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NOMINAL BONDS AND INTEREST RATES*

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In this article, I integrate the microfoundation of monetary theory with the model of limited participation to analyze the competition between nominal bonds and money. The market for government nominal bonds is centralized and Walrasian, whereas the goods market is modeled as random matches. The government imposes a legal restriction that requires all government goods to be purchased with money but not with bonds. By contrast, private agents can exchange between themselves with both money and bonds. I show that an arbitrarily small legal restriction is sufficient to prevent matured bonds from being a medium of exchange. I also analyze the effects of monetary policy with and without the legal restriction. Some of those effects differ significantly from traditional monetary models.

1. INTRODUCTION

The primary question to be addressed here is: Why do government-issued nominal bonds not circulate as a medium of exchange, whereas money does? A related question is: Why do such bonds bear positive interest even if they are default-free? In the process of answering these questions, I integrate the microfoundation of monetary theory with the influential work of Lucas (1990), so that the theory can be used to analyze standard policy issues.

The questions raised here are long-standing challenges to monetary theory (see Hicks, 1939). The lack of a consistent model, instead of informal arguments, is responsible for the unresolved status of these questions. Many monetary models give money a unique role by putting money into the utility function, the production function, or the transaction function. Because these functions are specified exogenously, the models are incapable of answering the above questions. To overcome this deficiency, a strong microfoundation for the role of money is necessary

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(see Wallace, 2001, for more arguments). Recently, monetary theorists have constructed one such microfoundation called monetary search theory (e.g., Kiyotaki and Wright, 1989, 1993). This theory uses decentralized exchange to support a role of a medium of exchange. With this development, it is now time to confront the above questions regarding nominal bonds.

Another purpose of this article is to develop an integrated framework for policy analysis. Monetary search theory has largely omitted nominal bonds. This omission is a major limitation of the theory for policy analysis. For example, one cannot use the theory to understand nominal interest rates or to examine open-market operations. By eliminating this limitation, I integrate monetary search theory with Lucas's (1990) model of limited participation. This integrated framework will allow me to examine the effects of monetary policy, without imposing the cash-in-advance constraint in the goods market as Lucas did.

The framework has a centralized bonds market and a decentralized goods market, which are separated from each other during each period. In the bonds market, the government sells and redeems nominal bonds. In the goods market, the exchange is modeled as bilateral matches. Agents trade with each other using media of exchange because there is no public record keeping of agents' transaction histories and it is difficult to have a double coincidence of wants in matches. Nominal bonds and money compete against each other to serve as such media. Both of them are fiat objects in the sense that they do not yield direct utility or facilitate production.

The only exogenous difference between money and bonds is that the government accepts money, but not bonds, as the means of payments. This legal restriction applies in matches with the government but not in matches between two private agents. The scope of the legal restriction is measured by the size of government sellers. Of particular interest is the case in which the scope of the legal restriction approaches zero.

I construct two models of this framework. In the first model, the government does not participate in the goods market and hence there is no legal restriction. In the second model, the government participates in the goods market as a positive measure of agents who follow exogenous trading rules. Both models generate positive discounting on newly issued bonds. The positive nominal interest rate is an outcome of the temporary separation between the bonds market and the goods market, as in Lucas (1990). The temporary separation implies that newly issued bonds cannot be used in the goods market in the issuing period, and hence they are not perfect substitutes for money. To compensate for this one-period loss of liquidity, newly issued bonds must be discounted.

The two models have very different predictions on whether matured bonds circulate as money. In the first model, where the government does not participate in the goods market, matured bonds circulate in the goods market as perfect substitutes for money. Agents are indifferent about how large a fraction of matured bonds to redeem and this fraction is indeterminate in the equilibrium. Such indeterminacy makes the price level, the ratio of matured bonds to money, and the asset values all indeterminate. However, all these equilibria have the same real output, consumption, and the nominal interest rate. In the second model, where the government participates in the goods market, the equilibrium is unique. In this equilibrium, matured bonds do not circulate as a medium of exchange. This is true no matter how small the legal restriction is, provided that it is positive. I explain this result by the frictions in the goods market.

This result in the second model provides a robust answer to the question of why matured bonds do not circulate as a medium of exchange. The key feature of this result is that even an arbitrarily small legal restriction is sufficient to drive matured bonds out of the circulation. A large legal restriction is not necessary for achieving the outcome. This result cannot be generated by a model with a centralized and Walrasian goods market. Although a large legal restriction in a centralized market may prevent matured bonds from circulating, bonds will start to circulate when the legal restriction becomes small. All the legal restriction does in that case is to shift money from the market for private goods to the market for government goods.

I also analyze two types of monetary policy. One is an increase in the money growth rate and the other is an increase in the amount of bond sales in the open market, both of which are deterministic changes. I will summarize the policy effects in Section 6. Not surprisingly, the policy effects are very different in the two models. Moreover, the policy effects in the second model contrast sharply with those in conventional models, although they share the feature that bonds do not circulate as money. Thus, it matters for policy analysis whether the outcome of no circulating bonds is endogenously generated or exogenously imposed.

The models constructed here share some features of Lucas's (1990) model, such as a positive discount on newly issued bonds. The key distinctions from Lucas's analysis are that the value of money is supported by the description of the trading environment and that nominal bonds are not precluded from circulation as a medium of exchange.

The structure of the model builds on two previous papers (Shi, 1997, 1999).[2] The way in which the government is modeled here is similar to that in Li and Wright (1998). In a search model, Aiyagari et al. (1996) first attempted to analyze the coexistence of money and government bonds. Their results have some resemblances to mine.[3] However, there are significant differences. First, I eliminate their assumption that money and bonds are indivisible, as indivisibility itself can generate spurious results on the coexistence of two nominal assets. Second, the models here are tractable for analyzing standard monetary policy, such as money growth and open-market operations. Third, Aiyagari et al. (1996) assume a decentralized bonds market, and so an agent who wants to redeem bonds may fail to do so with positive probability. This restriction on the redemption of matured bonds may force some agents to hold onto matured bonds, thus artificially making matured

[2] The original Kiyotaki–Wright search model assumes that goods and money are both indivisible. Shi (1995) and Trejos and Wright (1995) eliminate the assumption of indivisible money, Green and Zhou (1998) eliminate the assumption of indivisible money, and Molico (1997) and Shi (1997, 1999) eliminate both.

[3] In particular, for matured bonds to be perfect substitutes for money and yet newly issued bonds to be discounted in Aiyagari et al. (1996), the government must reject unmatured bonds with positive probability in trades.

bonds circulate. I eliminate this restriction by assuming a centralized market for issuing and redeeming bonds. As a result, there is a greater degree of competition between bonds and money, and the results are more robust.

Before going into the model, let me clarify two aspects of the analysis. First, the analysis in this article is positive instead of normative. In particular, the legal restriction is exogenously imposed, although it can be arbitrarily small. Whether the legal restriction can be welfare improving is an interesting question, but I do not address it here.[4] Second, I will restrict attention to one-period nominal bonds. This restriction will be relaxed in a sequel (Shi, 2003a).

With the restriction to one-period bonds, the timing of events described in this article implies that bonds can circulate only as matured bonds. Thus, the question is whether agents will choose to hold bonds beyond maturity and use them to buy goods in the future. One may wonder why any bond holder would choose to miss out on the redemption. This question is misguided by models where money is assumed to have an intrinsic value. In any model where money is intrinsically worthless, redemption of a nominal bond is a swap of one fiat object for another. Missing out on the redemption is not costly at all to a bond holder if bonds perform the role of a medium of exchange as well as money does.

In Section 2, I will describe an economy in which the government does not participate in the goods market. In Section 3, I will show that there are a continuum of equilibria in this economy and that bonds circulate at par with money. Then, in Section 4, I will introduce government agents into the goods market and establish the central result that there is no equilibrium in which bonds circulate in the goods market. The equilibrium in this economy will be characterized in Section 5, where I will also analyze the effects of money growth and open-market operations. I will conclude in Section 6 and supply necessary proofs in the Appendix.

2. AN ECONOMY WITH NOMINAL BONDS

In this section, I describe a search economy with nominal bonds. Because the integration of nominal bonds into a search model is relatively new, I abstract government agents from the goods market at the moment. This structure paves the way for the analysis in Section 4, where government agents will be introduced into the goods market.

2.1. *Households, Matches, and Timing.* The economy has discrete time and many types of households. Each type contains a large number of households that is normalized to 1. Households of each type are specialized in producing one type of good, which they do not consume but which they can sell for the goods they want to consume. The utility is $u(\cdot)$ from consumption goods and 0 from other

[4] To allow illiquid nominal bonds to improve welfare, one might follow Kocherlakota (2003) to introduce taste shocks. However, in Kocherlakota's model, the illiquidity of bonds is a physical feature, instead of an equilibrium outcome as in my article. He also assumes that matured bonds will perish if they are not redeemed immediately. These assumptions preclude matured bonds from circulating as money.

goods. The disutility of production is $\psi(\cdot)$. Assume $u' > 0$, $u'' \leq 0$, $\psi(0) = 0 = \psi'(0)$, $\psi'(q) > 0$ for all $q > 0$, and $\psi'' > 0$. Moreover, $u'(0) > \psi'(0)$ and $u'(\infty) < \psi'(\infty)$. Thus, for any given $k \in (0, \infty)$, there is a unique $q \in (0, \infty)$ such that $u'(kq) = \psi'(q)$.

Agents meet bilaterally and randomly in the market, as in Kiyotaki and Wright (1989, 1993). To emphasize the competition between money and nominal bonds, I assume that there is no chance of a double coincidence of wants in a meeting to support barter, or public record keeping of transactions to support credit trades. In addition, goods are perishable between periods. As a result, every trade requires a medium of exchange, which can be money or nominal bonds or both.[5] Money and bonds have no intrinsic value. Both can be stored without cost.

Nominal bonds are issued by the government and are default free. Each unit of a bond can be redeemed for one unit of money at maturity and the maturity is restricted to one period (see the introduction for a discussion on the maturity). For convenience, I make two auxiliary assumptions. First, bonds can be redeemed only at the maturity. Second, an agent can bring money and bonds separately into matches but not together into a match. These assumptions are not critical, as I will show in Section 4.4.

To specify the matching technology, let me call an agent in the goods market a *buyer* if he holds money or bonds and a *seller* if he holds neither money nor bonds. Let σ be the (fixed) fraction of agents in the goods market who are sellers and $(1 - \sigma)$ the fraction of agents who are buyers. I call a match between two agents a *trade match* if there is a single coincidence of wants, i.e., if one (and only one) of the two agents can produce the partner's consumption goods. A buyer encounters a trade match in a period with probability $\alpha\sigma$ and a seller does so with probability $\alpha(1 - \sigma)$, where $\alpha > 0$ is a constant.

Random matching can generate nondegenerate distributions of agents' money holdings and consumption. To maintain tractability, I assume that each household consists of a large number of members who share consumption each period and regard the household's utility function as the objective. This assumption makes the distribution of money holdings degenerate across households and hence allows me to focus on equilibria that are symmetric across households.[6]

A household consists of a measure σ of sellers and a measure $(1 - \sigma)$ of buyers. A seller produces and sells goods, whereas a buyer purchases consumption goods. There are two types of buyers: *money holders* who carry bonds and *bond holders* who carry bonds. Let n be the measure of money holders in the household and $(1 - \sigma - n)$ be the measure of bond holders. This division between the two types of buyers is endogenous, as n is chosen by the household. To focus on this choice, I assume that the division between buyers and sellers is fixed (see Shi, 1997, for an endogenous division).

[5] All these assumptions can be relaxed. See Shi (1995) and Trejos and Wright (1995) for the introduction of barter, Li (1998) for middlemen, Shi (1996) for bilateral credits, Kocherlakota and Wallace (1998) for imperfect public record keeping, and Shi (1999) for capital.

[6] The assumption of large households, used in Shi (1997, 1999), is extended from Lucas (1990). Lagos and Wright (2005) use a different set of assumptions to achieve essentially the same purpose of risk smoothing.

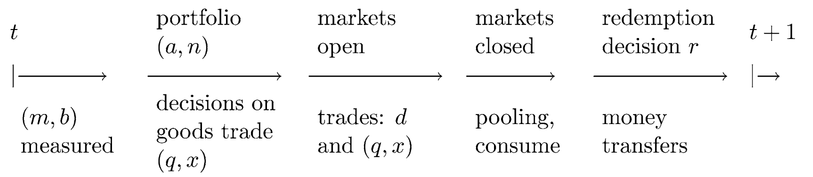

FIGURE 1

TIMING OF EVENTS IN A PERIOD

In contrast to the goods market, the bonds market is centralized and has a much lower transaction cost. To simplify, I abstract from such a transaction cost altogether by assuming that trades in the bonds market take zero measure of members. Also, following Lucas (1990), I assume that the government sells bonds only for money. Each newly issued bond is sold at a market price S and can be redeemed for one unit of money at (and only at) maturity. Thus, $1/S$ is the gross nominal interest rate.

The events in an arbitrary period t unfold as in Figure 1, where the subscript t is suppressed. At the beginning of the period, the household has an amount m of money and an amount b of matured bonds. These are the bonds that were matured in previous periods but not redeemed. The household divides money into a fraction a to be taken into the goods market and a fraction $(1 - a)$ into the bonds market. The household also divides buyers into moneyholders and bondholders, by choosing $n \in [0, 1 - \sigma]$. Thus, each moneyholder holds an amount of money, am/n, and each bond holderholds an amount of matured bonds, $b/(1 - \sigma - n)$. The household does not carry any matured bonds into the bonds market because new bonds are sold only for money.

At the time of choosing the portfolio divisions (a, n), the household also chooses the quantities of trade (q, x), which I will describe later.

Next, the goods market and the bonds market open simultaneously. In the bonds market, the household purchases an amount d of new bonds, using the money allocated to the bonds market. In the goods market, the agents trade according to the quantities (q, x) prescribed by the household. Then, the goods market closes. The household pools the receipts from the trades and allocates the same amount of consumption to every member.

After consumption, the bonds purchased at the beginning of the current period mature and the household chooses the fraction of such bonds to be redeemed (in the centralized market).[7] Let $r \in [0, 1]$ denote this *redemption fraction*. By the earlier auxiliary assumption, bonds that are not redeemed at maturity cannot be redeemed in the future. After redeeming bonds, the household receives a lump-sum monetary transfer L and time proceeds to the next period.

[7] An alternative timing structure is that bonds issued in this period become mature at the beginning of the next period at which time they can be redeemed for money, before other activities take place. This alternative structure does not make any difference in the results from the timing structure assumed here. In both cases, the amount of money obtained from redeeming the bonds cannot be used until the next period.

This timing sequence highlights the temporary separation of the bonds market and the goods market, as in Lucas (1990). A household cannot shift resources between the two markets within a period, although it can do so over time. This is a critical feature because it makes it costly to bring money into the bonds market.

This model differs from Lucas's in two aspects. First, the goods market is decentralized here in contrast to the centralized goods market. With decentralized trades, fiat objects such as money and nominal bonds can have positive values in equilibrium, even when the current model is extended to allow agents to barter. This endogenous role of a medium of exchange is absent when the goods market is Walrasian. Second, I allow households to use bonds, as well as money, to buy goods, whereas Lucas assumes that money is the only medium of exchange. Lucas's assumption amounts to imposing $r = 1$ a priori.

To focus on these new elements, I keep other aspects of the model as close as possible to Lucas's (1990). In particular, I adopt Lucas's assumption that monetary transfers maintain the money growth rate constant (in Part C of the Appendix, I will relax this assumption). Let M be the aggregate money holdings per household at the beginning of a period and M_{+1} the holdings in the next period. The (gross) rate of money growth between the two periods is $\gamma_{+1} = M_{+1}/M$. Monetary transfers maintain $\gamma_{+1} = \gamma$ as a constant. Thus, the effects of open-market operations are isolated from the effects of money growth.

2.2. *Quantities of Trade in the Goods Market.* Pick an arbitrary household as the representative household. Normalize the measure of members in the household to 1. Lowercase letters denote the decisions of this household, whereas uppercase letters denote other households' decisions or aggregate variables. Also, suppress the generic time subscript t, denoting the subscript $t \pm j$ as $\pm j$, for $j \geq 1$.

The household chooses the quantities of money and goods in each trade match. To simplify the analysis, I assume that the buyer in a trade match makes a take-it-or-leave-it offer. Consider a buyer holding asset i, where $i = m, b$. The offer specifies the quantity of goods that the buyer wants the seller to supply, q^i, and the quantity of asset i that the buyer gives, x^i. If a match is not a trade match, the household instructs its members to not trade.[8]

To describe the decisions on (q^i, x^i), let $\beta \in (0, 1)$ be the discount factor and $v(m, b)$ the household's value function, where I suppressed the dependence of v on aggregate variables. Let ω^i be the household's marginal value of asset i $(=m, b)$ in the next period, discounted to the current period. That is,

$$\omega^m \equiv \beta v_1(m_{+1}, b_{+1}), \quad \omega^b \equiv \beta v_2(m_{+1}, b_{+1}) \tag{1}$$

[8] I omit the possible trades between a money holder and a bond holder in the goods market because such trades are immaterial in the current model. As will become clear later, whenever there are matured bonds in the goods market, they are traded at par with money.

where the subscripts of v indicate partial derivatives and the subscript $+1$ indicates "the next period." Other households' values of the two assets are Ω^m and Ω^b, respectively.

When choosing the quantities (q^m, x^m), the household anticipates the following constraints:

$$x^m \leq am/n \tag{2}$$

$$\psi(q^m) \leq \Omega^m x^m \tag{3}$$

The first constraint says that the amount of money traded cannot exceed the amount the buyer carries into the trade. This is necessary because the matches are separated from each other. The second constraint says that the offer cannot make the seller worse off than not trading, where $\Omega^m x^m$ is the value of money that the seller's household gets by agreeing to the trade.

Similarly, the following constraints apply when the buyer holds bonds:

$$x^b \leq b/(1-\sigma-n) \tag{4}$$

$$\psi(q^b) \leq \Omega^b x^b \tag{5}$$

I will call (2) and (4) the *trading constraints* in the goods market. When (2) binds, I say that money yields *liquidity* or *service* in the goods market. Similarly, matured bonds yield liquidity or service in the goods market when (4) binds.

2.3. *The Representative Household's Decision Problem.* The household's choices in each period are the portfolio divisions (a, n), the quantities of trade (q^m, x^m, q^b, x^b), the amount of new bonds to purchase d, consumption c, the redemption fraction r, and future asset holdings (m_{+1}, b_{+1}). Taking other households' choices and aggregate variables (i.e., the uppercase letters) as given, the household solves the following problem:

$$(PH) \quad v(m,b) = \max\{u(c) - \alpha\sigma[N\psi(Q^m) + (1-\sigma-N)\psi(Q^b)] + \beta v(m_{+1}, b_{+1})\}$$

The constraints are as follows:

(i) the constraints in the goods market, (2)–(5), and

$$c = \alpha\sigma[nq^m + (1-\sigma-n)q^b] \tag{6}$$

(ii) the money constraint in the bonds market:

$$Sd \leq (1-a)m \tag{7}$$

(iii) the laws of motion of asset holdings:

$$m_{+1} = m + (r - S)d + \alpha\sigma(NX^m - nx^m) + L \tag{8}$$

$$b_{+1} = b + (1 - r)d + \alpha\sigma[(1 - \sigma - N)X^b - (1 - \sigma - n)x^b] \tag{9}$$

(iv) and other constraints:

$$0 \le a \le 1, \quad 0 \le n \le 1 - \sigma, \quad 0 \le r \le 1 \tag{10}$$

The disutility of production in the objective function is calculated as follows. The total number of trades that involve the household's sellers is $\alpha\sigma$. In a fraction N of such trades the trading partners are moneyholders, who ask for Q^m units of goods; in the remaining fraction of trades the trading partners are bondholders, who ask for Q^b units of goods. The amount of consumption given by (6) can be explained similarly. The constraint (7) in the bonds market is self-explanatory, with S being the price of newly issued bonds and d the amount of such bonds purchased by the household.

To explain the law of motion of money, (8), note that the household's money holdings can change between the current period and next period for three reasons: purchasing and redeeming newly issued bonds, trading in the goods market, and receiving monetary transfers. In particular, the household gets a net amount $(r - S)d$ from purchasing d units of money and redeeming a fraction r of them. Adding these three changes to the household's current money holdings gives the household's money holdings at the beginning of the next period.

The law of motion of matured bonds, given by (9), can be explained similarly. Note that bonds acquired in the goods market have already passed their maturity and hence, under an earlier auxiliary assumption, they cannot be redeemed for money.

To characterize optimal decisions, let ρ be the Lagrangian multiplier of the money constraint in the bonds market, (7), λ^m of the money constraint in the goods trade, (2), and λ^b of the bond constraint in the goods trade, (4). To simplify the equations, multiply λ^m by the number of trades that involve the household's moneyholders, $\alpha\sigma n$. Similarly, multiply λ^b by $\alpha\sigma(1 - \sigma - n)$. Incorporating these constraints, I can rewrite the objective function in (PH) as follows:

$$\begin{aligned} v(m, b) = \max\Big\{ &\beta v(m_{+1}, b_{+1}) + u(\alpha\sigma[nq^m + (1 - \sigma - n)q^b]) \\ &- \alpha\sigma[N\psi(Q^m) + (1 - \sigma - N)\psi(Q^b)] + \rho[(1 - a)m - Sd] \\ &+ \alpha\sigma n\lambda^m\left(\frac{am}{n} - x^m\right) + \alpha\sigma(1 - \sigma - n)\lambda^b\left(\frac{b}{1 - \sigma - n} - x^b\right)\Big\} \end{aligned}$$

where x^m and x^b satisfy (3) and (5) with equality (provided $\omega^m, \omega^b > 0$); that is,

$$x^i = \psi(q^i)/\Omega^i \quad \text{for} \quad i = m, b \tag{11}$$

The following conditions are necessary for the decisions to be optimal:

(i) For q^i:

$$u'(c) = (\omega^i + \lambda^i)\psi'(q^i)/\Omega^i, \quad i = m, b \tag{12}$$

(ii) For (r, a, d, n):

$$\omega^m = \omega^b \quad \text{if } r \in (0, 1) \tag{13}$$

$$\alpha\sigma\lambda^m = \rho \quad \text{if } a \in (0, 1) \tag{14}$$

$$r\omega^m + (1 - r)\omega^b = (\omega^m + \rho)S \quad \text{if } d \in (0, \infty) \tag{15}$$

$$q^m = q^b \quad \text{if } n \in (0, 1 - \sigma) \tag{16}$$

In each of these conditions, the variable attains the lowest value in the specified domain if the corresponding equality is replaced by "$<$," and the highest value if "$>$."

(iii) For (m, b) (envelope conditions):

$$\omega^m_{-1}/\beta = \omega^m + (1 - a)\rho + a\alpha\sigma\lambda^m \tag{17}$$

$$\omega^b_{-1}/\beta = \omega^b + \alpha\sigma\lambda^b \tag{18}$$

Condition (12) requires that the net gain to the buyer's household from asking for an additional amount of goods be 0. To induce the seller to produce one additional unit of good, the buyer must pay an additional amount $\psi'(q^i)/\Omega^i$ of asset i (see (11)). By making this payment, the buyer forgoes the discounted future value of the asset, ω^i, and causes the asset constraint in the trade to be more binding. Thus, $(\omega^i + \lambda^i)$ is the shadow cost of one unit of asset i to the buyer's household. The cost of getting an additional unit of good from the seller, given by the right-hand side of (12), must be equal to the marginal utility of consumption.

In (ii), (13) says that matured bonds must have the same future value as money if the household chooses not to redeem all bonds at maturity. Condition (14) says that for the household to allocate money to both the goods market and the bonds market, money must generate the same marginal liquidity in the two markets by relaxing the trading constraints. Condition (15) says that the expected future value of newly issued bonds must be equal to the cost of money used to acquire such bonds. Because the household may not redeem all bonds at the end of the period, the expected future value of newly issued bonds is the weighted average of the future values of money and matured bonds, where the weights are r and $(1 - r)$.

Condition (16) says that for the household to send both moneyholders and bondholders to the goods market, the two types of buyers must obtain the same quantity of goods in a trade match. To explain this condition, note that a buyer of type i (=m, b) obtains a surplus or net gain from a trade, $[u'(c)q^i - (\omega^i + \lambda^i)\, x^i]$, where $(\omega^i + \lambda^i)$ is the marginal cost of asset i to the buyer (as explained above). Since $(\omega^i + \lambda^i)x^i = u'\psi(q^i)/\psi'(q^i)$ by (11) and (12), a type-i buyer's surplus from trade is $u'(c)[q^i - \psi(q^i)/\psi'(q^i)]$. For the optimal choice for n to be interior, the two types of buyers must obtain the same surplus in a trade. This requires $q^m = q^b$ because $[q - \psi(q)/\psi'(q)]$ is a strictly increasing function.[9]

Finally, the envelope conditions require the current value of each asset to be equal to the sum of the future value of the asset and the expected service generated by the asset in the current markets. Take the condition for money, (17), for example. The current value of money is given by the left-hand side of (17), where ω^m_{-1} is divided by β because ω^m_{-1} is defined as the value of money discounted to one period earlier. The right-hand side of (17) consists of the future value of money, ω^m, the service generated by money in the current bond market, ρ, and the service generated by money in the current goods market, $\alpha\sigma\lambda^m$. The services in the two markets are weighted by the division of money into the two markets.

3. SYMMETRIC EQUILIBRIUM

3.1. *Definition, Focus, and the Nominal Interest Rate.* A (symmetric) monetary equilibrium consists of a sequence of a representative household's choices

$$\left(a_t, n_t, q_t^m, x_t^m, q_t^b, x_t^b, d_t, c_t, r_t, m_{t+1}, b_{t+1}\right)_{t=0}^{\infty}$$

the implied shadow prices $(\omega_t^m, \omega_t^b, \rho_t, \lambda_t^m, \lambda_t^b)_{t=0}^{\infty}$, and other households' choices such that the following requirements are met:

(i) Optimality: Given other households' choices, the household's choices solve (PH) with given initial holdings (m_0, b_0);
(ii) Symmetry: The choices (and shadow prices) are the same across households;
(iii) Clearing in the bonds market: $d_t = zM_t$, where M_t is money holdings per household at the beginning of period t and zM_t is the amount of new bonds issued in period t, with $z > 0$;
(iv) Positive and finite total value of each asset: $0 < \omega_t^m m_t < \infty$ and $0 < \omega_t^b b_t < \infty$ for all t if $m, b > 0$.

[9] Throughout this article, the term "surplus" or "net gain" is the marginal gain to the household, derived from a single trade. One should not confuse this surplus with the total surplus that the household obtains from all matches of a given type that involve the household's buyers. It is the marginal surplus that is relevant for the household's decisions, and hence for the assets' values in equilibrium, because the household has a large number of buyers each contributing only marginally to the household's utility. One way to explicitly distinguish the marginal surplus from the total surplus is to index each match by, say, j and express the choices in such a match as $(q^i(j), x^i(j))$ (see Shi, 1997). These choices across the index j interact with each other only through their effects on (c, ω^i), which are negligible when the household has a large number of trades in each period.

In this definition I have restricted bond issuing to be a constant fraction of the money stock. I have also restricted the total value of each asset to be positive and finite in order to examine the coexistence of money and bonds.[10]

For the issues in this article, I restrict attention further to equilibria that have the following features. First, money serves as a medium of exchange. This requires $a > 0$ and $n > 0$. (Note that $a < 1$ holds as a result of the bonds market clearing condition, and so $\rho = \alpha\sigma\lambda^m$.) Second, the household redeems a positive fraction of matured bonds, i.e., $r > 0$. Third, the equilibrium is stationary and, in particular, the bond–money ratio (b/m) and the value of the total stock of money $(\omega^m_{-1} m/\beta)$ are constant. Since the money stock grows at a constant rate γ, then $\omega^m_{-1}/\omega^m = m_+/m = \gamma$, which is also the gross rate of inflation in the stationary equilibrium.

Under these restrictions, the equilibrium is one of the following two types:

(a) $0 < r < 1$ and $0 < n < 1 - \sigma$: In this case, $\omega^m = \omega^b$, $q^m = q^b$, and bonds circulate in the goods market as perfect substitutes for money;

(b) $(1 - r)(1 - \sigma - n) = 0$: In this case, $q^m > q^b$ and bonds do not circulate.

In both cases, the gross nominal interest rate, $1/S$, can be derived from (15) as

$$1/S = 1 + \alpha\sigma\lambda^m/\omega^m \tag{19}$$

Note from (17) that $\alpha\sigma\lambda^m/\omega^m = \omega^m_{-1}/(\beta\omega^m) - 1$. Since $\omega^m_{-1}/\omega^m = \gamma$, the Fisherian relationship holds, that is,

$$1/S = \gamma/\beta \tag{20}$$

Newly issued bonds are sold at a discount and hence the net nominal interest rate is positive, provided $\gamma > \beta$. This result is reminiscent of Lucas's (1990), but it holds even when bonds circulate in the goods market as a medium of exchange. Although bonds can become perfect substitutes for money one period after they are issued, they are not perfect substitutes for money at the time of issuing. The temporary separation between the bonds market and the goods market prevents newly issued bonds from being used to purchase goods in the *same* period as they are issued. Thus, there is a one-period loss of liquidity to the amount of money allocated to purchase newly issued bonds. To compensate for this loss of liquidity, newly issued bonds must be discounted; otherwise the household will not allocate any money to the bonds market.

The liquidity value of money is the extent to which money relaxes the trading constraints in the goods market, as captured by the term $\alpha\sigma\lambda^m$ in (19). This liquidity value is positive only if $\gamma > \beta$. When $\gamma = \beta$, money generates a rate of return equal to the rate of time preference, in which case a household is indifferent between spending a marginal unit of money on goods and keeping it for its store of value. In the remainder of this article, I will impose $\gamma > \beta$.

[10] The bounded total value of each asset is necessary for the household's optimal decisions to be indeed characterized by the first-order conditions obtained in the previous section.

3.2. *Equilibria with Bonds Circulating in the Goods Market.* Matured bonds can circulate in the goods market as perfect substitutes for money. In fact, there is a continuum of such equilibria, as described below.

PROPOSITION 1. *Assume $z \in (0, \gamma/\beta)$. If $\gamma > 1$, the redemption fraction r is indeterminate. For each $r \in (0, 1)$, there exists a stationary equilibrium in which matured bonds circulate in the goods market as perfect substitutes for money. The price level and the bond–money ratio are lower in an equilibrium with a higher r than with a lower r. Also, the values of assets and the division n depend on r. However, the quantity of goods traded in a match, the level of consumption, the division of money a, and the nominal interest rate are all independent of r. If $\gamma \leq 1$, there is no equilibrium with a bounded and positive ratio b/m unless $r = 1$.*

It is useful to prove the proposition by constructing the continuum of equilibria in the proposition. Fix r at any arbitrary level in (0, 1). For this interior choice to be optimal, matured bonds and money must have the same value, i.e., $\omega^b = \omega^m$ (see (13)). In addition, the household must allocate a positive measure of buyers to trade with such bonds; i.e., $n \in (0, 1 - \sigma)$. For such a choice of n to be optimal, matured bonds and money must exchange for the same quantity of goods in a trade, i.e., $q^m = q^b$ (see (16)). Since the two assets have the same value and trade for the same quantity of goods, they are perfect substitutes in the goods market.

The quantities q^m and q^b are independent of r. To show this, suppress the superscripts of q^m and q^b since they are equal to each other. Then, (12) yields $\lambda^m = \lambda^b$ and $\lambda^m/\omega^m = u'(c)/\psi'(q) - 1$, where $c = \alpha\sigma(1-\sigma)q$. Combining these results with (19) and (20), I get

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma\left[\frac{u'(\alpha\sigma(1-\sigma)q)}{\psi'(q)} - 1\right] \tag{21}$$

This equation yields a unique solution for q for all $\gamma \geq \beta$ and the solution is independent of r. Thus, real consumption and real output are independent of r.

Also independent of r are the price of newly issued bonds, the nominal interest rate, and the division of money between the two markets, a. The price of newly issued bonds is $S = \beta/\gamma < 1$, as shown before, and the fraction a is given by

$$a = 1 - Sd/m = 1 - z\beta/\gamma \tag{22}$$

Clearly, $a \in (0, 1)$ if and only if $0 < z < \gamma/\beta$, a condition imposed in Proposition 1.

The redemption fraction affects other variables. First, the bond–money ratio, obtained from the law of motion of bonds (9), is

$$\frac{b}{m} = \frac{(1-r)z}{\gamma - 1} \tag{23}$$

Given $r \in (0,1)$, this ratio is positive and finite if and only if $\gamma > 1$. Also, it is a decreasing function of r when $\gamma > 1$. Second, to keep the money stock growing at the constant rate γ, (8) requires monetary transfers to satisfy

$$\frac{L}{m} = \gamma - 1 + z\left(\frac{\beta}{\gamma} - r\right) \tag{24}$$

Third, because $\gamma > \beta$, (21) implies $u'(c) > \psi'(q)$, and so $\lambda^m = \lambda^b > 0$. That is, $x^m = am/n$ and $x^b = b/(1-\sigma-n)$. Dividing these two equations and using $x^i = \psi(q)/\omega^i$, I get

$$n = (1-\sigma) \Big/ \left[1 + \left(\frac{1-r}{\gamma-1}\right)\left(\frac{z}{1 - z\beta/\gamma}\right)\right] \tag{25}$$

This fraction indeed lies in $(0, 1-\sigma)$ for $\gamma > 1$ and it is an increasing function of r. Fourth, the values of money and bonds are

$$\begin{aligned} \omega^m &= \psi(q)/x^m = n\psi(q)/(am) \\ \omega^b &= \psi(q)/x^b = (1-\sigma-n)\psi(q)/b \end{aligned} \tag{26}$$

Finally, nominal prices of goods are

$$p^m \equiv x^m/q = x^b/q \equiv p^b \tag{27}$$

Because $x^m = am/n$, $da/dr = 0$, and $dn/dr > 0$, the price level is a decreasing function of r. This completes the construction of the equilibrium for an arbitrary $r \in (0,1)$.

The redemption fraction is indeterminate because matured bonds are perfect substitutes for money in the goods market. By choosing not to redeem a unit of matured bond, the household misses out on the payment of one unit of money. However, the retained bond can purchase exactly the same amount of goods as does a unit of money. Thus, it does not matter to the household how much of the matured bonds to be redeemed.

Different redemption fractions lead to different levels of total "moneyness" in the economy, $(m + b)$. A higher redemption fraction reduces the bond–money ratio and hence reduces the total moneyness in the goods market. Since real output is the same for all r, the higher redemption fraction reduces the price level and increases the values of the two assets. Also, as the bond–money ratio falls, households shift some members from holding bonds to holding money; i.e., n increases. This shift is necessary for maintaining perfect substitutability between the two assets, by ensuring that money and bonds yield the same services in the goods market.

It is remarkable that (matured) nominal bonds circulate in the goods market in equilibrium, given that the model has imposed a few restrictions that seem to reduce the desirability of bonds relative to money. For example, the government does not accept matured bonds as payments for newly issued bonds and it refuses to redeem bonds that have passed their maturity. In the end, however, the fiat nature of bonds makes them equally acceptable in the goods market as the other fiat object—money.

The indeterminacy of equilibria in the current model differs from that in a conventional monetary model where the real money balance appears in the utility function. There, the price level is indeterminate when the utility function depends on the real money balance in particular ways. Here, the real money balance does not appear in the utility function. Instead, the substitution between money and other money-like assets in the goods market is the key to the indeterminacy.

Notice that the continuum of equilibria requires $\gamma > 1$. If $\gamma \leq 1$ and $r < 1$, then there is no equilibrium with a positive and bounded bond–money ratio (see (23)). However, this is an artifact of two assumptions. First, I have assumed that the government can borrow but not lend, i.e., $b > 0$. If the government can lend as well, then the continuum of equilibria can exist for $\gamma < 1$ as well as for $\gamma \geq 1$. Second, I have assumed that the government does not redeem bonds that have passed the maturity. If this assumption is eliminated, then one can show that a continuum of equilibria exist for $\gamma < 1$ (as well as for $\gamma \geq 1$), provided $1 > r > \max\{0, 1 - \gamma\}$.

3.3. *Effects of Open-Market Operations.* The continuum of equilibria makes policy analysis difficult because a change in a policy may induce the economy to switch from one equilibrium to another. To illustrate, let us examine a tightening open-market operation, modeled as an increase in z. Let $r < 1$ and $\gamma > 1$.

Assume first that the redemption fraction does not respond to the increase in z. Then, the tightening operation increases the bond–money ratio and *increases* the price level. To explain this seemingly paradoxical result, note that new bonds are sold for money at a discount. Each unit of bond is sold for less than one unit of money but, when the bond matures, it will circulate in the goods market as a perfect substitute for one unit of money. Thus, an increase in the bond sale increases the total moneyness in the goods market in the future and pushes up the price level.

The households absorb the increased quantity of bonds by allocating more buyers to transact with matured bonds, thus maintaining the perfect substitutability between the two assets in the goods market. The bond price and the nominal interest rate do not change. This is because the households fully anticipate the increase in z, and so they increase the amount of money allocated to the bonds market to completely offset the increase in the new issues.[11] Real consumption and output do not respond to the increase in bond issuing, either.

The effect on the price level can be quite different if the redemption fraction responds to the tightening operation, a possibility that can occur given the continuum of equilibria. For example, if the redemption fraction increases sufficiently with the tightening operation, then the bond–money ratio and the price level

[11] If the households must allocate money between the two markets before knowing the change in z, then the nominal interest rate will change, as in Lucas (1990).

can fall, instead of rise, with the increase in z. Thus, whether the tightening open-market operation will increase or decrease the price level depends on the direction and magnitude in which the redemption fraction responds to the operation.[12] Real consumption and output, however, are still invariant with respect to the operation.

4. GOVERNMENT IN THE GOODS MARKET

Now I introduce the government into the goods market and show that even an arbitrarily small legal restriction in the goods market can drive matured bonds out of the circulation as money.

4.1. *Government Traders and Private Households' Decisions.* The government has a measure N_g of buyers and a measure σ_g of sellers. Each government buyer holds M_g/N_g units of money. The total measure of buyers in the economy is $(1 - \sigma + N_g)$ and the total measure of sellers is $(\sigma + \sigma_g)$. With random matching, a buyer (private or government) encounters a trade match with a private seller with probability $\alpha\sigma$, and with a government seller with probability $\alpha\sigma_g$. Similarly, a seller gets a trade match in a period with a private moneyholder with probability αN, with a (private) bondholder with probability $\alpha(1 - \sigma - N)$, and with a government buyer with probability αN_g.[13] Assume that government goods are perfect substitutes for private goods. This assumption is not critical, as discussed later.

An important assumption is that the government sells goods only for money and buys goods only with money. More specifically, I assume that government agents trade in the following reasonable but exogenous ways:[14]

(i) *A government buyer*. A government buyer carries only money. In a trade match with a private seller, the government buyer buys goods at the price that a private moneyholder pays to a private seller. This price, denoted P^m, is the price averaged over all trades between two private agents who use money as payments. In addition, a government buyer spends all his money in the trade, and so the quantity of good he purchases is $M_g/(N_g P^m)$.[15]

[12] One can push the above argument further by introducing sunspots into the economy that serve as a device to coordinate the households' decisions on the redemption ratio. Then, it is possible that the price level can remain invariant with respect to open-market operations. Such invariant prices, of course, have nothing to do with any cost associated with adjusting nominal prices.

[13] To generate these matching probabilities, the aggregate number of matches is $\hat{\alpha}(\sigma + \sigma_g)(1 - \sigma + N_g)/(1 + \sigma_g + N_g)$, where $(1 + \sigma + N_g)$ is the total number of agents in the market and $\hat{\alpha} > 0$ is a constant. The matching probabilities described here are generated under random matching and the normalization $\alpha = \hat{\alpha}/(1 + \sigma_g + N_g)$.

[14] Another way to model government agents' trading behavior is to specify the government's objective function. I do not take this approach because it is not clear what the appropriate objective of the government is.

[15] In order for a private seller to be willing to sell a quantity as high as $M_g/(N_g P^m)$, his surplus must be nonnegative at this quantity, i.e., $\omega^m M_g/N_g \geq \psi(M_g/(N_g P^m))$. In a later proposition (Proposition 3), I will provide a condition under which this requirement is satisfied for all policy experiments conducted in this article.

(ii) *A government seller*. A government seller accepts only money for trade. In a trade match with a private moneyholder, a government seller sells goods only at the average price P^m, but leaves the quantity of goods for the private moneyholder to decide. Denote this quantity of goods as q^g.

Notice that all individual agents take P^m as given. Also, I abstract from the trades between two government agents, since such trades are inconsequential to private households.

The government's refusal to accept money as payments is a legal restriction, the extent of which can be measured by σ_g. Let me clarify two aspects of this restriction. First, the legal restriction is enforced only in trades with government agents. When two private agents trade with each other, they can choose to use both matured bonds and money as payments. Second, the legal restriction requires only that bonds should not be used *directly* to pay for government goods. Of course, agents can redeem the bonds for money first and then use the receipt to pay for government goods. However, this indirect payment takes one period of time in the model.

I now modify a household's consumption to incorporate the trades with government agents:

$$c = \alpha\{n[\sigma q^m + \sigma_g q^g] + \sigma(1-\sigma-n)q^b\} \tag{28}$$

The household also has a net receipt of money from trades with the government, $\alpha(\sigma M_g - \sigma_g n P^m q^g)$. Incorporating this additional term, the law of motion of the household's money holdings becomes

$$m_{+1} = m + (r-S)d + \alpha\sigma(NX^m - nx^m) + \alpha(\sigma M_g - \sigma_g n P^m q^g) + L \tag{29}$$

The law of motion of the household's bond holdings is still (9) and the money constraint in the bonds market is still (7).

The household's additional choice is q^g, the quantity of goods that a private buyer asks a government seller to supply. The corresponding money constraint in such a trade is

$$P^m q^g \leq am/n \tag{30}$$

Let λ^g be the Lagrangian multiplier of this constraint and multiply λ^g by the number of trades between a household's buyers and government sellers, $\alpha\sigma_g n$. The optimal choice of q^g satisfies

$$u'(c) = (\omega^m + \lambda^g)P^m \tag{31}$$

The quantities of goods and assets traded in matches between two private agents still satisfy (11) and (12). The amount of purchases of new bonds satisfies (15) and the redemption fraction satisfies (13). Moreover, the optimal choice of n still obeys (16).[16]

There are two revisions to the household's optimal conditions. First, when choosing the division of money between the two markets, a household takes into account the service generated by money in trades with government sellers, $\alpha\sigma_g\lambda^g$. So, the condition for a changes from (14) to the following equation:

$$\alpha\left(\sigma\lambda^m + \sigma_g\lambda^g\right) = \rho \quad \text{if} \quad a \in (0,1) \tag{32}$$

Second, for the same reason, I revise the envelope condition for money as follows:

$$\omega^m_{-1}/\beta = \omega^m + (1-a)\rho + a\alpha\left(\sigma\lambda^m + \sigma_g\lambda^g\right) \tag{33}$$

With these revisions, I can adapt the previous definition of an equilibrium to the current economy. Then, the symmetry between households requires an additional condition:

$$P^m = p^m \equiv x^m/q^m \tag{34}$$

As before, let γ be the gross rate of growth of the private sector's money holdings. Again, I focus on equilibria with $0 < a < 1$, $n > 0$, $0 < r \leq 1$, and $\gamma > \beta$. Then, the nominal interest rate is

$$\frac{1}{S} = \frac{\gamma}{\beta} = 1 + \alpha\left(\sigma\frac{\lambda^m}{\omega^m} + \sigma_g\frac{\lambda^g}{\omega^m}\right) \tag{35}$$

The net nominal interest rate is positive for all $\gamma > \beta$. Also, the constraint (7) binds and implies $a = 1 - z\beta/\gamma$.

Notice that

$$P^m = x^m/q^m = \psi(q^m)/(\omega^m q^m) < \psi'(q^m)/\omega^m \tag{36}$$

where the inequality follows from the fact that $\psi(q)/q < \psi'(q)$ for all $q > 0$. Then, (12) and (31) imply $\lambda^g > \lambda^m \geq 0$. That is, the money constraint between a private buyer and a government seller always binds. This yields

$$q^g = \frac{am}{nP^m} = q^m\frac{am}{nx^m} \tag{37}$$

[16] To see this, note that a private moneyholder's surplus from a trade with a government seller, $u'(c)q^g - (\omega^m + \lambda^g)P^m q^g$, is 0 under (31). Thus, a private buyer gets a positive surplus from trade entirely from matches with private sellers, just as a bondholder does. So, (16) continues to hold. Clearly, this result depends on the assumption that a government seller sells goods at a fixed price P^m. In addition to simplifying the algebra, this assumption strengthens the later result that matured bonds do not circulate in the goods market. If a moneyholder obtained a positive surplus from a trade with a government seller, as well as from a private seller, there would be an additional advantage to holding money relative to bonds.

If the money constraint also binds in a trade between two private agents, then $q^g = q^m$.

As in previous sections, monetary transfers keep the growth rate of per household money holdings constant at γ. I also assume that the government's money holdings grow at γ, so that the ratio between private and government money holdings remains constant. This is achieved by money creation. Let T be the amount (per household) of money creation at the end of a period. (If $T < 0$, then T is the amount of money destroyed.) The government's budget constraint is

$$T = (M_{g_{+1}} - M_g) + \left[(r - S)zM + \alpha\left(\sigma M_g - \sigma_g n P^m q^g\right) + L\right] \tag{38}$$

The term in brackets is equal to the change in the representative household's money holdings between two adjacent periods (see (29)). This change is $(\gamma - 1)M$. Therefore, to keep M_g and M both growing at γ, money creation is $T = (\gamma - 1)(M + M_g)$.

4.2. *Matured Bonds Do Not Circulate in the Goods Market.* Matured bonds do not circulate in this economy, as stated below (see Part A of the Appendix for a proof).

PROPOSITION 2. *For $\gamma > \beta$, there is no stationary equilibrium where* $0 < r < 1$.

This proposition provides a robust answer to the question of why matured bonds do not circulate in the goods market. That is, any degree of the legal restriction (σ_g) will drive matured bonds out of the goods market. In fact, the proposition is independent of σ_g, provided $\sigma_g > 0$. Even when σ_g is arbitrarily small, matured bonds will not act as a medium of exchange.

To explain why Proposition 2 holds, let me examine three necessary requirements for matured bonds to act as a medium of exchange together with money. These requirements cannot all be satisfied in the current model. First, matured bonds must have the same shadow value as money, i.e., $\omega^m = \omega^b$. Otherwise, a household would not hold both money and matured bonds; that is, the choice $0 < r < 1$ would not be optimal.

Second, the two assets must yield the same expected nonpecuniary return in terms of generating liquidity. Otherwise, there would be a net gain from changing the redemption fraction, which is feasible when $0 < r < 1$. Thus, $\alpha\sigma\lambda^b/\omega = \alpha\sigma\lambda^m/\omega + \alpha\sigma_g\lambda^g/\omega$, where ω is the common value of money and matured bonds in this case. Since bonds generate liquidity in fewer trades than money does, bonds can satisfy the requirement on the return only if they generate higher liquidity in every trade than money, i.e., only if $\lambda^b > \lambda^m$. Notice that the asset constraint in a match binds more severely if the quantity of goods traded in that match is lower. Therefore, $\lambda^b > \lambda^m$ if and only if $q^b < q^m$.

Third, each household must find it optimal to choose a positive measure of buyers to trade matured bonds for goods. When choosing buyers to carry an asset, the household considers the buyer's expected surplus from trade, which is the surplus in each trade times the probability of the trade. A positive measure of bondholders is optimal only if the expected surplus to a bondholder is equal to or greater than that to a moneyholder. Since a bondholder has a lower probability to trade than a money holder, a bond holder must obtain a higher surplus in each trade. Notice that a buyer's surplus in a trade increases in the quantity of goods in that trade. Thus, the requirement becomes $q^b \geq q^m$.[17] This contradicts the previous requirement $q^b < q^m$. Therefore, matured bonds do not circulate as a medium of exchange.

The first two requirements above should hold in any reasonable environment, including a Walrasian goods market. By contrast, the third requirement is a distinctive feature of decentralized exchange. With decentralized exchange, the number of trades (i.e., the extensive margin of trade) and the quantity in each trade (i.e., the intensive margin of trade) are both important to a household. These two margins cannot both be chosen optimally when buyers holding the two assets encounter trading opportunities at different rates as a result of the legal restriction.

Proposition 2 may seem striking, especially when compared with previous monetary search models. A common result in those models is that different assets can coexist as media of exchange (e.g., Shi, 1995; Aiyagari et al., 1996). The above proposition shows that this result can fail in special circumstances. Matured nominal bonds are not just any other asset; rather, they are assets whose exchange rate with money is fixed at one by the redemption process. Thus, the general message of Proposition 2 is that when two assets can be costlessly exchanged at a fixed exchange rate of unity, the one with a less favorable chance of trade in the decentralized goods market will not circulate as a medium of exchange.

4.3. *The Legal Restriction Does Not Drive Bonds Out of a Walrasian Market.* The legal restriction in the goods market is necessary, but *not* sufficient, for the result that matured bonds do not circulate as a medium of exchange. Decentralized exchange is also important, as I explained above. To appreciate this point, let me argue that the legal restriction does not prevent bonds from circulating in a Walrasian goods market.

To make the argument more general, let a household's utility function be $u(c, g)$, instead of $u(c + g)$, where g is consumption of government goods. So, government goods may be indispensable to the household. Suppose that there are two centralized markets, one for private goods and the other for government goods. Let the price level of private goods be P^m and of government goods P^g, measured in terms of money. Then, the household faces the following trading constraints in the goods market:

$$P^m c + P^g g \leq m + b$$

$$P^g g \leq m \tag{39}$$

The legal restriction, (39), does not bind, as long as the expenditure on government goods does not exceed the household's entire money holdings. In this case,

[17] The inequality need not be strict because the surplus a moneyholder obtains in a trade with a government seller is 0 under the particular description of government agents' trading strategies, as discussed earlier.

the legal restriction merely induces households to shift money from purchases of private goods to government goods. A household uses matured bonds, together with money, to purchase goods from other private agents.[18] Thus, in contrast to the case of decentralized exchange, a small legal restriction in a Walrasian market does not drive matured bonds out of the circulation as a medium of exchange.

Of course, matured bonds will be driven out of a Walrasian market if the legal restriction is so pervasive that (39) binds. However, it is not very interesting to rely on pervasive legal restrictions to explain why matured bonds do not act as a medium of exchange.

In the above formulation, I have assumed implicitly that each household knows the locations of the markets for private goods and government goods. One may wonder whether matured bonds will cease to circulate in the Walrasian markets if each buyer is randomly assigned to the two markets, a feature that is present in decentralized exchange. The answer is no. Even if a buyer with matured bonds is assigned to the government goods market, the buyer can promise to pay the government seller with money and obtain such money by selling the bonds to those buyers who are assigned to the private goods market. This arrangement is feasible since there is record keeping in the Walrasian markets. To prevent matured bonds from being used as a medium of exchange, one needs to restrict agents' ability to arrange delayed payments. Random, bilateral matching is the simplest form of such restrictions.

4.4. *Robustness.* I now show that Proposition 2 is robust to the relaxation of several auxiliary assumptions. First, Proposition 2 will continue to hold after the government's trading rules in the goods market are modified, provided that these modifications do not favor buyers who use bonds to trade. In fact, I have deliberately reduced the disadvantage of bonds relative to money by assuming that government sellers sell goods at the average price. This assumption reduces a private money holder's surplus to 0 in a trade with a government seller. Any reasonable modification of the government seller's trading rule, such as bargaining, would deliver a positive surplus of trade to a private holder of money. This would increase the advantage of money over matured bonds in the goods market and strengthen the argument for Proposition 2.

Second, Proposition 2 does not depend on the assumption that bonds can be redeemed only at maturity. Even if agents can redeem matured bonds in any period at or after maturity, it is still true that an interior choice of r is optimal to a household only if $\omega^m = \omega^b$. Then, the same argument following Proposition 2 leads to a contradiction to $r \in (0, 1)$ and $n \in (0, 1 - \sigma)$.

Third, Proposition 2 does not depend on the assumption that a buyer cannot carry both money and bonds into each match. Even if a buyer can hold both assets into a match, the buyer will still be unable to use the bonds to purchase

[18] The above argument is robust to allowing for arbitrage between money and bonds, either indirectly through goods or directly. Let b' be the amount of matured bonds that the household holds after the arbitrage, with the constraint $b' \geq 0$. Then, (39) becomes $P^g g \leq m + (b - b')$. This constraint reduces back to (39) in any symmetric equilibrium, because $b' = b$. Also, the argument is robust to the arbitrage between private and government goods, which merely determines the relative price P^g/P^m.

goods from a government seller. To see this formally, suppose that each buyer carries $am/(1-\sigma)$ units of money and $b/(1-\sigma)$ units of matured bonds into the goods market. For the sake of argument, let me retain the assumption that bonds can only be redeemed at maturity. To show that Proposition 2 still holds, suppose to the contrary that there is a symmetric equilibrium with $r \in (0, 1)$. Then, $\omega^m = \omega^b$. Denote this common value as ω. I show that this leads to a contradiction.

To proceed, I need to revise the asset constraints in trade. Use superscripts mb to indicate a trade match between two private agents (where the buyer can use both m and b), and a superscript g to indicate a trade match between a private buyer and a government seller (where the buyer can only use money). The asset constraint facing the buyer in these two matches is, respectively,[19]

$$\text{(40)} \qquad \frac{am+b}{1-\sigma} \geq x^{mb} = \frac{\psi(q^{mb})}{\Omega}$$

$$\text{(41)} \qquad \frac{am}{1-\sigma} \geq x^{g} = \frac{\psi(q^{g})}{\Omega}$$

Let λ^{mb} be the Lagrangian multiplier of (40) and λ^g of (41). Similar to (12), the optimal choice of q^{mb} satisfies

$$u'(c) = (\omega + \lambda^{mb})\psi'(q^{mb})/\Omega$$

The quantity q^g still satisfies (31), once the average price is redefined as $P^{mb} = X^{mb}/Q^{mb}$. As before, $\lambda^g > \lambda^{mb}$ and so $\lambda^g > 0$. The envelope conditions for money and matured bonds require that the two assets yield the same service in the market. When $\omega^m = \omega^b$, as supposed, this requirement becomes $\alpha\sigma\lambda^{mb} = \alpha(\sigma\lambda^{mb} + \sigma_g\lambda^g)$, which contradicts $\lambda^g > 0$. Therefore, there is no symmetric equilibrium where matured bonds circulate in the goods market.

5. EQUILIBRIUM AND POLICY EFFECTS

Now I characterize the equilibrium and examine its properties. I have shown so far that the equilibrium has $r = 1$, $n = 1 - \sigma$, and $b = 0$.

5.1. *Characterization.* There are two cases of the equilibrium, depending on whether the money constraint in a trade between a private money holder and a private seller binds, i.e., whether (2) binds. If this constraint does not bind, then $\lambda^m = 0$; if this constraint binds, then $\lambda^m > 0$.

[19] There is no need to distinguish money and bonds in the payment in a trade between two private agents. These two assets will be identical to each other, both to the seller who accepts them and to the buyer if he chooses to retain them. The two assets are exchanged at par with each other at the beginning of the next period.

Case 1. $\lambda^m = 0$: In this case, $q^g \geq q^m$ (see (37)) and $\psi'(q^m) = u'(c)$ (see (12)). Denote q^m in this case as q_1, q^g as q_1^g, and c as c_1. Then,

$$c_1 = u'^{-1}(\psi'(q_1)), \quad q_1^g = \frac{1}{\sigma_g}\left[\frac{c_1}{\alpha(1-\sigma)} - \sigma q_1\right]$$

To solve for q_1, use (35), (31), and (36) to derive the following equation:

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma_g\left(\frac{q_1\psi'(q_1)}{\psi(q_1)} - 1\right) \tag{42}$$

Case 2. $\lambda^m > 0$: In this case, $q^g = q^m$ (see (37)) and $\psi'(q^m) < u'(c)$ (see (12)). Denote q^m in this case as q_2, q^g as q_2^g, and c as c_2. Then,

$$q_2^g = q_2, \quad c_2 = \alpha(1-\sigma)(\sigma + \sigma_g)q_2$$

To solve for q_2, use (35), (12), (31), and (36) to derive the following equation

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma\left(\frac{u'(c_2)}{\psi'(q_2)} - 1\right) + \alpha\sigma_g\left(\frac{u'(c_2)q_2}{\psi(q_2)} - 1\right) \tag{43}$$

To describe the existence of the equilibrium, define q_0, γ_0, and $\underline{\gamma}$ as follows:

$$\psi'(q_0) = u'(\alpha(1-\sigma)(\sigma + \sigma_g)q_0) \tag{44}$$

$$\frac{\gamma_0}{\beta} - 1 = \alpha\sigma_g\left(\frac{q_0\psi'(q_0)}{\psi(q_0)} - 1\right) \tag{45}$$

$$\underline{\gamma} = \beta[1 + \alpha\sigma_g(\Psi - 1)] \quad \text{and} \quad \Psi \equiv \lim_{q\to 0} q\psi'(q)/\psi(q) \tag{46}$$

In both cases of the equilibrium, $\lambda^m \geq 0$, which requires $\psi'(q) \leq u'(c)$. Thus, the equilibrium must satisfy $q \leq q_0$ in both cases. The following proposition establishes the existence of the equilibrium (see Part B of the Appendix for a proof).

PROPOSITION 3. *Assume that $M_g/N_g < m/(1-\sigma)$ and that $q\psi'(q)/\psi(q)$ is a nondecreasing function for all $q > 0$. A stationary equilibrium exists iff $\gamma \geq \underline{\gamma}$ and $0 < z < \frac{\gamma}{\beta}(1 - \frac{M_g/N_g}{m/(1-\sigma)})$. The equilibrium is unique if $\gamma > \underline{\gamma}$. Moreover, if $[q\psi'(q)/\psi(q)]$ is strictly increasing for all $q > 0$, then $\underline{\gamma} < \gamma_0$ and both cases can occur: Case 1 occurs for $\underline{\gamma} < \gamma \leq \gamma_0$ and Case 2 for $\gamma > \gamma_0$. If $[q\psi'(q)/\psi(q)]$ is a constant, then $\gamma_0 = \underline{\gamma}$ and only Case 2 can occur (for $\gamma > \underline{\gamma}$).*

The condition $M_g/N_g < m/(1-\sigma)$ and the upper bound on z are required to ensure that a private seller is willing to accept the entire holdings of money from a government buyer, an assumption that I have maintained implicitly. To understand the assumption that the function $[q\psi'(q)/\psi(q)]$ is nondecreasing in q,

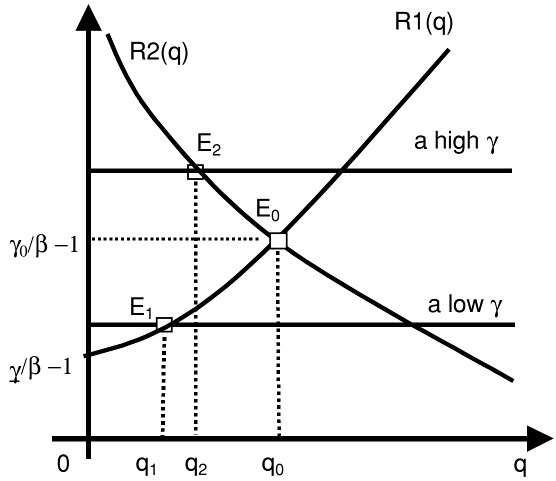

FIGURE 2

UNIQUE MONETARY EQUILIBRIUM

note that the real price of goods is $\omega^m P^m = \psi(q)/q$ (see (36)). Thus, the assumption requires that the marginal cost of production be at least as sensitive to changes in the quantity as the price function is. This assumption seems reasonable and it is satisfied by a wide class of functional forms, including the common specification $\psi(q) = q^{\Psi}(\psi_0 + \psi_1 q)$, where $\Psi > 1$, $\psi_0 > 0$, and $\psi_1 > 0$.

Figure 2 depicts the two cases for a strictly increasing function $[q\psi'(q)/\psi(q)]$. The two horizontal lines are $(\gamma/\beta - 1)$ at two levels of γ. The curve $R1(q)$ depicts the right-hand side of (42), which is increasing when $[q\psi'(q)/\psi(q)]$ is increasing. The curve $R2(q)$ depicts the right-hand side of (43), which is always decreasing. The two curves intersect at point E_0, where $\gamma = \gamma_0$ and $q = q_0$. As stated earlier, q must satisfy $0 < q \leq q_0$ in both cases. When $\gamma < \underline{\gamma}$, no such solution for q exists. When $\underline{\gamma} < \gamma \leq \gamma_0$, the solution is depicted by a point like E_1, which is Case 1. When $\gamma > \gamma_0$, the solution is depicted by a point like E_2, which is Case 2.[20]

In both cases of the equilibrium, $b = 0$ (since $r = 1$). The fraction of money taken to the goods market is $a = 1 - z\beta/\gamma$. The price level and the value of money are, respectively,

$$P^m = \frac{am}{(1-\sigma)q^g}, \quad \omega^m = \frac{(1-\sigma)\psi(q^m)}{am} \tag{47}$$

An interesting feature of the equilibrium is that $\lambda^m > 0$ if and only if $\gamma > \gamma_0$, although $\lambda^g > 0$ holds for all $\gamma > \beta$. That is, the money constraint does not always bind when a buyer buys goods from a private seller, whereas it always binds when

[20] If $[q\psi'(q)/\psi(q)]$ is constant for all $q > 0$, then the curve $R1(q)$ becomes a horizontal line going through E_0, and so an equilibrium exists only for $\gamma = \underline{\gamma}$.

he buys from a government seller. To explain why $\lambda^m = 0$ for $\gamma < \gamma_0$, recall that a buyer faces the supply schedule $x = \psi(q)/\omega^m$ in a trade with a private seller. In making the offer (x, q), the buyer takes into account the increasing marginal cost of production. By contrast, a government seller sells goods at the given price P^m and hence supplies goods with a linear schedule $x = P^m q$. Thus, it is possible that the government seller sells a larger quantity of goods than a private seller. Because the buyer has the same amount of money in these two types of trades, he is more likely to face a binding money constraint when trading with a government seller than with a private seller. However, this outcome occurs only when the money growth rate is low. When the money growth rate is high, the purchasing power of money is low, and hence the purchase of goods exhausts the buyer's money holdings in both types of trades.

5.2. *Effects of Monetary Policy.* I examine two types of monetary policy—an increase in the money growth rate and an increase in the amount of new bond sales. As stated before, the government's budget constraint is always satisfied and the ratio M_g/M is constant. To simplify the exposition here, let me assume that $[q\psi'(q)/\psi(q)]$ is a strictly increasing function, so that both Cases 1 and 2 of the equilibrium can occur in different intervals of the money growth rate. Before examining the effects of the policy, it is useful to distinguish private and public consumption/output, as listed below:

	Consumption	Output
Private	$c = \alpha(1-\sigma)\,(\sigma q^m + \sigma_g q^g)$	$y = \alpha\sigma(1-\sigma)(q^m + q^g \frac{M_g}{am})$
Government	$c_g = \alpha\sigma(1-\sigma) q^g \frac{M_g}{am}$	$y_g = \alpha(1-\sigma)\,\sigma_g q^g$
Aggregate	$c + c_g = y + y_g = \alpha(1-\sigma)(\sigma q^m + \sigma_g q^g + \sigma q^g \frac{M_g}{am})$	

Examine first the effects of an increase in the money growth rate γ. An increase in the money growth rate affects q^m in Cases 1 and 2 in opposite directions. This can be seen from Figure 2, where the solution for q^m is q_1 for $\gamma < \gamma_0$ (Case 1) and q_2 for $\gamma > \gamma_0$ (Case 2). When $\gamma < \gamma_0$, a buyer's money constraint binds in a trade with a government seller and does not bind in a trade with a private seller. In this sense, the cost of a trade for a buyer is higher with the government seller than with a private seller. An increase in money growth reduces the overall purchasing power of money and tends to reduce the total amount of consumption. To mitigate this fall in consumption, the household increases the amount of purchases in the trades that are less costly—the trades with private sellers. Thus, q^m increases with γ when $\gamma < \gamma_0$. When $\gamma > \gamma_0$, however, the money constraint binds in all trades. In this case, an increase in γ reduces q^m. The quantity q^m reaches the maximum at $\gamma = \gamma_0$.

Despite the above nonmonotonicity, an increase in the money growth rate always reduces the quantity of goods traded between a private buyer and a government seller, q^g, and hence reduces government output. This is because money growth reduces the real money balance and tightens the binding money constraint in such a trade. Moreover, the reduction in q^g dominates the change in private consumption. That is, even if a higher money growth rate increases the quantity of goods traded between private agents, q^m, it always reduces private consumption.

Higher money growth also reduces government consumption through a price effect. To understand this effect, notice that an increase in money growth increases a, the fraction of money that a household takes to the goods market. This reallocation of money drives up the price level relative to the money stock, i.e., drives up P^m/M. As a result, a government buyer can buy less with his money holdings, which reduces government consumption.

Because private consumption and government consumption both fall with a higher money growth rate, both consumption and output fall at the aggregate level.

By contrast, private output may either rise or fall. Private output consists of two parts. The first part is goods sold to private agents and the second part is goods sold to government agents. Because the first part responds to money growth nonmonotonically and the second part always decreases with money growth, the overall response of private output to money growth depends on both the level of money growth and the ratio between the two parts of private output. If the ratio of government money holdings to private holdings is low, i.e., if M_g/m is significantly smaller than $a\sigma_g/\sigma$, then most of private output is sold to private agents. In this case, private output responds to money growth in a hump-shaped pattern, although the hump does not necessarily occur at γ_0. If $M_g/m \geq a\sigma_g/\sigma$, private output decreases with money growth for all $\gamma > \underline{\gamma}$.

I now examine a tightening open-market operation, modeled as an increase in z with corresponding changes in transfers to maintain the money growth rate at γ. A tightening operation shifts money from the goods market to the bonds market, i.e., it reduces a. Given the money growth rate, however, this shift has no effect on (q^m, q^g). So, private consumption and government output remain the same as before. However, the price level falls because there is less money in the goods market. The lower price enables the government to purchase more goods from the private sector, leading to higher private output and aggregate output.

These real effects were absent in the previous model where the government does not participate in the goods market. They are also absent in models where the goods market is Walrasian, e.g., the certainty counterpart of Lucas (1990). There, a fully anticipated increase in open-market operations changes only the price level and other nominal variables, provided that monetary transfers keep the money growth rate constant.

Noticing that government consumption increases to respond to the tightening operation in this model, one may argue that a standard cash-in-advance model can also generate real effects of the operation if government spending is assumed to increase with the tightening open-market operation. The question, of course, is why government spending necessarily increases to accompany a tightening operation. The current model provides a mechanism that is consistent with the model's assumption that the government uses only money to purchase goods. Moreover, government spending in a standard model does not generate the same effects as in the current model. For example, government spending is likely to crowd out private consumption in a standard model, which does not occur in the current model.

6. CONCLUSION

In this article, I construct two search models to analyze the competition between fiat money and default-free nominal bonds. In both models, newly issued bonds are sold at a discount for money and thus they bear positive interest. However, the two models have different predictions on whether nominal bonds circulate as a medium of exchange. In the first model, the government does not participate in the goods market. In this case, there are a continuum of equilibria in which matured bonds circulate in the goods market as perfect substitutes for money. In the second model, the government participates in the goods market and refuses to accept bonds as payments, although private agents can choose to trade among themselves using both money and bonds. In this model, there is a unique equilibrium. Moreover, matured bonds do not circulate, even when the measure of government sellers in the goods market is arbitrarily small. This result provides a robust answer to the question why matured bonds do not circulate as money.

Matured bonds do not circulate in the second model because of the frictions in the goods market. With decentralized exchange, a buyer in a match cannot exchange bonds for money instantaneously, nor switch costlessly from a match with a government seller to a match with a private seller. Since a buyer holding bonds meets a government seller with positive probability who refuses to accept bonds, such a buyer has a smaller chance to trade than a buyer holding money. This wedge induces the households to redeem all matured bonds and use only money to buy goods, no matter how small the measure of government sellers in the goods market is. By contrast, the same legal restriction can drive bonds out of the circulation in a Walrasian goods market only if the legal restriction is enforced in a sufficiently large fraction of trades.

The two models in this article yield different policy effects. Take, for example, a permanent increase in issuing bonds in the open market, accompanied by a change in lump-sum monetary transfers to maintain the growth rate of the money stock constant. In both models, the open-market operation shifts money from the goods market to the bonds market. In the first model, this shift of money between the two markets has no effect on real output/consumption or the nominal interest rate. If households do not change the redemption fraction of matured bonds, the tightening operation increases the bond–money ratio and *increases* the total amount of media of exchange in the goods market, thus increasing the price level. If households change the redemption fraction to respond to the open-market operation, the response of the price level is difficult to predict. In the second model, however, the shift of money between the two markets reduces the price level. Also in contrast to the first model, real output and government consumption increase with the tightening operation, through a price effect, while private consumption and government output remain unchanged.

Money growth also has different effects in the two models. When bonds circulate as money, higher money growth reduces private output. When bonds do not circulate, higher money growth increases private output if the money growth rate is initially low, although it always reduces the sum of private and government output.

The results here cast doubts on the policy effects obtained in traditional models that assume money as the unique medium of exchange. When there is no restriction on bonds as payments for goods, as in the first model here, bonds will circulate as money and a continuum of equilibria will arise. The continuum presents a clear difficulty for predicting the effects of monetary policy. To avoid the continuum of equilibria, one can introduce the legal restriction in the goods market, as is done in the second model in this article. Although the legal restriction does induce the outcome that bonds do not circulate as money, an outcome that is exogenously imposed in conventional monetary models, the explicit model reveals some policy effects that differ from those in conventional monetary model. For example, an increase in bond issuing leads to higher private output through a price effect.

As stated in the introduction, one purpose of this article is to integrate the microfoundation of monetary theory with the fruitful analysis of Lucas (1990). The resulting framework combines a decentralized goods market with a centralized bonds market. It is easy to see that this framework can be used to examine any centrally traded asset, not just nominal bonds. However, there is still quite a distance to go toward a full integration of the two classes of models. In another paper (Shi, 2003a), I have incorporated bonds of longer maturity to allow unmatured bonds to circulate in the goods market as a medium of exchange. I have also introduced uncertainty in open-market operations (Shi, 2003b) and found that open-market operations generate a persistent liquidity effect on nominal interest rates and output.

APPENDIX

A. *Proof of Proposition 2.* For $0 < r < 1$ to be consistent with an equilibrium, matured bonds must be perfect substitutes for money in the private goods market, i.e., $\omega^b = \omega^m$ (see (13)). The constraint (4) must bind, i.e., $\lambda^b > 0$; otherwise (18) would require $\omega^m_{-1}/(\beta\omega^m) = 1$, which would in turn require $\gamma = \beta$. For $\lambda^b > 0$, it must be true that $n < 1 - \sigma$, because $b/(1 - \sigma - n) = \infty$ otherwise. The choice $n \in (0, 1 - \sigma)$ is optimal iff $q^m = q^b$ (see (16)). Denote this common value of q^m and q^b as q. Substituting λ^b from (12) into (18) and using $\omega^m_{-1}/\omega^m = \gamma$, I have

$$\text{(A.1)} \qquad \frac{\gamma}{\beta} - 1 = \alpha\sigma\left(\frac{u'(c)}{\psi'(q)} - 1\right)$$

Clearly, $\psi'(q) < u'(c)$ for all $\gamma > \beta$. So, (12) implies $\lambda^m > 0$, which in turn implies $q^g = q$ by (37). Substituting λ^m from (12) and λ^g from (31) into (35), I get

$$\text{(A.2)} \qquad \frac{\gamma}{\beta} - 1 = \alpha\sigma\left(\frac{u'(c)}{\psi'(q)} - 1\right) + \alpha\sigma_g\left(\frac{u'(c)q}{\psi(q)} - 1\right)$$

Because $\psi(q)/q < \psi'(q) < u'(c)$, (A.1) and (A.2) cannot both hold. Therefore, there does not exist a stationary equilibrium with $0 < r < 1$. ■

B. *Proof of Proposition 3.* Let me first validate the maintained assumption that a private seller in a match with a government buyer is always willing to produce enough goods to exchange for all the money the government buyer holds. This behavior is indeed optimal for the private seller iff $\omega^m M_g/N_g \geq \psi(M_g/(N_g P^m))$. Substituting $P^m = \psi(q^m)/(\omega^m q^m)$, I write the required condition as $k \geq \psi(kq^m/\psi(q^m))$, where $k = \omega^m M_g/N_g$. For any given $A > 0$, let $k^*(A)$ be the positive solution to the equation $k = \psi(kA)$. It is easy to verify that this solution exists and is unique. Moreover, $k \geq \psi(kA)$ iff $k \leq k^*(A)$. Notice that, when $A = q^m/\psi(q^m)$, $k = \psi(q^m)$ satisfies the equation $k = \psi(kq^m/\psi(q^m))$. Since the solution $k^*(A)$ is unique for given A, then $k^*(q^m/\psi(q^m)) = \psi(q^m)$. Therefore, it is optimal for a private seller to produce enough goods for all the money that a government buyer has if and only if $\omega^m M_g/N_g \leq \psi(q^m)$. Substituting the equilibrium results $\omega^m = (1-\sigma)\psi(q^m)/(am)$ and $a = 1 - z\beta/\gamma$, I can rewrite this condition as

$$z \leq \frac{\gamma}{\beta}\left(1 - \frac{M_g/N_g}{m/(1-\sigma)}\right)$$

This upper bound on z is imposed in Proposition 3, together with the condition that ensures its positivity, $M_g/N_g < m/(1-\sigma)$. Note that $a = 1 - z\beta/\gamma > 0$.

Next, I show that the solutions to (42) and (43) exist under the conditions described in the proposition. Notice that a unique solution to (43), denoted q_2, exists for all $\gamma > \beta$. By contrast, a unique (positive) solution to (42), denoted q_1, exists only when $\gamma \geq \underline{\gamma}$ and when $q\psi'(q)/\psi(q)$ is strictly increasing for all $q > 0$. If $q\psi'(q)/\psi(q)$ is constant over q, then (42) is satisfied only at $\gamma = \underline{\gamma}$, in which case $\gamma_0 = \underline{\gamma}$ and $q = q_0$. To determine which of the two cases characterizes the equilibrium, recall that the equilibrium requires $0 < q \leq q_0$. Suppose first that $[q\psi'(q)/\psi(q)]$ is strictly increasing for all $q > 0$. For $\gamma \leq \gamma_0$, the solutions to (42) and (43) satisfy $q_1 \leq q_0 < q_2$, and so Case 1 characterizes the equilibrium. For $\gamma > \gamma_0$, the solutions to (42) and (43) satisfy $q_2 < q_0 < q_1$, and so Case 2 characterizes the equilibrium. Suppose now that $[q\psi'(q)/\psi(q)]$ is constant over $q > 0$. Then, at $\gamma = \underline{\gamma}$, the equilibrium is Case 1 with $q = q_0$. For $\gamma > \underline{\gamma}$, the equilibrium is given by Case 2.

To show that these solutions constitute a unique equilibrium, I need to verify that an individual household does not have incentive to deviate to $r < 1$ when all other households choose $r = 1, n = 1 - \sigma$, and trade according to the described quantities. Consider the following deviation by an individual household. The household keeps an amount, ε, of bonds that matured in the *previous* period and uses them to trade for goods in the current period. All other households continue to use the equilibrium strategies and so, if they receive any bond in the trade, they will value it at $\Omega^b < \Omega^m$.

In order to trade bonds for goods, the deviating household must allocate some buyers to carry the bonds to the goods market. Let Δ be the measure of the household's buyers who carry the bonds into the goods market, each carrying an amount ε/Δ. Each of these bondholders has a trade match with a private seller with probability $\alpha\sigma$.

The deviation yields a benefit from potential trades with private sellers. When meeting a private seller in a trade match, a bondholder can offer the amount of bonds for q_ε units of goods. Because the seller will accept the trade as long as $\Omega^b\varepsilon/\Delta \geq \psi(q_\varepsilon)$, the bondholder will offer q_ε such that $\Omega^b\varepsilon/\Delta = \psi(q_\varepsilon)$, i.e., $q_\varepsilon = \psi^{-1}(\Omega^b\varepsilon/\Delta)$. Because each bondholder gets such a trade with probability $\alpha\sigma$, the total number of such trades is $\alpha\sigma\Delta$ and the total utility generated from such trades is $\alpha\sigma\Delta\, u'(c)q_\varepsilon$. The amount of bonds that the household fails to trade away is $(1 - \alpha\sigma)\Delta(\varepsilon/\Delta) = (1 - \alpha\sigma)\varepsilon$. Thus, the total benefit of the deviation is

$$\alpha\sigma\,\Delta u'(c)q_\varepsilon + (1-\alpha\sigma)\varepsilon\omega^b$$

The deviation has two costs. The first cost is the value of money that the household would have if it did not deviate in the previous period. In that case, the amount ε of bonds would be redeemed for ε units of money, which would have a value $\varepsilon\omega^m_{-1}/\beta$. The second cost is that allocating Δ buyers to hold bonds takes them away from trading with money. Because each trade by a moneyholder generates a surplus $u'(c)\ [q^m - \psi(q^m)/\psi'(q^m)]$ (recall that the surplus from trading with a government seller is 0), the second cost of the deviation is $\alpha\sigma\,\Delta$ times this amount. Therefore, the total cost of the deviation is

$$\varepsilon\frac{\omega^m_{-1}}{\beta} + \alpha\sigma\,\Delta u'(c)\left(q^m - \frac{\psi(q^m)}{\psi'(q^m)}\right)$$

The net gain from the deviation, divided by Δ, is less than the following amount:

$$\alpha\sigma u'(c)q_\varepsilon - \left(\frac{\gamma}{\beta} - 1 + \alpha\sigma\right)\psi(q_\varepsilon) - \alpha\sigma u'(c)\left(q^m - \frac{\psi(q^m)}{\psi'(q^m)}\right)$$

where I have substituted $\omega^b\varepsilon/\Delta = \psi(q_\varepsilon)$, $\omega^b < \omega^m$, and $\omega^m_{-1}/\omega^m = \gamma$. Denote the above expression temporarily as $f(q_\varepsilon)$. For the deviation to be not profitable, it is sufficient that $f(q_\varepsilon) \leq 0$ for all $q_\varepsilon \geq 0$. Since f is concave, it is maximized at q^*_ε, which solves

$$\frac{\gamma}{\beta} - 1 = \alpha\sigma\left(\frac{u'(c)}{\psi'(q^*_\varepsilon)} - 1\right)$$

Using this definition to substitute $(\gamma/\beta - 1)$ in $f(q^*_\varepsilon)$, I can show that $f(q^*_\varepsilon) \leq 0$ iff $q^*_\varepsilon - \psi(q^*_\varepsilon)/\psi'(q^*_\varepsilon) \leq q^m - \psi(q^m)/\psi'(q^m)$. Because $[q - \psi(q)/\psi'(q)]$ is an increasing function of q, $f(q^*_\varepsilon) \leq 0$ iff $q^*_\varepsilon \leq q^m$. In turn, $q^*_\varepsilon \leq q^m$ iff

$$\frac{\gamma}{\beta} - 1 \geq \alpha\sigma\left(\frac{u'(c)}{\psi'(q^m)} - 1\right)$$

This is satisfied in both Cases 1 and 2 (see (42) and (43)). Therefore, the deviation is not profitable. ■

C. *A Different Assumption on Monetary Transfers.* In the text I have assumed that, when there is an open-market operation, the government adjusts monetary transfers to keep the money growth rate unchanged. This helped me in isolating the effects of the open-market operations from the effects of a change in money growth. Still, one may want to know the features of equilibria when the transfers do not neutralize the effect of the operation on money growth. I examine this alternative transfer process in this part of the Appendix. To economize on space, I analyze only the economy where the government does not participate in the goods market (i.e., the one examined in Section 3).

Let monetary transfer be a constant fraction, l, of the money stock. The price of newly issued bonds is still $S = \beta/\gamma$. Equations (21)–(23) continue to hold. The gross rate of money growth, still defined as $\gamma = m_{+1}/m$, is now endogenous. With $L/m = l$, (24) implies

$$\gamma - 1 - l = z\left(r - \frac{\beta}{\gamma}\right) \tag{A.3}$$

This equation solves for γ for given (l, z, r). In the following proposition, I detail the existence of the solution and the implied properties of the equilibria. A discussion on the proposition will follow the proof.

PROPOSITION C.1. *Assume $z > 0$ and $l > -1$. The condition $l > (\beta - r)z$ is necessary and sufficient for there to be a solution for γ that satisfies $\gamma > 1$ and induces $a > 0$. Under this condition, the solution for γ is unique for given (l, z, r). There are a continuum of equilibria that differ from each other in the redemption fraction r. Between equilibria, the following properties hold*

$$\frac{d\gamma}{dr} > 0, \quad \frac{dS}{dr} < 0, \quad \frac{da}{dr} > 0, \quad \frac{dn}{dr} > 0, \quad \frac{dq}{dr} < 0, \quad \frac{d(b/m)}{dr} < 0$$

In each equilibrium, i.e., for each given r, an increase in z reduces the money growth rate γ if r is not too large. In this case,

$$\frac{dS}{dz} > 0, \quad \frac{da}{dz} < 0, \quad \frac{dn}{dz} < 0, \quad \frac{dq}{dz} > 0, \quad \frac{d(b/m)}{dz} > 0$$

Also, if either u is sufficiently concave or ψ sufficiently convex, then p/m is lower in an equilibrium with a higher r than with a lower r, and an increase in z increases p/m.

PROOF. Rewrite (A.3) as follows:

$$\gamma^2 - (1 + l + rz)\gamma + \beta z = 0 \tag{A.4}$$

Temporarily denote the left-hand side as $f(\gamma)$. Recall that $a > 0$ iff $\gamma > \beta z$. Also, the bond–money ratio is interior iff $\gamma > 1$. So, I look for the solution that satisfies $\gamma > \gamma_A \equiv \max\{1, \beta z\}$.

Notice that $f(1) = (\beta - r)z - l$ and $f(\beta z) = \beta z[(\beta - r)z - l]$. So, $f(\gamma_A) < 0$ iff $l > (\beta - r)z$. Also, by computing $f'(1)$ and $f'(\beta z)$, I have $f'(\gamma_A) = 2\gamma_A - 1 - l - rz$. Consider the case $l \leq (\beta - r)z$ first. In this case, $f(\gamma_A) \geq 0$. Also,

$$f'(\gamma_A) = 2\gamma_A - 1 - l - rz \geq 2\gamma_A - 1 - \beta z \geq 0$$

This implies that, for all $\gamma > \gamma_A$, $f(\gamma) > f(\gamma_A) \geq 0$. There is no solution to $f(\gamma) = 0$ that satisfies $\gamma > \gamma_A$. Thus, a necessary condition for the desired solution to exist is $l > (\beta - r)z$.

The condition $l > (\beta - r)z$ is also sufficient for a unique solution $\gamma > \gamma_A$ to exist. To see this, note that $f(\gamma_A) < 0$ when $l > (\beta - r)z$. Since $f(\gamma)$ is a convex, quadratic expression, its two roots lie on the opposite of γ_A. The unique solution that satisfies $\gamma > \gamma_A$ is

$$\gamma = \frac{1}{2}\{1 + l + rz + [(1 + l + rz)^2 - 4\beta z]^{1/2}\}$$

It is easy to verify that this solution satisfies $d\gamma/dl > 0$ and $d\gamma/dr > 0$. Under this solution for γ, there are a continuum of values of r each leading to an equilibrium.

Because $d\gamma/dr > 0$, (20) implies $dS/dr < 0$; (23) implies $d(b/m)/dr < 0$; (25) implies $dn/dr > 0$; (22) implies $da/dr > 0$; and (22) implies $dq/dr < 0$. After substituting a from (22), n from (25), and $x^b = am/n$, the price level of goods normalized by the money stock is

$$\frac{p}{m} = \frac{1}{(1 - \sigma)q(\gamma - 1)}\left[\gamma - 1 + (1 - r - \beta)z + \frac{\beta z}{\gamma}\right]$$

From (A.4), I get $\beta z/\gamma = 1 + l + rz - \gamma$. Thus,

$$\gamma - 1 + (1 - r - \beta)z + \frac{\beta z}{\gamma} = l + (1 - \beta)z$$

Therefore,

$$\frac{p}{m} = \frac{l + (1-\beta)z}{(1-\sigma)(\gamma-1)q}$$

The normalized price level falls with r iff $(\gamma - 1)q$ increases with r. This requires that either the cost function ψ be sufficiently convex or the utility function u be sufficiently concave. More precisely, computing $dq/d\gamma$ from (21), I have

$$\frac{d(p/m)}{dr} < 0 \Longleftrightarrow \frac{q\psi''}{\psi'} + \frac{(-u'')c}{u'} > \frac{\gamma - 1}{\gamma - \beta + \alpha\sigma\beta}$$

The effects of z can be analyzed similarly. From the above solution for γ, I have $d\gamma/dz < 0$ iff $\gamma < \beta/r$. Because γ is an increasing function of r, then $\gamma < \beta/r$ is satisfied for at least small r (and possibly for all $r \leq 1$.) Thus, for small r, $d\gamma/dz < 0$. In this case, (20) implies $dS/dz > 0$; (23) implies $d(b/m)/dz > 0$; (25) implies $dn/dz < 0$; (22) implies $da/dz < 0$; and (21) implies $dq/dz > 0$. The normalized price level increases with z if either the cost function ψ or the utility function u has sufficient curvature. ■

As in Section 3, there are still a continuum of equilibria distinguished from each other in the redemption fraction. Also similar to Section 3, different equilibria have the following differences. First, a higher redemption fraction reduces the ratio of matured bonds to money in the economy. That is, the total moneyness in the goods market, normalized by the money stock, is lower when households redeem a larger fraction of matured bonds. Thus, a higher redemption fraction exerts a downward pressure on the price level normalized by the money stock. Second, a higher redemption fraction induces each household to allocate more buyers to carry money in order to absorb the increased money stock relative to bonds.

In contrast to Section 3, a higher redemption fraction increases the money growth rate here, and hence generates the following additional effects. First, the bond price is lower and the nominal interest rate higher. Second, the fraction of money taken to the goods market is higher. Third, the quantity of goods exchanged in a trade is lower because higher money growth reduces the value of the real money balance. The last effect provides an upward pressure on the price level, but it is weak when the utility function is sufficiently concave or the cost function sufficiently convex. In this case, the dominating effect on the price level is the reduction in the total moneyness. That is, a higher redemption fraction reduces the price level normalized by the money stock, as in Section 3.

A tightening open-market operation has effects similar to those in Section 3. In particular, the bond–money ratio increases, the total moneyness in the goods market increases, and the fraction of buyers (bondholders or moneyholders) falls. However, because the operation reduces the money growth rate, the nominal interest rate falls and the quantity of goods in a trade increases. Again, the change in the quantity of goods affects the price level only slightly, and so the price level normalized by the money stock increases with the operation.

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