. This makes the assignment problem much less interesting, since the gist of the assignment problem is to find how differences in one factor price can be amplified by differences in other factors assigned to it. Allowing both machines and workers to have different qualities is necessary for addressing this issue and generates a stronger efficiency result: the wage-posting framework ensures not only the efficient division of the match surplus between the two sides of the match, as in Moen (1997), but also the efficient allocation of machine qualities to different skills.

The frictional assignment model is also closely related to two-sided matching models analyzed recently by Burdett and Coles (1997), Shimer and Smith (1998), Sattinger (1995), and Burdett and Wright (1998). In particular, Shimer and Smith have also reached the conclusion that sufficient complementarity in production between the two sides of the market is necessary for a

[2] Another strand of the search literature, surveyed by McMillan and Rothschild (1994), assumes that agents only know the distribution of wages before search and must incur the search cost to find any particular wage. With this type of search the market assignment is also unlikely to be efficient.

frictional assignment to be positive.[3] Focusing on equilibrium outcomes rather than efficiency, these models follow the footsteps of earlier random-matching models to employ exogenous matching functions and/or exogenous rules of surplus division between matched agents. In contrast, the current model uses the wage-posting setup to endogenize both the matching function and the surplus division which, as described above, are essential for the market mechanism to deliver efficiency. In comparison with Shimer and Smith (1998), in particular, the additional difference is that the number of firms here is determined by free-entry rather than being fixed. Allowing for endogenous entry is necessary for the market to provide the correct tightness for each skill. Also, if matching is time-consuming and the ratio of agents on the two sides of the market is fixed at an arbitrary number, a non-positive assignment may not be surprising. By allowing for free-entry, I establish a stronger result: Even when the market tightness adjusts efficiently, the assignment is not necessarily positive.

The remainder of this paper is organized as follows. Section 2 analyzes the frictionless assignment. Section 3 characterizes the efficient assignment with matching frictions. Section 4 describes the decentralization mechanism. Section 5 examines the properties of the frictional assignment. Section 6 extends the analysis to a dynamic setting and calibrates the model. In particular, the wage distribution and its responses to technological progress are calculated. Section 7 concludes the paper and the appendix provides some proofs.

2. Frictionless assignment

For the moment let us consider a simple economy where the time horizon is one period and agents are all risk neutral. There are a large number, N, of workers who differ in skills. To make things simple, skills can be observed and measured by a one-dimensional object s, which lies in a compact set S with a minimum $s_L > 0$ and a maximum s_H. Skills are distributed among workers in the labor force according to $G_0(\cdot)$ with a density function $g_0(\cdot)$. The number of workers with

[3] There is also a fair amount of work that analyzes centralized matching with frictions and/or decentralized matching without prices (see Roth and Sotomayor (1990)). In contrast, the focus of this paper is on how prices/wages can induce efficiency in a decentralized matching framework.

skill s is $n(s) = Ng_0(s)$, which is a large exogenous number.

Machines differ in qualities that are denoted $k \in K \subseteq R_+$. A machine of quality k costs $C(k)$ to make. In contrast to the fixed distribution of workers, the distribution of machines is endogenously determined by firms' entry. A machine can be operated by only one worker at a time. Workers and firms derive income solely from their production. A worker of skill s operating a machine of quality k produces output $F(k, s)$. Machine qualities and skills are complementary, i.e., $F_{ks} > 0$. The assignment problem is to find a mapping ϕ: $S \rightarrow K$, that assigns a machine quality $\phi(s)$ to each skill s. The assignment is called *positive* if higher skills are assigned better machines, i.e., if $\phi_s(s) > 0$.

Assumption 1. *(i)* $C(0) \geq 0$, $C_k(0) = 0$, $C_k(k) > 0$ *and* $C_{kk}(k) \geq 0$ *for all* $k > 0$*;*
(ii) $F_k(k, s) > 0$, $F_{kk}(k, s) < 0$, $F_s(k, s) > 0$ *and* $F_{ss}(k, s) < 0$ *for all* s *and* k*;*
(iii) $F_{ks} > 0$, $F(0, s) = F(k, 0) = 0$*;*
(iv) There exists a non-empty subset of K *such that* $F(k, s_L) - C(k) > 0$*;*
(v) $(F_k C_{kk} - C_k F_{kk})F > (F_k - C_k)F_k^2$.

Conditions (i) and (ii) are standard for cost and production functions. (iii) requires skills and machine qualities to be complementary and, for unmatched machines and workers, output to be zero. (iv) says that even the lowest skill can produce positive net output with some machine qualities. Since $F_s > 0$, there are positive match surpluses to be made for all skills. (v) is a concavity condition necessary for the assignment problem to have a maximum.

Consider first a perfect world without matching frictions so that every worker can be matched instantaneously with a machine. The efficient assignment in this world, denoted ϕ^p, maximizes the net output, $F(k, s) - C(k)$. That is, for each s, $\phi^p(s)$ satisfies:

$$F_k(\phi^p(s), s) = C_k(\phi^p(s)). \tag{2.1}$$

Under Assumption 1, the assignment ϕ^p exists and is unique. Moreover, complementarity between skill and machine quality implies $\phi_s^p(s) > 0$, i.e., the assignment is positive.

The efficient assignment can be decentralized as follows. Imagine that for every pair of machine quality and skill, (k, s), including those pairs that are not observed in equilibrium, there is a wage $W(k, s)$. This wage schedule must satisfy two requirements: it must be non-negative for every pair (k, s), and whenever $F - C \geq 0$ it must deliver zero net profit for the firm. The zero net profit requirement comes from the assumption that there is free entry by firms into the economy. Thus, for every pair (k, s),

$$W(k,s) = \begin{cases} F(k,s) - C(k), & \text{if } F(k,s) - C(k) \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Given the wage schedule and for any given skill s, a firm chooses k to solve:

$$\max_{k} \ \{W(k,s)\colon (2.2)\}.$$

Thus, the choice of k maximizes the net output and the solution is the efficient assignment given in (2.1). This assignment implies the following equilibrium wage for skill s:

$$w^p(s) \equiv W(\phi^p(s), s) = F(\phi^p(s), s) - C(\phi^p(s)) > 0.$$

The equilibrium wage satisfies $w^p_s(s) = F_s(\phi^p(s), s) > 0$. Thus, as in a standard framework with homogeneous machines, a skill is rewarded at the margin with its marginal product and higher skills get higher wages. In contrast to a framework with homogeneous machines, the wage (or earning) function is not necessarily concave: Since the assignment is positive, a higher skill uses a better machine and so the marginal product of skill may increase with skill levels.

The competitive assignment problem can be formulated as a dual problem where firms compete in offering machines to workers so as to minimize the net cost $W + C - F$. That is, given the wage schedule and for any given skill s, the choice of k solves:

$$\max_{k} \ \{F(k,s) - C(k) - W(k,s)\colon \ W(k,s) \geq w^p(s)\}.$$

The dual formulation illustrates that firms cannot increase the net profit by bringing into market s a machine that is different from $\phi^p(s)$.

It is important that the wage schedule $W(k,s)$ specifies a wage for every possible pair (k,s), even though in equilibrium only the pair $(\phi^p(s),s)$ is observed for each skill s. For all $k' \neq \phi^p(s)$, (2.2) is a restriction off the equilibrium path and it is rationalized as follows.[4] Other pairs (k',s), where $k' \neq \phi^p(s)$, are not observed in equilibrium not because no firm has ever thought about pairing k' with s but because k' is inferior to the chosen one $\phi^p(s)$. If a firm offers $k' \neq \phi^p(s)$ to workers with skill s and makes a non-negative profit, the firm should expect other firms to enter and drive the wage to $W(k',s) = F(k',s) - C(k')$. In this case, the firm will not attract any worker, since $W(k',s) < W(\phi^p(s),s)$ for all $k' \neq \phi^p(s)$.

The same off-the-equilibrium-path restriction implies that workers of skill s have no incentive to apply to firms with any other machine quality $k' \neq \phi^p(s)$. If they did, they would receive the wage $W(k',s)$ which is strictly less than the wage they receive if they stay with firms with $\phi^p(s)$.

3. Efficient assignment with frictions

Now consider an economy with frictions that not all workers and machines can be matched instantaneously. The simplest way to do this is to assume that a worker can apply to at most one firm in the period. Let the efficient assignment be $\phi^o(\cdot)$ and, for brevity, refer to the group of machines with quality $\phi^o(s)$ and workers with skill s as market s, although at this point I am not concerned with the market assignment. If there are $m(\phi^o(s))$ machines in market s (and $n(s)$ workers), the matching probability is $1 - e^{-b^o(s)}$ for each firm and $[1 - e^{-b^o(s)}]/b^o(s)$ for each worker, where $b^o(s) \equiv n(s)/m(\phi^o(s))$ is the tightness of market s. These probabilities are derived later in Section 4. At this point let us notice that the matching probabilities depend on the tightness in an intuitive way and they imply a linearly homogeneous matching technology.

The efficient assignment in this frictional world maximizes the sum of expected net product of all pairs of machine qualities and skills. The social planner cannot achieve this objective by choosing machine qualities alone. This is because the sum of expected net output also depends on the number of matches. The planner must also choose the number of firms for each market s,

[4] Similar restrictions off the equilibrium path are imposed in Gale (1992).

$m^o(s)$, in order to control the number of matches. That is, the planner solves:

$$\mathcal{L} \equiv \max_{(b^o(s),\phi^o(s))_{s\in S}} \int_{s\in S} m^o(s)\left[\left(1-e^{-b^o(s)}\right)F(\phi^o(s),s) - C(\phi^o(s))\right]ds, \tag{3.1}$$

subject to $m^o(s) = n(s)/b^o(s)$. The first term in the square brackets of the integrand is expected output of each firm and the second term is the cost of the machine. Note that a machine must be put in place in order to be matched with a worker and so the cost of the machine must be incurred regardless of the outcome of the match.

The first-order conditions for (ϕ, b) are:

$$1 - e^{-b^o(s)} = \frac{C_k(\phi^o(s))}{F_k(\phi^o(s),s)}; \tag{3.2}$$

$$1-(1+b^o(s))e^{-b^o(s)} = \frac{C(\phi^o(s))}{F(\phi^o(s),s)}. \tag{3.3}$$

(3.2) states that the assignment ϕ^o equates the expected marginal product of the machine quality, $(1-e^{-b})F_k$, to the marginal cost. To explain (3.3), note that adding one more firm to the market increases the expected net output by $(1-e^{-b})F - C$ but also increases congestion to existing firms. The increased congestion reduces each existing firm's matching probability by $\frac{b}{m}e^{-b}$ and hence reduces aggregate output in that market by $be^{-b}F$. (3.3) requires that the number of firms be such that at the margin expected net output from an additional firm is equal to expected crowding out caused by that firm.

Proposition 3.1. *Under Assumption 1, there exist an efficient assignment ϕ^o and an associated market tightness b^o. Moreover, $\phi^o(s) < \phi^p(s)$ for every $s \in S$.*

Proof. Under (v) in Assumption 1, the solution to (3.2) and (3.3) is a local maximum of the efficient assignment problem if it exists. Since the LHS of (3.2) is less than one for all $b < \infty$, the assignment must satisfy $C_k < F_k$. Since the frictionless assignment $\phi^p(s)$ satisfies $C_k = F_k$ and since $C_k - F_k$ is an increasing function of k, the condition $C_k < F_k$ is equivalent to $\phi^o(s) < \phi^p(s)$ for all s. Also, the LHS of (3.2) is greater than that of (3.3) for all $b > 0$ and so the assignment must satisfy $C_k/F_k > C/F$. This is equivalent to $k > k_{\min}(s)$, where $k_{\min}(s)$ is defined as follows:

$$[F(k,s)C_k(k) - C(k)F_k(k,s)]_{k=k_{\min}(s)} = 0. \tag{3.4}$$

Under Assumption 1, $k_{\min}(s)$ is well defined and $0 \le k_{\min}(s) < \phi^p(s)$.

Eliminating b^o from (3.2) and (3.3) yields:

$$\frac{C_k(k)}{F_k(k,s)} - \frac{C(k)}{F(k,s)} + \left(1 - \frac{C_k(k)}{F_k(k,s)}\right) \ln\left(1 - \frac{C_k(k)}{F_k(k,s)}\right) = 0. \tag{3.5}$$

Temporarily denote the left-hand side of the above equation by $LHS(k)$ for any given s. Then $LHS(k_{\min}(s)) < 0$, $LHS(\phi^p(s)) > 0$ and so, for each $s \in S$, there is at least one solution for k lying in the interior of $(k_{\min}(s), \phi^p(s))$. QED

For general functional forms uniqueness of the efficient assignment is difficult to be established but will be assumed throughout the following analysis. Examples in Section 5.2 show that uniqueness is guaranteed for some popular functional forms. Note that the frictional assignment assigns a lower machine quality to each skill than the frictionless assignment does. This is because the possibility of failing to get matched in the frictional economy reduces the expected profit from any machine quality. Let me delay the discussion on other properties of the efficient assignment and turn now to the decentralization of the efficient assignment.

4. Market assignment with frictions

To decentralize the efficient assignment described above, consider the following markets. For each pair (k, s), there is a market tightness schedule $B(k, s)$ determined by a zero-profit condition for entry. Taking this schedule as given, each firm selects a machine quality and each worker chooses a quality k to target the application. Before each worker selects any particular firm in the targeted group, firms post wages simultaneously. Observing all the posted wages, workers decide which firm to apply to, possibly with mixed strategies, and then each firm chooses a worker among its applicants. Note that, in contrast to the frictionless world, what is taken as given by firms here is not the wage schedule but the schedule of market tightness.

The problem can be solved backward. First, for given machine quality k and the tightness $B(k,s)$, I determine the equilibrium wage in market s, $W(k,s)$. With the wage I can compute firms' expected profit and workers' expected wage. Second, the market tightness schedule $B(k,s)$ must be such that firms in each market earn a zero expected net profit, i.e., the expected profit equals the cost of the machine. Third, taking the schedule $B(k,s)$ as given, firms solve for the assignment by choosing k to maximize workers' expected wage.

4.1. Wage $W(k,s)$

Isolating the market where the machine quality is k and workers' skill is s, I determine the wage emerging from the wage posting game. Variables here are indexed by (k,s) which are suppressed. It is useful to express the wage as a share A of output, i.e., $A = W/F$, and formulate firms' wage posting decision as one that determines the wage share. Throughout this paper, I am interested only in the equilibrium that is symmetric within each market, i.e., in equilibrium all firms in a market post the same wage share A for the same skill.

All firms announce their wage shares simultaneously and workers apply to the firms after observing all posted wage shares. If a firm gets only one worker, the worker is rewarded the job. If the firm gets more than one worker, each worker is selected with equal probability. In either case, production begins immediately after the match and output is divided between the worker and the firm according to the posted share. If a firm fails to recruit any worker, output is zero.[5]

Since workers observe all posted wages and then choose which firm to apply, firms can directly influence workers' application strategies through the posted wages. To be more specific, let all $(m-1)$ firms in the market post a wage share A and the remaining one firm post a wage share A^d. Call this firm the deviator and other firms non-deviators. If $A^d > A$, the deviator can

[5] The qualitative results will be similar if each worker observes only two independently drawn wages, but the exercise is more cumbersome (see Acemoglu and Shimer (1998)). Similarly, one can allow firms to post the reserve wage rather than the actual wage and then hold an auction after receiving two or more applications. With this setup the actual wage equals the reserve wage if the firm receives only one application and equals zero if the firm receives at least two applications. The reserve wage serves a role very much like the actual wage in the current framework but there is a dispersion in actual wages (see Julien et al. (1998)). Such a dispersion complicates the analysis without contributing much to the main issue.

attract more workers than non-deviators do. However, not all workers go to the deviator with probability one – if they did so, the probability for each to be selected by the firm would be very small ($1/n$) and workers could improve the expected wage by applying to a non-deviator. Let p^d be the probability with which each worker applies to the deviator. Then $p = (1 - p^d)/(m - 1)$ is the probability with which the worker applies to each of the non-deviators.

To find how p^d depends on A^d, let us compute the expected wage of an arbitrary worker in that market, say worker 1, who applies to the deviator. When worker 1 applies to the deviator, there might be k other workers applying to the same firm, which occurs with probability $C_{n-1}^k (p^d)^k (1 - p^d)^{n-1-k}$. In this case worker 1 is chosen by the firm with probability $1/(k+1)$. Since k can be any integer from 0 to $n-1$, worker 1 gets the job from the deviator with the following probability

$$\sum_{k=0}^{n-1} \frac{1}{k+1} C_{n-1}^k (p^d)^k (1 - p^d)^{n-1-k} = \frac{1 - (1 - p^d)^n}{np^d}.$$

If worker 1 gets the job from the deviator, his ex post gain (wage) is $A^d F$. The expected gain must be the same as that obtained from applying to a non-deviator, i.e.,

$$\frac{1 - (1 - p^d)^n}{np^d} A^d = \frac{1 - (1 - p)^n}{np} A. \tag{4.1}$$

This equation implicitly defines a function $p^d = p^d(A^d)$, which can be shown to be an increasing function. Therefore, by posting a higher wage share the deviator can obtain a higher expected number of workers. Since the function $p^d(\cdot)$ is continuous, workers respond to a marginal increase in the offer by only a marginal increase in the application probability.

The deviator chooses A^d to maximize the expected profit, taking the dependence $p^d(A^d)$ into account but taking other firms' wage shares as given. Since the probability with which the deviator has at least one worker is $1 - (1 - p^d)^n$, A^d solves:

$$\max_{A^d} \; \{[1 - (1 - p^d)^n](1 - A^d)F \colon \, (4.1)\}.$$

The above problem can be solved directly but the algebra is messy. The complexity arises from the fact that a single firm's deviation affects both p^d and p. That is, the expected wage that a worker gets from applying to any other firm, the right-hand side of (4.1), is also affected by A^d. By posting a high wage share the deviator attracts workers away from other firms, reduces the congestion that workers face in other firms and hence increases workers' expected wage from applying to those firms. To simplify algebra, I assume that n and m are sufficiently large. In this case, a single firm's deviation has a negligible effect on the congestion which workers face in other firms and so the deviator can treat the expected wage that a worker gets from other firms as exogenous (see Burdett et al. (1996)).[6] Denote this expected market wage as a share EA of output. Then, for large n and m, the solution to the deviator's problem can be approximated arbitrarily closely by the solution to the following problem:

$$\max_{A^d} \left\{[1-(1-p^d)^n](1-A^d) \colon \frac{1-(1-p^d)^n}{np^d} A^d = EA\right\}.$$

The above deviation cannot be profitable in equilibrium and so the solution to the above problem must be $A^d = A$, which implies $p^d = p = 1/m$. In the limit $n, m \to \infty$ (but $n/m \to B$), $np^d = np \to B$ and $(1-p)^n \to e^{-B}$. The limit of the first-order condition becomes:

$$A = \frac{B}{e^B - 1}. \tag{4.2}$$

It is evident that A is in the interior of $(0,1)$ for all finite B and is a decreasing function of B. Thus, workers get a smaller share of the output when the ratio of workers to firms is larger.

Moreover, the matching probability for a firm is endogenously determined as $1-(1-p)^n \to 1-e^{-B}$ and the matching probability for a worker is

$$\frac{1-(1-p)^n}{np} \to \frac{1-e^{-B}}{B}.$$

These matching probabilities were used in Section 3 for the efficient assignment. Each worker's expected wage, EW, and each firm's expect profit, EP, are as follows:

$$EW = e^{-B}F; \quad EP = [1-(1+B)e^{-B}]F. \tag{4.3}$$

[6] More precisely, when n and m are both large with a finite ratio n/m, the effect of a single firm's deviation on the probability that each worker applies to the deviator, p^d, is of order $1/n$, but the effect on the probability that each worker applies to a non-deviator, $p = (1-p^d)/(m-1)$, is of order $1/n^2$. Thus, the deviation has a non-negligible effect on the probability that a worker obtains a job from the deviator, $[1-(1-p^d)^n]/(np^d)$, but a negligible effect on the probability that a worker obtains a job from a competing firm, $[1-(1-p)^n]/(np)$.

These results are intuitive. If a firm did not face any risk of failing to obtain a worker, i.e., if $e^{-B} = 0$, the firm would have all the monopoly power and would demand the entire output; if a worker did not face any risk of failing to get a job, the worker would have all the monopoly power and would demand the entire output. More generally, as the market gets tighter, i.e., as B increases, the worker's expected wage as a share of output decreases and the firm's share increases. Note that EW and EP do not add up to the value of output: the remainder, $Be^{-B}F$, is the expected loss in output due to congestion and is borne by the firm.

4.2. Market tightness and the choice of machine quality

The market tightness schedule $B(k, s)$ must be such that the expected net profit of operating machine k with skill s is zero, i.e., $EP(k, s) = C(k)$, where EP is given above. This is true for such (k, s) that output is at least as high as the cost of the machine. For all other (k, s) such that $C(k) > F(k, s)$, no firm will adopt k for s and so $B(k, s) = \infty$. That is,

$$\begin{cases} 1 - [1 + B(k, s)]e^{-B(k,s)} = \frac{C(k)}{F(k,s)}, & \text{if } C(k) \leq F(k, s) \\ B(k, s) = \infty, & \text{otherwise.} \end{cases} \tag{4.4}$$

Taking the schedule $B(k, s)$ as given, each firm chooses k to maximize workers' expected wage, i.e., $\max_k e^{-B(k,s)}F(k, s)$. The solution yields the assignment $k = \phi(s)$, which induces a market tightness for market s, $b(s) = B(\phi(s), s)$. Since the choice of k affects the value of B, to depict the solution in a two-dimensional diagram we can alternatively formulate the problem as such that each firm directly chooses (B, k):

$$(P) \quad \max_{(B,k)} \left\{ e^{-B}F(k, s) \colon 1 - (1 + B)e^{-B} = \frac{C(k)}{F(k, s)} \right\}.$$

The solution is illustrated in Figure 4.1. For any given s and the worker's expected wage, the worker's indifference curve, $B = IND(k)$, is increasing and concave. The constraint in (P) gives an upward sloping curve, $B = ZNP(k)$. The place where the two curves are tangent to each other gives the solution to the first-order conditions of (P). Condition (v) in Assumption 1 ensures $ZNP_{kk} > IND_{kk}$ around the solution and so the solution is a local maximum of (P).

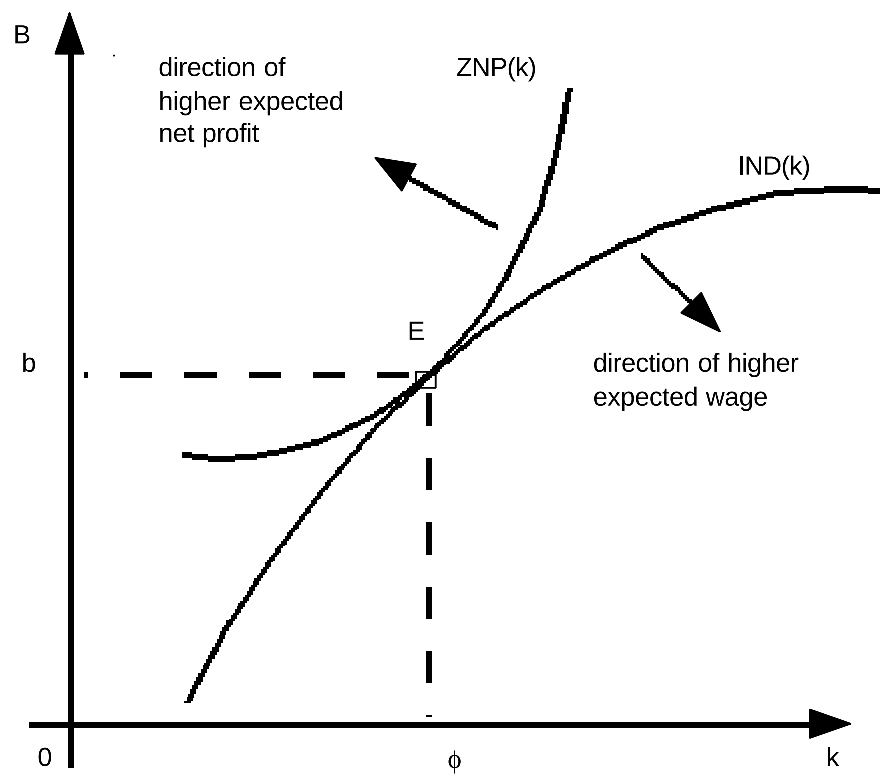
B
direction of
higher expected
net profit
ZNP(k)
IND(k)
E
b
direction of higher
expected wage
0
ϕ
k

Figure 4.1:

I now argue that the assignment leaves no incentive for firms or workers to deviate. As in the frictionless assignment, the assignment problem (P) can be alternatively written as a dual problem where each firm maximizes the expected net profit, subject to the constraint that the worker gets at least the equilibrium expected wage. Therefore, if a firm brings into market s a different machine quality $\phi' \neq \phi(s)$, it will make a negative net profit.

Note that (4.4) imposes off-equilibrium-path restrictions on the tightness which can be rationalized as follows. If a firm offers $k' \neq \phi(s)$ to worker s and makes a non-negative profit, the firm should expect other firms to enter to offer k' to s as well, which will drive the expected net profit to zero and the market tightness to $B(k', s)$ described in (4.4). Similarly, for a worker s who tries to target a machine quality k', both the firms and the worker must perceive a tightness $B(k', s)$. This off-the-equilibrium restriction makes it irrational for worker s to target any machine quality other than $\phi(s)$: Given the tightness schedule, $\phi(s)$ maximizes the expected wage of worker s.

I am now ready to present the central result:

Proposition 4.1. *The assignment given by the solution to (P) is an equilibrium (market) assignment. The market assignment ϕ and the market tightness b are efficient.*

Proof. I have already argued that the solution to (P) forms an equilibrium. To show that it is efficient, let λ be the Lagrangian multiplier of the constraint in (P). The first-order condition for B yields $\lambda = F/B$. Substituting λ into the first-order condition for k yields: $Be^{-B} = \frac{C_k}{F_k} - \frac{C}{F}$. This combined with the constraint in (P) gives $1 - e^{-B} = C_k/F_k$, which is the same as (3.2). The constraint in (P) is the same as (3.3). Therefore, the solutions for (k, B) are (ϕ^o, b^o). QED

Remark 1. *Since the market assignment is efficient, the expected wage for a worker with skill s must be equal to the social marginal benefit of such workers. That is,*

$$Ew(s) = \frac{d\mathcal{L}}{dn(s)}, \tag{4.5}$$

where $\mathcal{L}$ is the maximized value of the social welfare defined in (3.1). Then $w(s)$ can be recovered from the relation $Ew(s) = w(s)(1 - e^{-b(s)})/b(s)$.

The market assignment is efficient because equilibrium wages are tied endogenously to the market tightness in a special way. As (4.2) reveals, the wage share for a pair (k,s) is a decreasing function of the market tightness, $B(k,s)$. For any given skill, if there are fewer firms in the market, i.e., if $B(k,s)$ is larger, the wage posting scheme enables firms to explore the relatively large supply of labor and retain a larger share of output. Thus, for given s, a higher quality and more expensive machine is rewarded with a larger share of the surplus. This higher share does two things for a more expensive machine. First, it compensates the diminishing marginal product of machine quality and enables the (expected) social marginal product of machine quality, $(1-e^{-B})F_k$, to be equal to the social marginal cost, C_k. Second, it ensures that the number of firms using the machine to be equal to the efficient one, i.e., the one given by (3.3).

Another way to look at the connection between the wage share of output and the market tightness is to tie the wage share to the matching function. The matching function is endogenously generated by the wage-posting game and the total number of matches in a period in market s is $m(1-e^{-n/m})$. The elasticity of the total number of matches with respect to the number of workers in that market is

$$\frac{n}{m(1-e^{-n/m})}\cdot\frac{\partial[m(1-e^{-n/m})]}{\partial n}=\frac{B}{e^{B}-1}=A. \tag{4.6}$$

Therefore, the wage-posting outcome rewards each factor by precisely the factor's contribution to the match, defined as the elasticity of matches to the factor. This is not a sheer coincidence. Rather, it arises from the fact that agents (firms) who actively create matches by creating vacancies are given the "property rights" to choose the split of the match surplus (by choosing wages). Also, they are the ones who bear the congestion from their choices, as (4.3) shows.

It should then be clear that efficiency cannot be guaranteed by arbitrary wage determination schemes. One such scheme that fails is the Nash bargaining framework in Diamond (1982), Mortensen (1982), and Pissarides (1990). In these models, both the matching function and the weights in Nash bargaining are exogenously fixed. When the wage is determined, there is only one firm and one worker on each side of the bargain and each side has local monopoly power.

This one-to-one situation does not accurately reflect the composition of firms and workers before the match and so the Nash bargaining outcome typically fails to deliver efficiency. The only exception is when the worker's bargaining weight is exogenously set to be equal to that in (4.2), but then it cannot be constant as it necessarily varies with the market tightness. This efficiency requirement is, of course, the one obtained by Hosios (1990). The current analysis generalizes it to a richer environment. Not only is the condition necessary for the match surplus to be divided efficiently between workers and firms, but also it is necessary for stimulating firms to choose the "right" machine quality for each skill.

The second scheme that fails to deliver efficiency is the search models surveyed by McMillan and Rothschild (1994), where workers know only the distribution of wages and must costly search to find a specific firm's wage offer. Since any specific firm's wage is not observed by workers before search, it does not influence workers' search decisions directly; only the distribution of wages does. That is, individual firms' wages do not have the ex ante allocative role as they do in the wage-posting framework and so the market tightness for each skill is unlikely to be efficient.

5. Properties of the frictional assignment and wages

5.1. Properties

An important feature of the frictional assignment is that each skill level is associated with a market tightness as well as a machine quality. One implication is that the assignment is not always positive. If high-skill workers get matched sufficiently more quickly than low-skill workers, they may get lower quality machines than do low-skill workers and yet still enjoy a higher expected wage. That is, the machine assignment can be negative if $b(s)$ is sufficiently decreasing in skill. These properties are stated below:[7]

Proposition 5.1. *The assignment ϕ is positive, i.e., $\phi_s(s) > 0$ for all $s \in S$, if and only if*

$$\frac{FF_{ks}}{F_k F_s} > \frac{CF_k(F_k - C_k)}{C_k(FC_k - CF_k)}.$$

[7]The proof is omitted, since it involves tedious algebra of differentiating (3.2) and (3.3) and substitution.

A higher skill has a higher matching rate, i.e., $b_s(s) < 0$, *if and only if*

$$\frac{FF_{ks}}{F_k F_s} < \frac{CF(F_k C_{kk} - C_k F_{kk})}{C_k F_k (FC_k - CF_k)}.$$

Thus, $b_s \geq 0$ *implies* $\phi_s > 0$ *and so* $\phi_s \leq 0$ *implies* $b_s < 0$. *Under (v) in Assumption 1, there is a non-empty parameter region in which both* $\phi_s > 0$ *and* $b_s < 0$.

Although a positive assignment cannot be guaranteed in general, it does emerge when skills and machine qualities are sufficiently complementary to each other in production. Exactly how complementary should the two factors be to generate a positive assignment depends also on features of the cost function. On the other hand, if the two are extremely complementary to each other, an increase in skill requires a large increase in machine quality, which at the margin is very costly for firms to make. In this case there will be fewer firms using high qualities machines, i.e., $b_s(s) > 0$. When skills and machines qualities are strongly but not extremely complementary to each other, high skills are assigned high quality machines and find jobs more easily.

These properties can be illustrated in Figure 4.1 by increasing the skill level from s to s'. When s increases to s', the zero-net-profit curve $ZNP(k)$ shifts down. The new tangency point in Figure 4.1 can be either on the left or on the right side of the original tangency point and so $\phi(s')$ can be either smaller or greater than $\phi(s)$. Similarly, the new tangency point can be either above or below the original tangency point and so a higher skill is not necessarily associated with a less tight market or a higher matching probability. Nevertheless, if the new solution is at least as high as the original one along the vertical axis, then it must be on the right side of the original one. That is, for $s' > s$, $b(s') \geq b(s)$ implies $\phi(s') > \phi(s)$ and so $\phi(s') \leq \phi(s)$ implies $b(s') < b(s)$. With a negative assignment, higher skills must be compensated by a higher matching rate.

I now turn to equilibrium wages. First, let us examine the expected wage for worker s:

$$Ew(s) = e^{-b(s)} F(\phi(s), s) = e^{-B(\phi(s),s)} F(\phi(s), s).$$

The fact that the worker's indifference curve $IND(k)$ in Figure 4.1 moves toward southeast when s increases shows that a higher skill gets a higher expected wage, regardless of the signs of ϕ_s

and b_s. A close inspection of the expression for the expected wage reveals that skill might affect the expected wage in three ways. An increase in s (i) affects the machine quality assigned to it; (ii) increases output directly; and (iii) attracts more firms to the market and hence reduces the workers' congestion. The effect of (i) on the expected wage depends on whether the assignment is positive, but the effects of (ii) and (iii) are unambiguously positive. Moreover, the effect of (i) on the expected wage vanishes when the machine quality is chosen optimally by the firm. In the current setting, in particular, the output increased by a better machine is exactly canceled by the increased congestion that the better machine creates for workers (since, for any given skill, the higher cost of the better machine make fewer firms choose it).

The observed wage for skill s is $w(s) \equiv W(\phi(s), s)$. Direct computation yields:

Proposition 5.2. *$Ew_s(s) > 0$ for all s. If $\phi_s \geq 0$ then $w_s > 0$. That is, a higher skill is rewarded a higher wage if the assignment is positive. Moreover, $w_s < F_s(\phi(s), s)$ if and only if $b_s(s) < 0$.*

The result that wages increase with skills for positive assignments is not obvious ex ante. As stated in Proposition 5.1, a positive assignment may be accompanied by an increasing matching rate for high skills. Since what matters to workers' decisions is not the actual wage but rather the expected wage, the outcome $w_s < 0$ can be consistent with $\phi_s > 0$, a priori, if $b(s)$ is sufficiently decreasing. The proposition shows that this does not happen in equilibrium.

The proposition also states that the marginal reward to skill, w_s, is less than the marginal product of skill if and only if the matching rate increases with skill. This is because an increase in skill is compensated not by the actual marginal product but by the expected marginal product of skill which takes into account of the matching rate. If a higher skill comes with a lower matching rate, i.e., if $b_s > 0$, the additional skill must be compensated by more than the marginal product of skill in order to make up for the reduced matching probability. This is in contrast with the result in the frictionless economy, where w_s is always equal to the marginal product of skill.

The case where $b(s)$ is constant over S is an important special case. What economies give rise to a constant b? The following proposition provides an answer (see Appendix A for a proof) and the next subsection provides some examples.

Proposition 5.3. *Suppose* $b(s) = constant$. *Then the assignment* ϕ^o *is positive. If* $F(k,s)$ *is linearly homogeneous in* (k,s), *then there exist constants* $(\delta 1, \delta 2)$ *with* $\delta 1 < \delta 2 < 1$ *such that* $\phi(s)$ *is implicitly given by the following equation:*

$$s = s_L \exp\left[\int_{\phi(s_L)/s_L}^{\phi(s)/s} \left(\frac{\delta 1}{\delta 2 - \delta 1} f(y) - y\right)^{-1} dy\right], \tag{5.1}$$

where $f(k/s) \equiv F_s/F_k$. *If the production function* F *is the CES type,* $F = F_0[\alpha k^\rho + (1-\alpha)s^\rho]^{1/\rho}$, *then the assignment is*

$$\phi(s) = \left[[\phi(s_L)]^\rho + \frac{(1-\alpha)\delta 1}{\alpha(\delta 2 - \delta 1)}(s^\rho - s_L^\rho)\right]^{1/\rho}, \tag{5.2}$$

and the cost function must have the following form:

$$C(k) = \delta 1 \cdot F_0 \left[\frac{\alpha \delta 2}{\delta 1} k^\rho + (1-\alpha) s_L^\rho - \frac{\alpha(\delta 2 - \delta 1)}{\delta 1}[\phi(s_L)]^\rho\right]^{1/\rho}. \tag{5.3}$$

In this case the assignment is concave if and only if $C_{kk} > 0$.

5.2. Examples

Example 5.4. $C(k) = C_0 k^\gamma$ *and* $F(k,s) = F_0 k^\alpha s^{1-\alpha}$, $\alpha \in (0,1)$.

This is a special case described in Proposition 5.3, with $\rho = 0$. In fact, taking the limit $\rho \to 0$ on (5.3) shows that $C(k) = C_0 k^{\alpha \delta 2/\delta 1}$ for some $C_0 > 0$. Therefore, $\alpha\delta 2/\delta 1 = \gamma$ and the unique assignment in (5.2) becomes

$$\phi(s) = \phi(s_L)\left(\frac{s}{s_L}\right)^{(1-\alpha)/(\gamma-\alpha)}.$$

The assignment is concave if and only if $\gamma > 1$, an implication of Proposition 5.3. Since $b_s = 0$, Proposition 5.2 implies that the marginal wage w_s is equal to the marginal product of labor.

Example 5.5. $C(k) = C_0 k$ *and* F *is the CES form with* $\rho < 1$.

This is another special case of Proposition 5.3, with $C_{kk} = 0$. Setting $C_{kk} = 0$ in (5.3) yields a restriction on $\phi(s_L)$. Substituting such $\phi(s_L)$ into (5.2) yields $\phi(s) = \phi_0 s$ for some constant $\phi_0 > 0$. The assignment is positive and linear. This example is interesting because Jovanovic (1998) shows that, in a frictionless assignment, non-degenerate distributions of skills and machine qualities are consistent with positive long-run growth in per-capita income only when the cost of machine is linear in quality, at least at the aggregate level. This example indicates that a similar result can be established when the assignment is frictional.

Example 5.6. *$F(k, s)$ is the CES type with $\rho \neq 0$ and $C(k) = C_0 k^\gamma$ with $\gamma > 1$.*

The assignment is no longer linear and $b_s \neq$ constant. The assignment can even be decreasing in skill. To see this, let $C_0 = 0.2$, $\gamma = 3$, $F_0 = 1$, $\alpha = 0.35$ and $\rho = 0.8$. Then $\phi(2) = 0.254 > 0.25 = \phi(3)$. Skills are compensated for the negative assignment by a higher matching rate, as $b(2) = 0.075 > 0.064 = b(3)$.

6. Dynamic assignment

I now extend the analysis to a dynamic setting where unmatched workers and firms keep trying to get matched over time. The extension is necessary for three reasons. First, it is important to show that the central results are robust to workers' and firms' dynamic concerns. Second, the one-period model is difficult to be calibrated to check the quantitative results. Third, it is important to examine how steady state wage distribution responds to disturbances in the production technology. I characterize the efficient allocation below.

6.1. Characterization and existence

Firms and workers live forever and discount future with a factor $\beta \in (0, 1)$. Quality k machine costs $C(k)/(1 - \beta)$ to produce and so $C(k)$ is the cost per period. Now N is the total number of workers in the entire labor force rather than the ones who are looking for jobs. Let $n(s)$ be the exogenous number of skill s workers in the labor force and $g_0(s) \equiv n(s)/N$ be the density of the skill distribution in the labor force. A number $u_t(s)$ of skill s workers are unemployed at the beginning of period t and $E_t(s)$ are employed after recruiting in period t. Since $u_t(s)$ and $E_t(s)$ are measured at different points of time in period t, they do not add up to $n(s)$. Let $m_t(s)$ now stands for the number of firms that recruit in period t in market s rather than the total number of firms in market s. In any period t, the labor market tightness in market s is $b_t(s) = u_t(s)/m_t(s)$; the matching rate is $\mu_t(s) \equiv (1 - e^{-b_t(s)})/b_t(s)$ for each unemployed worker and $1 - e^{-b_t(s)}$ for each recruiting firm. Those firms that already have a worker at the beginning of a period do not recruit in that period and, similarly, those workers who already have a job at the beginning of the period do not look for a job. Also as before, the machine cost per period is incurred the moment a firm posts a vacancy.

If an unemployed worker and a vacant job get matched in period t, they produce immediately.[8] After production, some matches separate. Since the focus here is on recruiting rather than separation, I simply assume that each match separates with an exogenous probability σ. Thus, the number of workers remaining matched at the beginning of period t is $(1-\sigma)E_{t-1}(s)$. Then,

$$u_t(s) = n(s) - (1-\sigma)E_{t-1}(s); \tag{6.1}$$

$$E_t(s) = (1-\sigma)E_{t-1}(s) + \mu_t(s)u_t(s). \tag{6.2}$$

Eliminating $u_t(s)$ gives employment dynamics:

$$E_t(s) = \mu_t(s)n(s) + (1-\sigma)[1-\mu_t(s)]E_{t-1}(s). \tag{6.3}$$

In the steady state the market tightness is $b(s)$ and so E and u are:

$$E(s) = \frac{n(s)}{1-\sigma+\sigma/\mu_t(s)}; \;\; u(s) = n(s) - (1-\sigma)E(s). \tag{6.4}$$

The social planner now maximizes the sum of net present value added in each period. Consider the cost of machines first. Each time a vacancy is posted, the cost for that period is that of the vacant machine, which is $C(k)$ for machine k. The number of vacancies in period t is $u_t(s)/b_t(s)$.

[8] Often it is assumed that new matches start to produce in the next period. The immediate production assumed here has no particular implications on the analytical results. It is used here because it is used in the one-period setting. This makes it easy to interpret the one-period result as a special case of the dynamic result when $\beta \to 0$.

Using (6.1), aggregate vacancy cost in period t in market s is

$$C \cdot \frac{n(s) - (1-\sigma)E_{t-1}(s)}{b_t(s)}.$$

Next, consider the value produced by new matches in t. The number of new jobs created in period t in market s is $E_t(s) - (1-\sigma)E_{t-1}(s)$. Each new job generates the following present value:

$$\frac{F(k,s) - C(k)}{1-\beta(1-\sigma)} + C(k) = \frac{F(k,s) - \beta(1-\sigma)C(k)}{1-\beta(1-\sigma)}.$$

The first term on the left-hand side is the present value of the net product from the new match, where the discount rate takes into account of both subjective discounting and job separation. Since the machine cost in period t is counted both in the vacancy cost and in the net product in period t, it is added back onto the present value of the new match to avoid double accounting.

The efficient assignment assigns a machine quality $k_t = \phi_t(s)$ and a market tightness $b_t(s)$ for each s in each period t to solve the following problem:

$$\begin{aligned} \mathcal{L}D \equiv \max_{(k_t, b_t, E_t)} \sum_{t=0}^{\infty} \beta^t \int_{s\in S} \Bigg\{ & \frac{F(k_t,s) - \beta(1-\sigma)C(k_t)}{1-\beta(1-\sigma)}[E_t(s) - (1-\sigma)E_{t-1}(s)] \\ & - C(k_t)\frac{n(s) - (1-\sigma)E_{t-1}(s)}{b_t(s)} \Bigg\} ds \end{aligned} \tag{6.5}$$

subject to (6.3). In the steady state, the first-order conditions for (b_t, E_t, k_t) yield:

$$1 - e^{-b(s)} = \frac{[1-\beta(1-\sigma)]C_k(k)}{F_k(k,s) - \beta(1-\sigma)C_k(k)}; \tag{6.6}$$

$$C(k) = \left\{1 - [1+b(s)]e^{-b(s)}\right\} \cdot \frac{F(k,s) - \beta(1-\sigma)C(k)}{1-\beta(1-\sigma)(1-e^{-b(s)})}. \tag{6.7}$$

The above conditions have similar interpretations to the one-period counterparts, (3.2) and (3.3), and approach the one-period counterparts when either $\beta \to 0$ or $\sigma \to 1$.

Remark 2. *Focusing on the steady state assignment does not lose much insight. In fact, the first-order conditions for (k_t, b_t, E_t) can be used to obtain a closed sub-system of dynamic equations involving only k and b. Thus, when a shock hits the system, the values of k_t and b_t must immediately jump to the new steady state levels, while E_t gradually adjusts.*

Rewriting (6.6) and (6.7) to express C_k/F_k and C/F as functions of b, as in (3.2) and (3.3), one can use the same procedure for proving Proposition 3.1 to show that the steady state assignment exists. Therefore, the features of a one-period assignment can be easily translated into the dynamic environment for the steady state assignment. Also, the dynamic efficient allocation can be decentralized by a market assignment following a similar route to that in Section 3. The exercise is much more involved because agents now consider the payoffs from infinitely many periods. There is no need to repeat the procedure here, which is available upon request.

Equilibrium wages can be computed directly from (6.5) using Remark 1 in Section 4. That is, since the equilibrium allocation is efficient, wages must be such that the expected gain to an unemployed worker with skill s is equal to the social marginal benefit of increasing $n(s)$. Let us focus on the steady state and compute the expected gain to the worker. Let $V_u(s)$ be the steady state value function of an unemployed worker with skill s and $V_e(s)$ be the corresponding value function when the worker is employed. Then,

$$V_u(s) = \mu(s)V_e(s) + [1-\mu(s)]\beta V_u(s);$$

$$V_e(s) = w(s) + \sigma\beta V_u(s) + (1-\sigma)\beta V_e(s).$$

If an unemployed worker gets hired, the value is V_e, but if he/she fails to find a job, the value is the discounted value from the next period, βV_u. The expected gain to the worker is $\mu(s)[V_e(s) - \beta V_u(s)]$. Efficiency then requires

$$\sum_{t=0}^{\infty} \beta^t \mu(s)[V_e(s) - \beta V_u(s)] = \frac{d\mathcal{L}D}{dn(s)},$$

where $\mathcal{L}D$ is defined in (6.5). From the value functions one can obtain:

$$\begin{aligned} \mu(s)[V_e(s) - \beta V_u(s)] &= (1-\beta)V_u(s) \\ &= w(s) \cdot \left[\beta(1-\sigma) + \frac{1-\beta(1-\sigma)}{\mu(s)}\right]^{-1}. \end{aligned}$$

Computing $d\mathcal{L}D/dn(s)$ and using (6.7), we have:

$$w(s) = \frac{C(\phi(s))}{e^{b(s)} - 1 - b(s)} \left[\beta(1-\sigma) + \frac{1-\beta(1-\sigma)}{\mu(s)}\right]. \tag{6.8}$$

Given market tightness $b(s)$, the wage $w(s)$ is an increasing function of the machine quality assigned to it; given the machine quality $\phi(s)$, the wage $w(s)$ is a decreasing function of the market tightness. Of course, the assigned machine quality and the market tightness cannot be separated because they are a package of the deal.

6.2. Steady state distributions

Let us denote the steady state density of the skill distribution in employment by $h(\cdot)$ and the skill distribution in unemployment at the beginning of a period by $g(\cdot)$. The density of the wage distribution is then $h(w^{-1}(\cdot))$, where $w^{-1}(\cdot)$ is the inverse function of the wage function $w(\cdot)$. With (6.4), the total number of workers employed after recruiting in the steady state is

$$TE = \int_{s \in S} \frac{n(s)}{1 - \sigma + \sigma/\mu(s)} ds.$$

The number of workers unemployed at the beginning of a period is $N - (1 - \sigma)TE$. Then,

$$h(s) = \frac{n(s)/TE}{1 - \sigma + \sigma/\mu(s)}; \quad g(s) = \frac{n(s) - (1 - \sigma)E(s)}{N - (1 - \sigma)TE}.$$

Naturally, both distributions depend on the skill distribution in the labor force through $n(s)$.

More important, the skill distributions in employment and unemployment depend on equilibrium tightness in each market, $b(s)$. Compared with the exogenous distribution of skills in the labor force, $g_0(s)$, the skill distribution in employment is more skewed toward high skills and the skill distribution in unemployment is more skewed toward low skills if and only if $b_s(s) < 0$. Since the market tightness depends on the assignment $\phi(s)$, one must have a good idea about the nature of the assignment in order to make inferences on the skill distribution in each status of employment. In contrast, in a frictionless economy every worker finds a match instantaneously and so the skill distribution in employment is identical to the one in the labor force.

Compared with a frictionless world, in a frictional world it is also more unreliable to directly draw inferences on the shape of the skill distribution from the wage distribution or vice versa. In the frictionless world, the assignment is always positive and so observing the wage distribution tells us something useful about the skill distribution in the labor force. For example, if the density of the wage distribution is uniform, then the skill distribution in the labor force must also be uniform; if the density of the wage distribution has a peak at some level w^*, then the skill distribution density in the labor force must also have a peak at some skill level s^*. Such inference cannot be made in a frictional world without knowing the assignment. For example, unevenly distributed wages over skills can be consistent with uniform distribution of skills in the labor force if fewer high-skill workers are unemployed than low-skill workers.

The market assignment and the market tightness both depend on the production function and the cost function of producing machines. To say more about the wage distribution and the skill distribution in employment, I calibrate the model next. The exercise also shows whether the differential tightness is quantitatively significant for wage inequality.

6.3. Numerical exercise

In the following numerical exercise, skill levels are discrete points ranging from $s_L = 1$ to $s_H = 5$, with a mean $s_M = (s_H + s_L)/2$ and a grid 0.08. The total number of skill levels is 51. Skills in the entire labor force are assumed to be uniformly distributed over the 51 levels, i.e., $g_0(s) = 1/51$. The uniform distribution is used here because the distributions of wages and employment in the frictional economy can be easily compared with those in a frictionless economy. The production function and the cost function are:

$$F(k, s) = F_0 k^\alpha s^{1-\alpha}; \quad C(k) = C_0(k^\gamma + C_1).$$

The constant $C_1 > 0$ captures the cost of machines that are independent of the quality.

The parameter values are as follows:

$$\begin{aligned} N &= 1,\ F_0 = 1,\ \alpha = 0.3,\ \beta = 0.99,\ \sigma = 0.06, \\ \gamma &= 2.8565,\ C_0 = 27.0596,\ C_1 = 0.01519. \end{aligned}$$

The total number of workers, N, and the multiplier in the production function, F_0, are normalized to one. α is set to 0.3, which gives a wage share of output as 66% for $s = s_H$.[9] The length of

[9] The wage share of output is commonly calibrated to 0.64 (see Christiano, 1998). When labor were paid with

a period is interpreted as a quarter and so the discount factor, β, is chosen to imply a quarterly real interest rate around 1%. The quarterly job separation rate $\sigma = 6\%$ is a realistic number, as documented in Davis and Haltiwanger (1991). To identify γ, I set the unemployment rate to 12.5% for workers with skill s_L and to 6.5% for workers with skill s_M, where the unemployment rate is the average of the rate before recruiting, $u(s)/n(s)$, and the rate after recruiting, $1 - E(s)/n(s)$. These rates are realistic. For example, the U.S. unemployment rate in 1991 is 12.3% for workers who had less than 4 years of high school and 6.7% for workers who had 4 years of high school only (Bureau of the Census, 1997). With (6.6) and (6.7), these two restrictions solve for γ. They also give a restriction between C_0 and C_1. Finally, I set $C_1 = 0.1 \times k_M$ to solve for C_0 and C_1, where k_M is the machine quality allocated to workers with skill s_M.[10]

The main characteristics of the frictional assignment are illustrated in Figures 6.1 − 6.4 with circles, together with the characteristics of the frictionless assignment (solid lines). I discuss them below. First, the assignment is positive and concave, as shown in Figure 6.1. This result is useful because a positive assignment cannot be guaranteed a priori, as stated in Proposition 5.1.

Second, Figure 6.1 confirms the result in Proposition 3.1 that the machine quality assigned to each skill is lower in the frictional economy than in the frictionless economy. The difference between the two gets larger as skill increases. Figure 6.2 reveals similar differences in wages between the two economies. It is remarkable that even with significant unemployment the machine quality and wage for each given skill in the frictional economy are very close to those in the frictionless economy. Do we then conclude that the frictions do not matter much for machine quality assignments or wages? Of course not, and we come to the third feature.

Third, the *distributions* of machine qualities and wages are more skewed toward high levels in the frictional economy than in the frictionless economy. Figure 6.3 plots the employment density for each skill in the two economies, $h(s)$ and $g_0(s)$, against the corresponding machine quality, $\phi(s)$

the marginal product, this would imply $\alpha = 0.36$. A lower value of α is chosen here because wages are strictly below the marginal product of labor.

[10] Not surprisingly, machine qualities and wages vary sensitively with the value chosen for C_1/k_M. However, this amounts to a shift in the supports of the wage and machine quality distributions but not much change in the shape of these distributions.

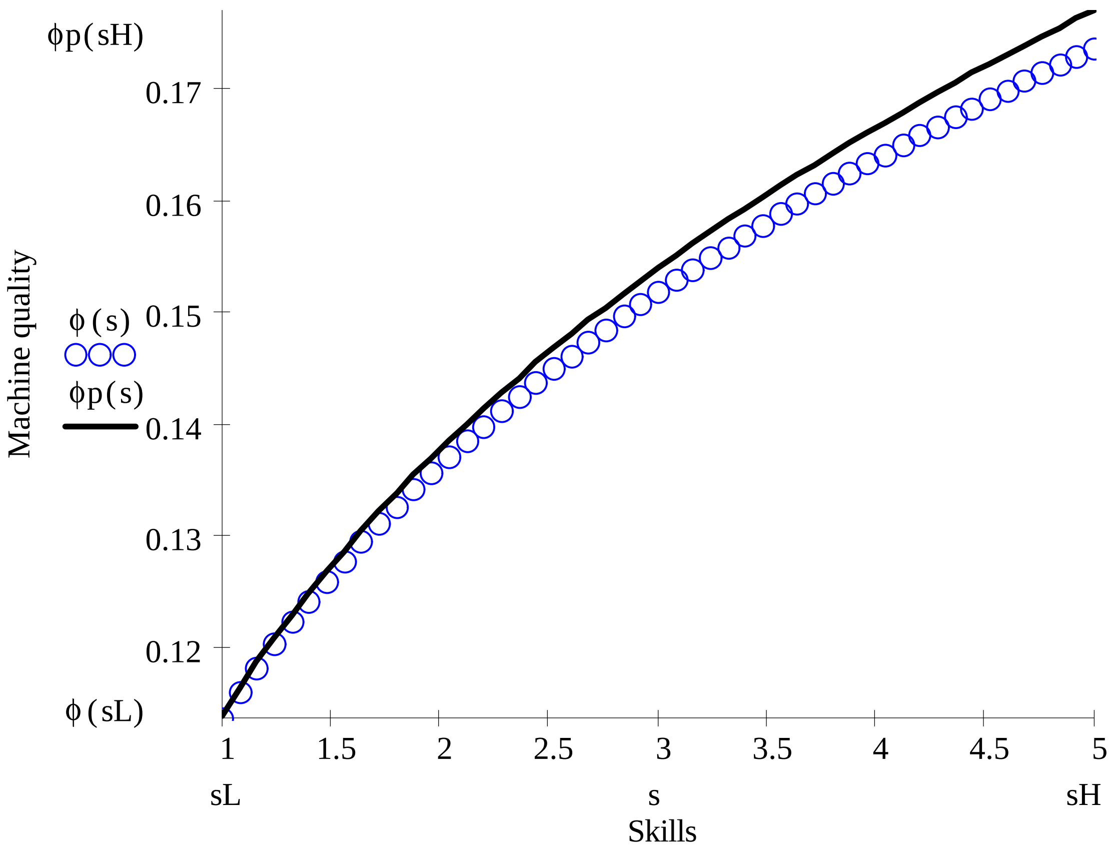

Figure 6.1: Machine quality assignment

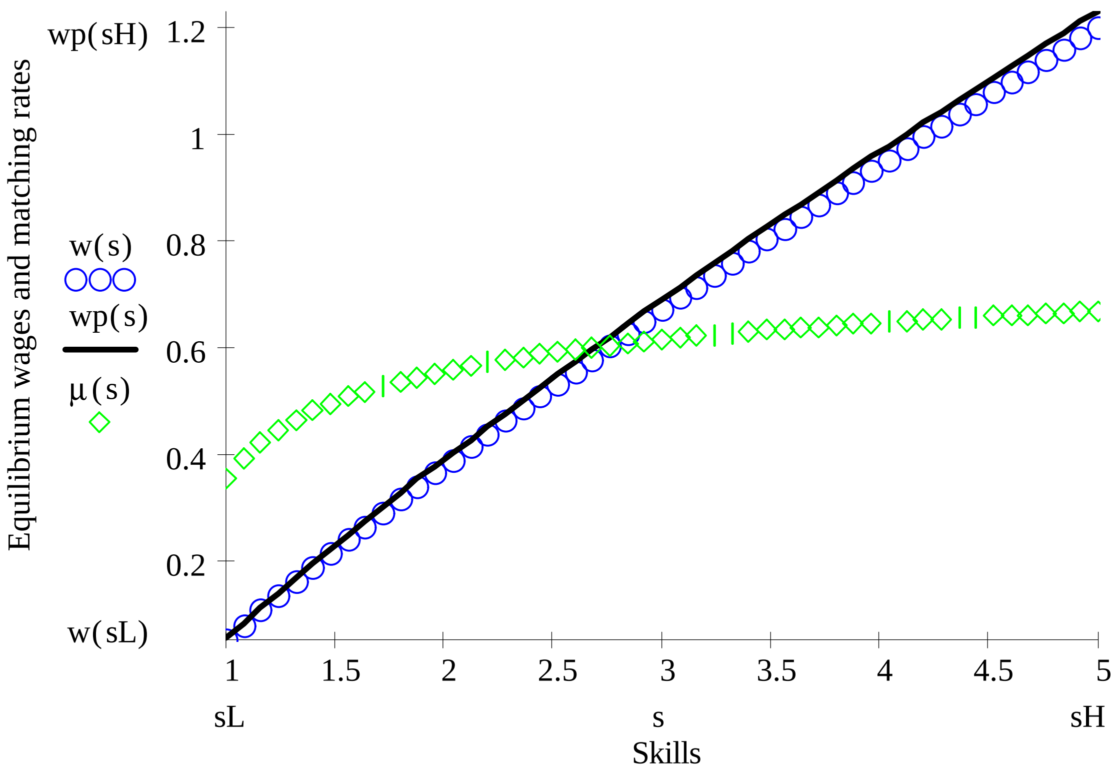

Figure 6.2: Wages and matching rates

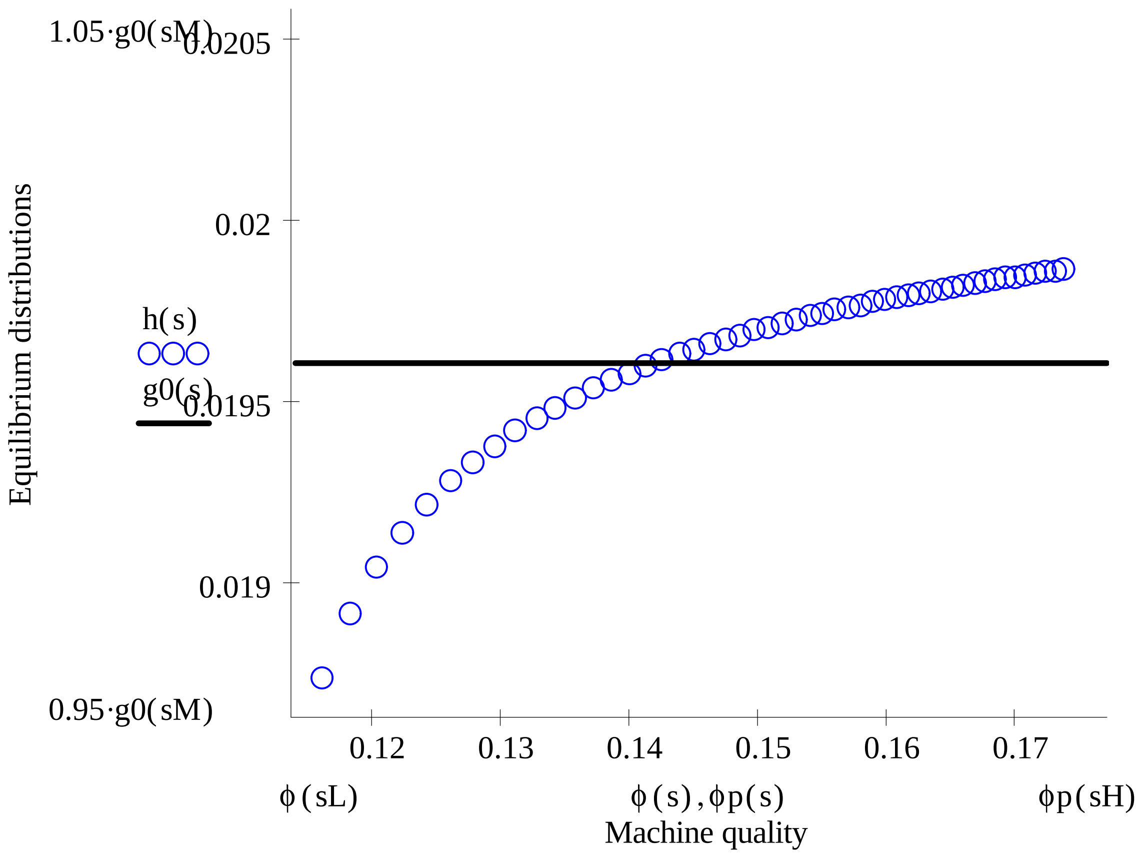

Figure 6.3: Distributions of machine qualities

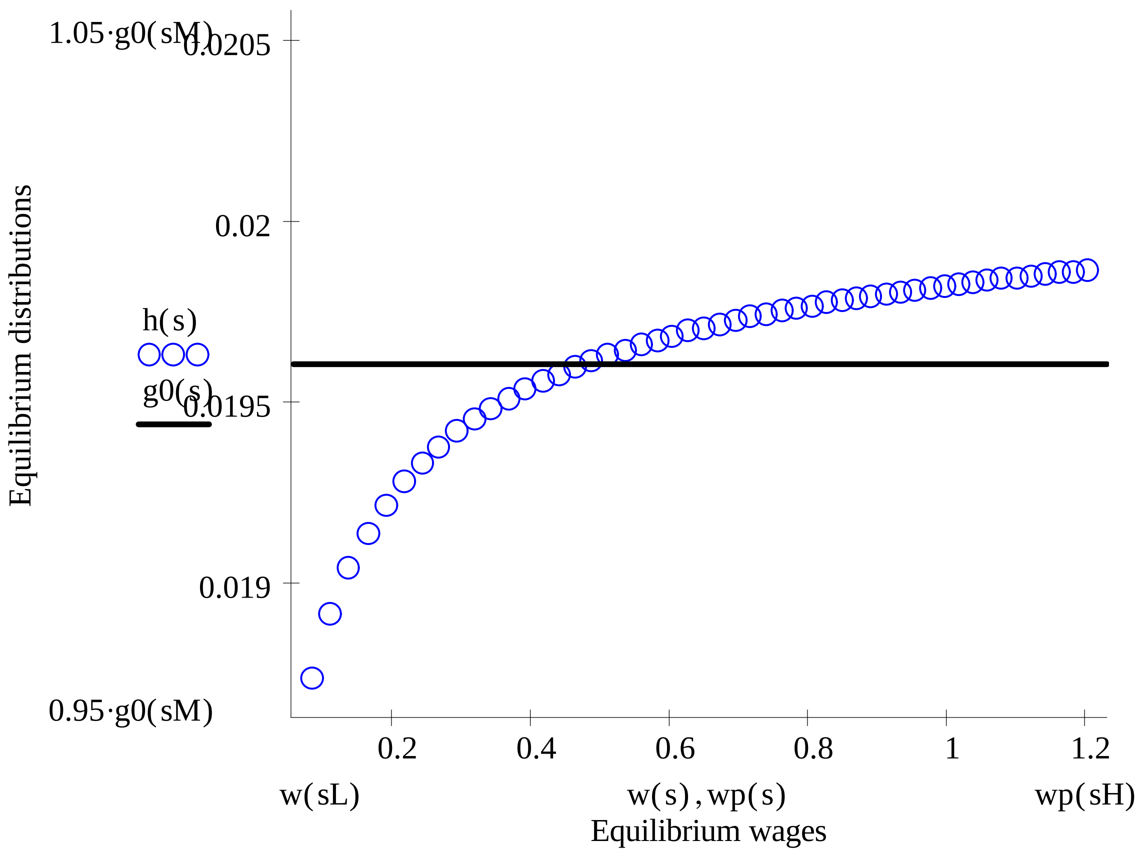

Figure 6.4: Wage distributions

and $\phi^p(s)$. The diagram gives the densities of machine qualities in the two economies. The figure shows that about 6% fewer machines with the lowest quality and about 2.5% more machines with the highest quality are used in the frictional economy than in the frictionless economy. Similarly, Figure 6.4 shows a similar skewness of the wage density in the frictional economy. Thus, there is a sense that the wage distribution is more unequal in the frictional economy.

The additional inequality arises here not because a higher skill is assigned with an even higher machine quality than that suggested by the traditional assignment literature. On the contrary, the opposite occurred, as discussed above for Figure 6.1. Rather, the additional inequality arises because the assignment induces changes in the employment distribution. That is, it reduces the proportion of low-skill workers in employment by making the market tighter for them. This additional channel for the assignment to affect wage inequality can be seen from Figure 6.2, where the matching rate $\mu(s)$ increases with skill. The wage share (not shown) also increases with skill, as firms compete for high-skill workers.

6.4. Responses to a reduction in machine costs

I now briefly examine the response of the assignment to a technological progress that makes machines cheaper to make. In particular, consider the case where the costs for all machines fall in the same proportion. I examine this type of technological progress not because I believe it is the most realistic one but because its effects on skill and wage distributions are not clear relative to the effects of, say, a skill-biased technological progress. Let C_0 fall by 5%. The percentage changes in machine assignment, wages and matching rates are reported in Figures 6.5 and 6.6 for both the frictional economy (circles) and the frictionless economy (lines).

Not surprisingly, the reduction in machine costs increases the machine quality assigned to each skill. This is true for both economies, but the percentage increase in machine qualities is slightly smaller in the frictional economy than in the frictionless economy. The increases in quality are almost uniform across skills in both economies. In contrast, the percentage wage increases are far from uniform, as shown in Figure 6.6. The cost reduction benefits low-wage earners much more

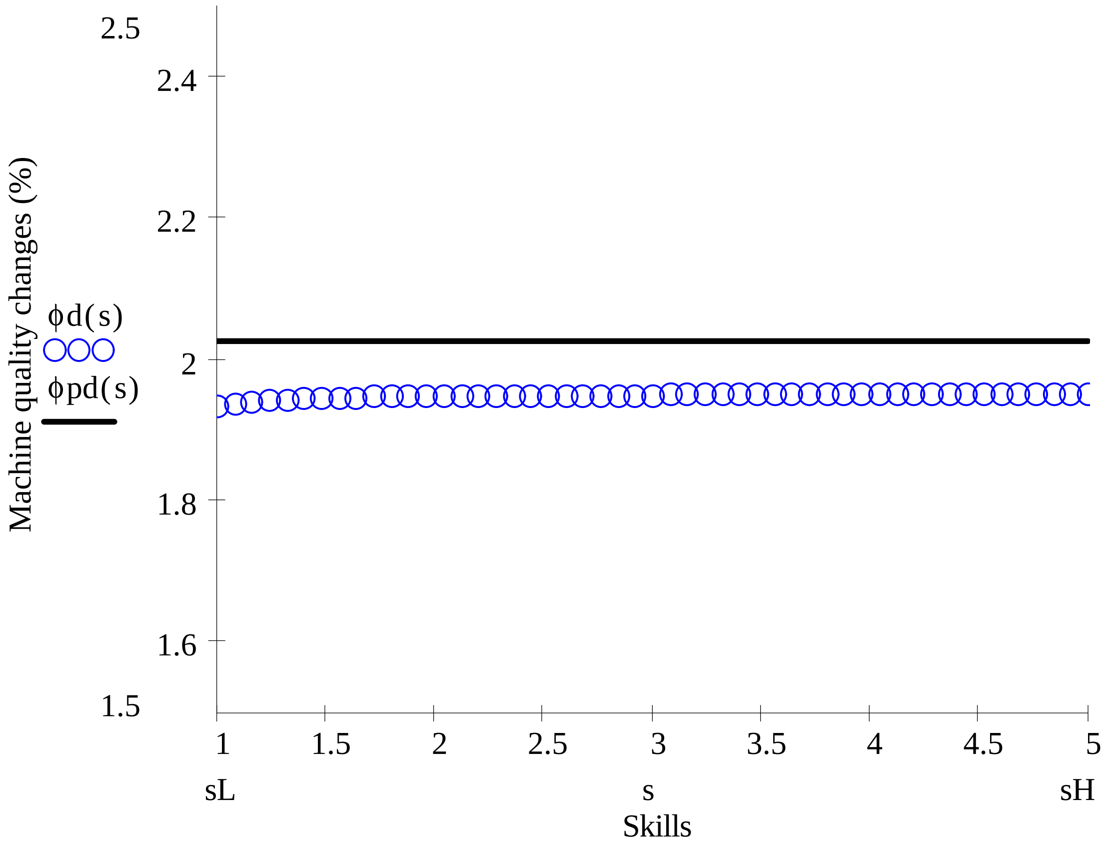

Figure 6.5: Responses in machine qualities

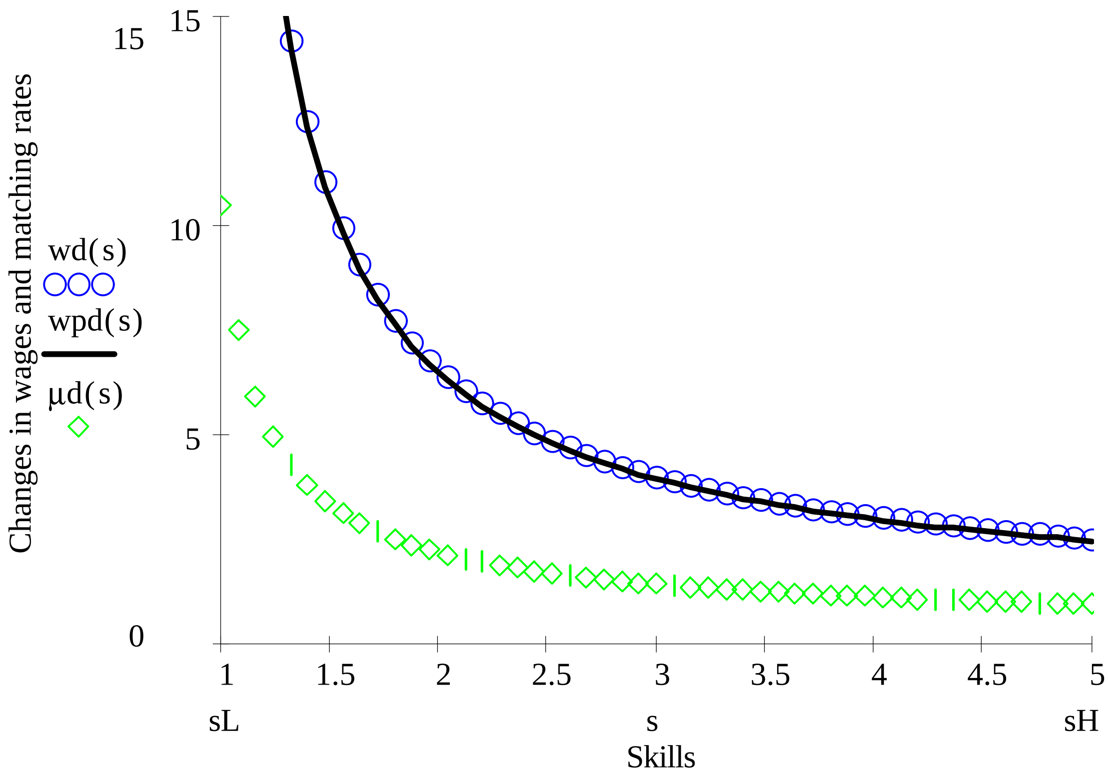

Figure 6.6: Responses in wages and matching rates

than high-wage earners. In both the frictional economy and the frictionless economy, wages at the bottom increases by more than 15% but wages at the top increases by less than 3%. Thus, a uniform reduction in machine costs reduces wage inequality in both economies.

The reduction in wage inequality is larger in the frictional economy than in the frictionless economy, because there is a simultaneous shift in employment distribution in the frictional economy. Although the matching rate increases for all skills when machine costs fall, it increases by roughly 10% for bottom-wage earners but by less than 2% for top-wage earners. The employment distribution becomes less skewed toward high skills than before.

A general interpretation of the result is that a uniform technological progress benefits low-skill workers more than high-skill workers by increasing the relative employment of low-skill workers. However, technological progress is not always uniform and in many cases it reduces the cost of advanced machines proportionally more than reducing low quality machines (see Greenwood and Yorukoglu (1997)). In this case, employment and wage distributions will become more rather than less skewed toward high-skill workers (see Shi (1998) for a comparison).

7. Conclusion

A wage-posting framework is shown to induce an efficient assignment between diverse skills and diverse machines when the matching is time-consuming. The (efficient) frictional assignment assigns each skill with a market tightness as well as a machine quality. It has several features in contrast with a frictionless assignment. First, higher skills are not necessarily assigned higher machine qualities even when machine qualities and skills are complementary in production – sufficient complementarity is required for a positive assignment. This is because a higher skill can be sufficiently compensated by a less tight market without a higher machine quality. Second, differences in skill prices reflect differences not only in skills and machine qualities assigned to them but also in matching rates. If skills and machine qualities are sufficiently but not extremely complementary with each other, higher skills are assigned better machines and experience higher matching rates. In this case the wage distribution is more skewed toward high wages in the frictional economy than in the frictionless economy. Calibration exercises show that the matching friction increases inequality more through the differential tightness than through wages.

As indicated in the introduction, the use of the framework is not restricted to the labor market. Rather, it is applicable to any market that has the following features: a large number of participants on each side of the market who have diverse characteristics, a matching process that is time-consuming, and a high turnover rate. For example, the loan market can be analyzed in a similar way, where projects can differ vastly in size and probability of success and loans come also in different sizes and terms.

Even if we restrict the discussion to the labor market, the analysis should be useful for macroeconomists who like to know how technological progress affects productivity and the allocation of skills. It should also be useful for labor economists who estimate the earning function. The central implication of the analysis is that a worker's wage depends on market characteristics, such as the market tightness, in addition to the worker's characteristics, such as skill, and the firm's characteristics, such as machine quality and capital intensity. An earning function that is estimated without much attention to market characteristics is likely to be biased, as reality is that skilled workers are more likely to find a job than unskilled workers. The estimates are also likely to be unstable when there is rapid technological progress, which changes the differential market tightness for different skills and the wage distribution.

The analysis is only a first step to analyzing frictional assignments and some important aspects of the labor market are ignored, such as multi-dimensional skills, match-specific productivity, private information and/or uncertainty in productivity. These elements will enrich the model by increasing wage inequality between similar workers and hence allow a better match between the model and the data. They will also illustrate the ex post role of wages in retaining workers and revealing productivity, in addition to the ex ante role of attracting workers analyzed here.

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Appendix

A. Proof of Proposition 5.3

When b is a constant, Proposition 5.1 immediately implies $\phi_s(s) > 0$. Also, (3.2) and (3.3) imply that $C_k/F_k = \delta 2$ and $C/F = \delta 1$ are constants, with $\delta 1 < \delta 2 < 1$. Totally differentiating the equation $C/F = \delta 1$ with respect to s, substituting C_k by $\delta 2 F_k$ and writing F_s/F_k as $f(k/s)$ yields:

$$\phi_s(s) = \frac{\delta 1}{\delta 2 - \delta 1} \cdot f\left(\frac{\phi(s)}{s}\right). \tag{A.1}$$

Making a transformation $z(s) = \phi(s)/s$ and substituting ϕ yields

$$\frac{ds}{s} = \left(\frac{\delta 1}{\delta 2 - \delta 1} f(z) - z\right)^{-1}.$$

Integrating from s_L to s yields the solution in the proposition.

If F is the CES type, then $f(y) = \frac{1-\alpha}{\alpha} y^{1-\rho}$ and integrating (A.1) gives (5.2). Substituting $s = \phi^{-1}(k)$ into the function F and using $C = \delta 1 F$, one recovers the cost function (5.3). With $\rho < 1$, the cost function is strictly convex if and only if

$$[\phi(s_L)]^\rho > \frac{(1-\alpha)\delta 1}{\alpha(\delta 2 - \delta 1)} s_L^\rho,$$

which is also necessary and sufficient for $\phi_{ss}(s) < 0$. QED

Complementary note for "Frictional assignment"

(note for publication)

B. Dynamic market assignment

In this appendix, a market assignment is described to decentralize the dynamic efficient assignment in Section 6. In the competitive framework, machine producers produce machines and rent them at market rates to firms. Let the machine producing sector be perfectly competitive. Since the cost of producing a quality k machine is constant over time, $C(k)/(1-\beta)$, the machine must yield a present value of the rental income equal to $C(k)/(1-\beta)$, no matter when it is produced. This implies that the rental cost of a quality k machine must be $C(k)$ in each period.

To find the market assignment in this dynamic setting, first suppose that firms use machine k to combine with skill s in market s. Let us find the wage share in market s, $A_t(k,s) = W_t(k,s)/F(k,s)$. Suppress the indexes (k,s) and denote $A_t^c = \{A_\tau\}_{\tau \geq t}$. Let $J_{ft}(A_t^c)$ be the present value of a job (to the firm) that is already filled at the beginning of period t and that pays a path of wage shares A_t^c. Let J_{vt} be the present value of a vacancy posted in period t. Similarly, let $V_{et}(A_t^c)$ be the present value of a job (to the worker) that is already filled the beginning of period t paying a path of wage shares A_t^c and V_{ut} be the present value of an unemployed worker in period t. (The symbols V_{et} and V_{ut} are the dynamic counterparts of those used in the text.) Note that, unlike J_{ft} and V_{et}, J_{vt} and V_{ut} depend not on a specific firms' wage offers but on all firms' wage offers and so the dependence is suppressed.

The value functions are given by the following Bellman equations:

$$J_{ft}(A_t^c) = (1-A_t)F - C + \sigma\beta J_{vt+1} + (1-\sigma)\beta J_{ft+1}(A_{t+1}^c); \tag{B.1}$$

$$J_{vt} = -C + \left(1 - e^{-B_t}\right)\left[J_{ft}(A_t^c) + C\right] + e^{-B_t}\beta J_{vt+1}; \tag{B.2}$$

$$V_{et}(A_t^c) = A_t F + \sigma\beta V_{ut+1} + (1-\sigma)\beta V_{et+1}(A_{t+1}^c); \tag{B.3}$$

$$V_{ut} = \frac{1-e^{-B_t}}{B_t} V_{et}(A_t^c) + \left(1 - \frac{1-e^{-B_t}}{B_t}\right)\beta V_{ut+1}. \tag{B.4}$$

These equations are standard. For example, (B.2) equates the present value of vacancy to the expected value from hiring minus the vacancy cost C. With probability e^{-B} the vacancy fails to be filled, in which case the present value is the discounted value of a vacancy in the next period, βJ_{vt+1}. With probability $1-e^{-B}$ the job is filled, in which case the job yields a present value $J_{ft} + C$. The cost C is added to J_{ft} because J_{ft} is not the present value of a newly created job but rather the value of a job that was filled before t, in which the machine rental cost C is already deducted (see (B.1)).

Now consider the determination of the equilibrium wage share. Suppose that in market (k, s) all other firms post the path of wage shares A_t^c and a single firm is contemplating a deviation to $A_t^{cd} \equiv \{A_\tau^d\}_{\tau \geq t}$. If the firm succeeds in hiring a worker in t, it pays the wages according to the share path A_t^{cd} until the job is separated. The present value of this filled job is $J_{ft}(A_t^{cd}) + C$, where $J_{ft}(A_t^{cd})$ is calculated as in (B.1) with A^{cd} replacing A^c. If the firm fails to hire any worker in t, it reverts to the share path A_{t+1}^c that other firms will post from next period onward, which yields a discounted present value βJ_{vt+1}. The surplus to the firm from filling the job is $J_{ft}(A_t^{cd}) + C - \beta J_{vt+1}$. Similarly, if a worker is hired by the firm in period t, the present value is $V_{et}(A_t^{cd})$, which is calculated as in (B.3) with A^{cd} replacing A^c, and the surplus to the worker is $V_{et}(A_t^{cd}) - \beta V_{ut+1}$. Using the counterparts of (B.1) and (B.3) for $J_{ft}(A_t^{cd})$ and $V_{et}(A_t^{cd})$, I can compute:

$$J_{ft}(A_t^{cd}) = \sum_{\tau=0}^{\infty} [\beta(1-\sigma)]^\tau \left[(1 - A_t^d)F - C + \sigma\beta J_{vt+1+\tau}\right]; \tag{B.5}$$

$$V_{et}(A_t^{cd}) = \sum_{\tau=0}^{\infty} [\beta(1-\sigma)]^\tau \left[A_t^d F + \sigma\beta V_{ut+1+\tau}\right]. \tag{B.6}$$

Let p_t^d be the probability that each worker applies to the deviating firm. Then the firm's decision problem is:

$$\max_{A_t^{cd}} \left[1 - (1 - p_t^d)^n\right] \left[J_{ft}(A_t^{cd}) + C - \beta J_{vt+1}\right]$$

subject to

$$\frac{1 - (1 - p_t^d)^n}{np_t^d} \left[V_{et}(A_t^{cd}) - \beta V_{ut+1}\right] = ES,$$

where ES is the expected surplus that the worker gets from that market. With (B.5) and (B.6), the problem yields the following division of the match surplus in a symmetric equilibrium:

$$V_{et} - \beta V_{ut+1} = \frac{B_t}{e^{B_t} - 1} \left[(J_{ft} + C - \beta J_{vt+1}) + (V_{et} - \beta V_{ut+1})\right]. \tag{B.7}$$

The equation states that the worker's surplus in terms of the present value is a share $B_t/(e^{-B_t} - 1)$ of the total match surplus. Note that this share is identical to that given by (4.2) in the one-period case.

(B.7) implicitly determines the wage share path A_t^c but recovering the wage shares is complicated. To simplify, I impose the free entry condition $J_{vt} = 0$ for every pair (k, s) and consider only the steady state. In the steady state, (B.5) and (B.6) (without the superscript d) can be used to solve for J_{ft} and V_{et} as functions of steady state w and V_u. Substituting these functions into (B.7) and putting back the indexes (k, s) yields:

$$\begin{aligned} W(k,s) &= \frac{B(k,s)}{e^{B(k,s)}-1}\left[F(k,s)-(1-\sigma)\beta C(k)\right] \\ &\quad + \left(1-\frac{B(k,s)}{e^{B(k,s)}-1}\right)(1-\beta)(1-\sigma)\beta V_u, \end{aligned} \tag{B.8}$$

where

$$V_u = V_u(k,s) \equiv \frac{e^{-B(k,s)}\left[F(k,s)-\beta(1-\sigma)C(k)\right]}{(1-\beta)\left[1-\beta(1-\sigma)(1-e^{-B(k,s)})\right]}. \tag{B.9}$$

Substituting V_u into (B.8) and using (6.7) yields the same wage equation as that in (6.8) when k is set to $\phi(s)$.

One can compute the worker's steady state expected surplus in a match, denoted $ES(k, s)$, and the firm's expected surplus, denoted $EP(k, s)$, as follows:

$$ES(k,s) = \frac{e^{-B(k,s)}}{1-\beta(1-\sigma)}\left\{F(k,s)-\beta(1-\sigma)\left[C(k)+(1-\beta)V_u\right]\right\}; \tag{B.10}$$

$$EP(k,s) = \frac{1-[1+B(k,s)]e^{-B(k,s)}}{1-\beta(1-\sigma)}\left\{F(k,s)-\beta(1-\sigma)\left[C(k)+(1-\beta)V_u\right]\right\}. \tag{B.11}$$

Notice that the wage, the worker's expected surplus and the firm's surplus all approach to their one-period counterparts when $\beta(1-\sigma) \to 0$. Finally, with (B.2), the entry condition ($J_v = 0$) requires $J_f = C/(e^B - 1)$, which is equivalent to

$$EP(k,s) = (1-\beta)C(k). \tag{B.12}$$

In the above formulas for W, ES and EP, I have deliberately kept the notation V_u rather than substituting it with (B.9). The purpose is to make its easier to describe firms' choices on machine quality, to which I now turn. For any given s, the choice of k solves:

$$(PD) \quad \max_k \; \{ES(k,s)\text{: (B.12) holds}\},$$

where the value V_u is taken as given in the formulas for ES and EP. The solution can be written as $k = \Phi(s, V_u)$. This choice of machine quality is the fixed point of $\phi(s) = \Phi(s, V_u(\phi(s), s))$.

The reason why V_u should be taken as given in (PD) is that a single firm has a negligible influence on the value function of an unemployed agent, which depends all firms' decisions. To see this more clearly, consider the decision problem associated with the dual of (PD). That is, imagine a single firm's decision on entering market s using a machine k' that is not necessarily the same as what every other firm in market s uses, $\phi(s)$. Since all other firms in market s use machine $\phi(s)$, the entry by a single firm using a different machine has negligible influence on the value function of an unemployed worker, which is $V_u(\phi(s), s) \equiv V_u^*(s)$. After the firm enters the market with machine k', it announces a wage W' which could differ from $w(s) = W(\phi(s), s)$. Since both (k', W') are potentially different from $(\phi(s), w(s))$, each worker may apply to the firm with a probability that is different from what he uses to apply to other firms. Let this probability be p' and denote $B' \equiv \lim_{n,m\to\infty} np'$. One can repeat the firm's wage posting decision to show that the firm's wage offer, W', is given by precisely (B.8) but with $B(k, s)$ being replaced by B', V_u being replaced by $V_u^*(s)$ and k in the functions F and C being replaced by k'. Similarly, this firm with k' gets an expected surplus $EP'(k', s)$ and any worker s who applies to this firm gets an expected surplus $ES'(k', s)$, which are given by (B.11) and (B.10), respectively, by replacing $B(k, s)$, V_u and k in the way described above. The choice of k' solves

$$(PD') \quad \max_{k'} \ \{EP'(k', s) - (1-\beta)C(k') \colon \ ES'(k', s) \geq ES(\phi(s), s)\}.$$

This is, of course, the dual problem of (PD) and has the same solution. Since V_u here is taken as given at a value $V_u(\phi(s), s)$, the solution is $k' = \Phi(s, V_u(\phi(s), s))$. For the assignment to be consistent with equilibrium, I must have $\Phi(s, V_u(\phi(s), s)) = \phi(s)$, as I stated for (PD).

It is now straightforward to show that the first-order conditions of either (PD) or (PD') after substituting V_u by (B.9), lead to (6.6) and (6.7). Therefore, the market assignment is efficient.