Frictional Assignment. I. Efficiency

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This paper examines the two-sided matching problem where the agents on each side of the market are heterogeneous and the matching process is time-consuming. This is cast in a labor market setting where workers of different skills match with different machine qualities. I characterize the efficient allocation and then show that it can be decentralized by a market mechanism. The efficient assignment is not always positively assortative, despite the fact that machine qualities and skills are complementary in production. To decentralize the efficient allocation, the market mechanism requires the firms to post wages and commit each machine quality to a particular skill. Implications on wage inequality are briefly examined.

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1. INTRODUCTION

This paper analyzes the frictional assignment, which refers to the following two-sided matching problem. Two sides of a market must be matched with each other in order to produce "output" and the matching process is time-consuming. Agents on each side differ in characteristics and the level of output depends on the pair’s characteristics. The market can be the labor market, where workers of different skills need be matched with machines of different qualities, the loan market, where projects of different qualities need be financed by different loan provisions, or the marriage market, where men and women look for marriages. I characterize the efficient allocation in this frictional world, show that it can be decentralized by a competitive mechanism, and then explore the features of the frictional

1 This is the first part of a previously circulated paper entitled "Frictional assignment" [22]. I thank Ken Burdett, George Neumann, and a referee for comments. I have also benefited from comments by workshop and conference participants at Toronto, Tilburg, Laval, Illinois at Urbana-Champaign, Canadian Economic Theory meeting (Toronto, 1998), Canadian Macro Study Group meeting (London, 1999), and American Economic Association meeting (Boston, 2000). Financial support from the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged. All remaining errors are mine alone.

assignment. To be specific, I focus on the matching problem between machine qualities and worker skills.

Matching frictions are an important cause of persistent unemployment and under-utilized machines, but they have been ignored in the assignment literature until recently. The classic pieces by [3, 26] deal exclusively with frictionless matching environments, i.e., economies where a match can be formed instantaneously if it is advantageous for the two sides to do so.\(^\text{2}\)

A central result there, that the market yields efficient outcomes, fails in general when the matching process is time-consuming (see [9, 16, 18]). This failure makes one wonder whether there are market mechanisms to capture the unrealized gains in the frictional environment.

Besides efficiency, two other issues in the assignment literature need be re-examined in a frictional world. One is whether the assignment is positively assortative (or positive for short), i.e., whether high-quality machines are allocated to high skills. The other is how wage inequality between skills is affected by the factors that are matched with them. In a frictionless world, the equilibrium assignment is always positive, provided that the two sides are complementary in the production function (see [3]). The positive assignment implies that the wage differential between a high-skill worker and a low-skill worker is attributed to the difference in machine qualities assigned to them as well as to the skill difference itself. With matching frictions it is far from obvious whether the assignment is positive and so, to understand wage inequality, one must know what the frictional assignment looks like.

To address these issues I examine a large market where workers differ in skills and skills are complementary with machine qualities in production. To emphasize the difference between a frictional assignment and a frictionless one, skills are assumed to be perfectly observable so as to make it straightforward to characterize the frictionless assignment. The matching friction is modelled by the assumption that each worker can match with at most one firm in a period.

The main departure of the frictional assignment from a frictionless one is that it is not always positive, even though machine qualities and worker skills are complementary in production. This is because a frictional assignment must assign a “right” number of firms or market tightness as well as a “right” machine quality to each skill in order to ensure efficiency. When skills and machine qualities are not very complementary in production, assigning high skills and high-quality machines to match with each other results in under-utilization of at least one of them, without much gain in output from the complementarity. A non-positive assignment can increase expected net output in this case by increasing the utilization rate of both

high skills and high quality machines: Each high-skill worker is assigned to many low-quality machines and each high-quality machine is assigned to many low-skill workers. Only when skills and machine qualities are sufficiently complementary in production is a positive assignment efficient, in which case the increase in output from the complementarity between high skills and high quality machines outweighs the cost of under-utilization created by a positive assignment.

There is a market mechanism that decentralizes the efficient assignment, as follows. With free entry firms choose machine qualities and post wages to match with workers. They can commit to the posted wages and the skills that their chosen machines are intended for. This mechanism achieves efficiency by (i) properly allocating the decision rights on surplus division to agents who actively organize the market, (ii) allowing for full-fledged competition among these agents, and (iii) featuring commitments by these agents to the surplus division method and to the particular types of agents whom they choose to match with. For (i), in particular, allocating the decisions rights to firms generates endogenous matching functions and wages that internalize the externalities in the matching process. In contrast, improper allocation of the decisions rights, as characterized by an exogenous matching function and an exogenous division of the match surplus, contributes to the inefficiency in the standard random matching models [9, 16, 18].

1.1. Comparisons with the Literature

This paper builds on price/wage posting models by Peters [17] and Montgomery [15], who analyze an environment where agents on one side of the market post prices/wages to attract matches. This framework has been employed in [1, 2, 5, 7, 14, 24]. Burdett et al. [5] and Cao and Shi [7] examine the equilibrium in a finite economy and its convergence. Shi and Wen [24] examine the effects of taxes and subsidies with wage-posting and other wage determination schemes. Moen [14] shows that the wage-posting framework achieves efficiency when each side of the labor market is homogeneous. Maintaining a homogeneous labor force, Acemoglu and Shimer [1] allow firms to choose the level of investment and show that the wage-posting framework solves the so-called hold-up problem in investment. Extending this framework to an economy with risk-averse workers, they [2] show that an unemployment insurance scheme must be used to accompany the wage-posting framework in order to achieve efficiency.

Another strand of the search literature, surveyed by [13], assumes that agents only know the distribution of wages before search and must incur the search cost to find any particular wage. The market assignment in that environment is also inefficient because competition between firms is limited.
My model is in common with these models in the use of the wage-posting setup. The main difference is that in my model agents are heterogeneous on both sides of the market, as opposed to heterogeneity on at most one side in the above models. With heterogeneity on both sides of the market, the assignment problem is much more interesting and I can address the important issue of how agents with different attributes are matched in equilibrium and in the efficient allocation. For example, one can examine how differences in skill prices can be amplified by differences in machine qualities matched to these skills. Moreover, with heterogeneity on both sides of the market, efficiency is more difficult to be achieved than if workers are homogeneous. As will be argued in Subsection 4.3, to achieve efficiency firms must be able to commit to the particular skill that a machine quality is chosen for as well as to the posted wages.

There are several other papers that have analyzed the frictional assignment with heterogeneous agents on both sides, e.g., [4, 21, 25]. Burdett and Coles [4] have shown that the assignment is positively assortative in the sense that agents on each side of the market voluntarily segregate themselves into subgroups according to their attributes and higher subgroups on the two sides are matched with each other. In contrast to a frictionless environment, each subgroup contains heterogeneous agents. This result of a positive assignment, as it turns out, depends sensitively on the particular payoff structure used in the Burdett–Coles paper, namely, an agent’s utility is equal to the partner’s attribute. With a more general “production” function and Nash bargaining, Shimer and Smith [25] have shown that the assignment is not necessarily positive.

These models focus on equilibrium assignments. The main characteristics of these models are exogenous matching functions and exogenous surplus sharing rules (Nash bargaining). Like the standard random matching models mentioned earlier, these exogenous elements imply that the market does not internalize the externalities in the matching process and so the equilibrium is inefficient. In contrast, both the matching function and the surplus sharing rule are endogenous in my model, generated by the wage-posting framework. A main contribution of my paper to the assignment literature is then to show that non-positive assignments can sometimes be efficient.

My paper also reveals a different reason why the frictional assignment can be non-positive. In the Shimer–Smith paper, one cause of the non-positive assignment is the exogenous matching function. When agents are exogenously matched, an agent cannot afford to limit one’s acceptance set.

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4 There is also a fair amount of work that analyzes centralized matching with frictions and/or decentralized matching without prices (see [19]). In contrast, the focus of this paper is on how prices/wages can induce efficiency in a decentralized matching framework.
to only one particular type of partners. If he or she did so, the probability with which he or she will be paired with a desired partner by the exogenous matching function would be low and the agent's expected life-time utility would be low; the agent could improve his or her utility by instead accepting any match that provides a positive surplus. Thus, an agent with a particular attribute is willing to be matched with a set of types on the other side. This acceptance set can fail to be convex and the non-convexity generates non-positive assignments.

The non-convexity does not exist in my model. With the ability to commit to hire a particular skill and to offer a particular wage, firms can direct workers' search and attract only the desired applicants. In fact, it is ex ante optimal for a firm to commit a particular machine quality to a single skill. The set of skills that a firm likes to be matched with is a singleton and, by definition, it is convex. A non-positive assignment arises instead from the fact that a higher skill can be compensated by a better machine or a less tight market; when machines and skills are not very complementary with each other, it is efficient to compensate high skills with a less tight market but not with high-quality machines.

A non-standard element of my model is that there is free entry on one side of the market (the firms' side), in contrast to a standard assignment model where the number of agents is fixed on both sides of the market. Free entry is important for efficiency, because it ensures that firms choose machine qualities to maximize skills’ expected social marginal product. With free entry, I will show in the next section that Becker's [3] central result continues to hold in a frictionless environment. I then switch to a frictional economy, characterizing the efficient assignment in Section 3 and the market assignment in Section 4. In Section 5 I examine the properties of the frictional assignment.

2. FRICTIONLESS ASSIGNMENT

Consider an economy where agents are all risk neutral and live for one period (see [22] for a dynamic setting with infinitely-lived agents). Workers differ in skills and, to make things simple, skills can be observed and measured by a one-dimensional object \( s \), which lies in a set \( S \) of

\(^5\) The free-entry assumption is more appropriate for the labor market than for the marriage market. But, even for the marriage market, the numbers of men and woman seeking marriages are not fixed in reality, because some men or women can choose to stay out of the market.
discrete points with a minimum $s_L > 0$ and a maximum $s_H$. There are a large, exogenous number of workers of each skill $s$, $n(s)$.

Machines differ in qualities that are denoted $k \in R_+$. A machine of quality $k$ costs $C(k)$ to make. In contrast to the exogenous distribution of workers, the distribution of machines is endogenously determined by firms. A machine can be operated by only one worker at a time. Workers and firms derive income solely from their production. A worker of skill $s$ operating a machine of quality $k$ produces output $F(k, s)$. Machine qualities and skills are complementary, i.e., $F_{ks} > 0$. In accordance with a standard assignment model like [3], utility is assumed to be transferrable between firms and workers and so the social surplus from a pair $(k, s)$ is $F(k, s) - C(k)$.

The assignment problem is to find a mapping $\phi$ that assigns a subset of machine qualities $\phi(s)$ to each skill $s$. The assignment is positive if higher skills are assigned to better machines. This corresponds to $\phi(s) > 0$ when $\phi(s)$ is a singleton for each $s$.

Assumption 1. (i) $C(0) \geq 0$, $C_k(0) = 0$, $C_k(k) > 0$ and $C_{kk}(k) \geq 0$ for all $k > 0$;

(ii) $F_k(k, s) > 0$, $F_{kk}(k, s) < 0$, $F_s(k, s) > 0$ and $F_{ss}(k, s) < 0$ for all $s$ and $k$;

(iii) $F_{ks} > 0$, $F(0, s) = F(k, 0) = 0$;

(iv) There exists a non-empty subset $K \subset R_+$ such that $F(k, s_L) - C(k) > 0$ for all $k \in K$;

(v) $(F_kC_{kk} - C_kF_{kk})F > (F_k - C_k)F_k^2$.

Conditions (i) and (ii) are standard. Condition (iii) requires skills and machine qualities to be complementary and, for unmatched machines and workers, output to be zero. Condition (iv) says that even the lowest skill can produce positive net output with some machine qualities. Since $F_s > 0$, for all skills there are machines that produce positive net output. Condition (v) is a concavity condition that guarantees uniqueness of the efficient assignment in a frictional environment.

Consider first a perfect world without matching frictions so that every worker can be matched instantaneously with a machine. The efficient
assignment in this world, denoted \( \phi^p \), maximizes the sum of net output, \( \sum_{s \in S} \sum_{k \in \Phi^*} [F(k, s) - C(k)] \). Thus, for every \( k \in \phi^p(s) \),

\[
F_k(k, s) = C_k(k). \quad (2.1)
\]

Under Assumption 2.1, the solution exists and is unique for each \( s \). So, \( \phi^p(s) \) is a singleton for each \( s \). Moreover, complementarity between skills and machine qualities implies \( \phi^p(s) > 0 \), i.e., the assignment is positive.

The efficient assignment can be decentralized as follows. Imagine that for every pair of machine quality and skill, \( (k, s) \), including those pairs that are not observed in equilibrium, there is a wage \( W(k, s) \). This wage schedule is non-negative for every pair \( (k, s) \) and, whenever \( F - C \geq 0 \), the wage delivers zero net profit for the firm. That is, for every pair \( (k, s) \),

\[
W(k, s) = \begin{cases} 
F(k, s) - C(k), & \text{if } F(k, s) - C(k) \geq 0 \\
0, & \text{otherwise.} 
\end{cases} \quad (2.2)
\]

Since not all the pairs \( (k, s) \) will appear in equilibrium, this wage schedule contains restrictions both along the equilibrium path and off the equilibrium path. These restrictions are rationalized by potential entry. If a firm contemplates the choice of a machine quality \( k \) for skill \( s \), the firm should expect that other firms can do the same and wipe out any positive profit.

Given the wage schedule and for each skill \( s \), a firm chooses machine qualities to solve

\[
\max_{k \in \mathbb{R}} \{ W(k, s) : (2.2) \}.
\]

That is, for each skill, firms choose machine qualities that pick the highest wage subject to the zero-profit condition. Since wage is maximized by machine qualities that maximize the net output, the equilibrium assignment is the same as the efficient one, \( \phi^p(s) \), and is a singleton.

The equilibrium wage for skill \( s \) is:

\[
w^p(s) \equiv W(\phi^p(s), s) = F(\phi^p(s), s) - C(\phi^p(s)) > 0.
\]

Thus, \( w^p(s) = F_s(\phi^p(s), s) > 0 \). As in a standard framework with homogeneous machines, a skill is rewarded at the margin with its marginal product and higher skills get higher wages. In contrast to a framework with homogeneous machines, the wage (or earning) function is not necessarily concave: Since the assignment is positive, a higher skill uses a better machine and so the marginal product of skill may increase with skill levels. The skill premium is magnified by higher machine qualities assigned to higher skills.
Choosing machine quality $\phi^s(s)$ for only skill $s$ workers is ex ante rational for firms, since firms’ decision can be equivalently formulated as the following dual problem:

$$\max_{k \in \mathbb{R}_+} \{ F(k, s) - C(k) - W(k, s) : W(k, s) \geq u^s(s) \}.$$  

That is, firms that choose the machine quality $\phi^s(s)$ do not, ex ante, want to hire skills other than skill $s$: If a firm planned to hire skill $s' \neq s$ for machine $\phi^s(s)$, it could improve its net profit by using machine $\phi^{s'}(s')$ for skill $s'$ instead.

Firms’ ex post incentives are different from the above ex ante ones. Once the cost of machine is sunk, a firm with a machine $\phi^s(s)$ may be willing to accept skills $s' \neq s$. Therefore, for the above decentralization of the efficient allocation to work, firms must be able to commit to the particular skill that their chosen machines are intended for and make such commitment known to workers. The same kind of commitment is necessary in a frictional environment.

3. EFFICIENT ASSIGNMENT WITH FRICTIONS

Now consider an economy with frictions that prevent workers and machines from being matched instantaneously. The simplest way to do this is to assume that a worker can have a match with at most one firm in the period. Let $\phi^s(s)$ be the set of machine qualities that the planner chooses for skill $s$. If $\phi^s(s)$ is not a singleton, the planner can divide skill $s$ workers into subgroups and match each subgroup to a machine quality in $\phi^s(s)$. For each $k \in \phi^s(s)$, let $M^s(k, s)$ be the number of machines of quality $k$ chosen by the planner and $B^s(k, s)$ be the ratio of skill $s$ workers to quality $k$ machines. The number of skill $s$ workers assigned to machines $k \in \phi^s(s)$ is $M^s(k, s) B^s(k, s)$. For brevity, I refer to this subgroup of skill $s$ workers and the corresponding machines $k$ as unit $(k, s)$ and to the number $B^s(k, s)$ as the tightness in this unit.

The planner in this frictional world maximizes the sum of expected net product of all pairs of machine qualities and skills. The planner is constrained by the same matching friction that agents face in a decentralized environment described later in Section 4. In particular, for unit $(k, s)$, the matching probability is $1 - e^{-B^s(k, s)}$ for each firm and $[1 - e^{-B^s(k, s)}] / B^s(k, s)$ for each worker. These matching probabilities depend on the tightness in an intuitive way and they imply a linearly homogeneous matching technology. Because not all workers and machines are matched, the sum of expected net output depends on the number of matches as well as on the
machine quality. To achieve the objective the planner must then choose the number of firms and a division of workers into units as well as the machine quality for each unit.

**Definition 3.1.** An efficient allocation in the frictional environment is a triple \((\phi^*(s), M^*(k, s), B^*(k, s))\), where \(\phi^*(s)\) assigns a set of machine qualities to skill \(s\) and \(M^*(k, s), B^*(k, s)\) assign a number of machines (firms) and a tightness to unit \((k, s)\), which solves the following problem:

\[
\mathcal{L} = \max_{\phi(s), M(k, s), B(k, s) \in \Phi(s)} \sum_{s \in S} \sum_{k \in \Phi(s)} M(k, s) \left[ (1 - e^{-B(k, s)} F(k, s) - C(k)) \right],
\]

subject to

\[
\sum_{k \in \Phi(s)} M(k, s) B(k, s) = n(s).
\]

The constraint (3.2) states that skill \(s\) workers allocated to different units sum up to the available number \(n(s)\). In (3.1) the first term in the square brackets is expected output of each firm in the unit \((k, s)\) and the second term is the machine cost. Since a machine must be put in place in order to be matched with a worker, the machine cost is sunk at the time of matching.

The following lemma simplifies the planner’s problem and will be useful later for comparing equilibrium allocation with the efficient one.

**Lemma 3.2.** The efficient assignment \(\phi^*(s)\) solves the following problem,

\[
(P^*) \max_{k \in R_s} e^{-B(k, s)} F(k, s)
\]

subject to

\[
1 - [1 + B(k, s)] e^{-B(k, s)} = C(k)/F(k, s).
\]

**Proof.** If \(k \in \Phi(s)\), the planner must choose a positive number of machines with quality \(k\) and assign a positive number of skill \(s\) workers to each of these machines. That is, \(M^*(k, s) > 0\) and \(B^*(k, s) > 0\). The first-order conditions for \(M^*(k, s)\) and \(B^*(k, s)\) of the planner’s problem are

\[
\lambda(s) B^*(k, s) = [1 - e^{-B(k, s)}] F(k, s) - C(k),
\]

\[
\lambda(s) = e^{-B(k, s)} F(k, s),
\]

where \(\lambda(s)\) is the Lagrangian multiplier of (3.2), i.e., the shadow price of skill \(s\). Eliminating \(\lambda\) from these two conditions yields (3.3). Since (3.4) and (3.5)
are satisfied by all $k \in \phi^o(s)$, the social surplus created by all units involving skill $s$ workers is

$$
\sum_{k \in \phi^o(s)} M^o(k, s)[(1 - e^{-B^o(k, s)}) F(k, s) - C(k)]
$$

$$
= \sum_{k \in \phi^o(s)} M^o(k, s) \lambda(s) B^o(k, s) = \lambda(s) n(s).
$$

The first equality follows from (3.4) and the second equality from (3.2).

Now suppose, contrary to the Lemma, that there is $k_1 \in \phi^o(s)$ that does not solve $(P^o)$. Then (3.5) implies that none of the machine qualities in $\phi^o(s)$ solves $(P^o)$. Let $\hat{k}(s)$ be a solution to $(P^o)$ and $B(\hat{k}(s), s)$ satisfy (3.3). Consider an alternative allocation that chooses $n(s)/B(\hat{k}(s), s)$ number of machines $\hat{k}(s)$ and allocates all skill $s$ workers to such machines. This allocation is feasible (as (3.2) is satisfied) and generates the following social surplus from skill $s$ workers:

$$
\frac{n(s)}{B(\hat{k}(s), s)} [(1 - e^{-R\hat{k}(s), s}) F(\hat{k}(s), s) - C(\hat{k}(s), s)]
$$

$$
= n(s) e^{-R\hat{k}(s), s} F(\hat{k}(s), s) > \lambda(s) n(s).
$$

The equality follows from (3.3) and the inequality from the supposition that $k \in \phi^o(s)$ does not maximize $e^{-Bk, s} F(k, s)$. A contradiction. Q.E.D.

Lemma 3.2 states that the efficient machine quality choices for skill $s$ workers maximize the marginal social value of skill $s$ workers, subject to the constraint that the social marginal value of a machine equals the cost. The objective function in $(P^o)$ is the social marginal value of skill $s$ workers. To see this, note that if a marginal and exogenous increment in skill $s$ workers is allocated to match with machines of type $k \in \phi^o(s)$, each machine $k$'s matching probability increases by $e^{-Rk, s} M(k, s)$, yielding a total increase in expected output in the unit $(k, s)$ as $e^{-Rk, s} F(k, s)$. The constraint (3.3) requires the social marginal value of a machine to be equal to the cost. To see this, note that adding one more machine $k \in \phi^o(s)$ increases the expected net output by $(1 - e^{-Rk, s}) F(k, s) - C(k)$ but also increases congestion to existing firms in the unit $(k, s)$. The increased congestion reduces the matching probability of each existing machine in that unit by $B(k, s) e^{-Rk, s}$ and hence reduces aggregate output by $B(k, s) e^{-Rk, s} F(k, s)$. Condition (3.3) requires that the number of machines $k$ be such that expected net output from an additional machine $k$ is equal to expected crowding out caused by that machine to the unit $(k, s)$. 

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The proposition below states existence and uniqueness of the efficient assignment (see Appendix A for a proof):

**Proposition 3.3.** Under Assumption 1, the efficient assignment \( \phi^*(s) \) exists and is a singleton for each \( s \). Let \( k_{\text{min}}(s) \) be defined by:

\[
F(k, s) C_k(k) - C(k) F_k(k, s) = 0. \tag{3.6}
\]

Then \( k_{\text{min}}(s) < \phi^*(s) < \phi^*(s) \) for every \( s \in S \).

The efficient assignment is illustrated in Fig. 1. For any given \( s \), the curve \( B = \text{IND}(k) \) depicts combinations of \( (k, B) \) that give the same social marginal value of skill \( s \) workers, denoted \( \lambda(s) \), where \( \text{IND}(k) \equiv \ln F(k, s) - \ln \lambda(s) \). This is an increasing and concave curve. The curve \( B = \text{ZNP}(k) \) depicts the constraint (3.3), along which expected net output created by a marginal machine equals output this machine crowds out. This is an upward sloping curve. The efficient assignment is given by point \( E \), where the two curves are tangent to each other.\(^8\) The tangency point is characterized

\(^8\) Condition (v) in Assumption 2.1 ensures that \( \text{ZNP}_{i,k} > \text{IND}_{i,k} \) whenever \( \text{ZNP}_k = \text{IND}_k \), which implies that the two curves are tangent to each other only once.
by the first-order condition of \((P^o)\) and (3.3), which can be written as follows, respectively:

\[
1 - e^{-b^o(s)} = \frac{C_k(\phi^o(s))}{F_k(\phi^o(s), s)};
\]

\[
1 - (1 + b^o(s)) e^{-b^o(s)} = \frac{C(\phi^o(s))}{F(\phi^o(s), s)},
\]

where \(b^o(s) \equiv B^o(\phi^o(s), s)\) is the efficient tightness for skill \(s\) and \(m^o(s) \equiv m(s)/b^o(s)\) is the efficient number of machines \(\phi^o(s)\). Eq. (3.7) states intuitively that the efficient assignment \(\phi^o\) equates the expected marginal product of the machine quality, \((1 - e^{-b}) F_k\), to the marginal cost.

The frictional assignment assigns a lower machine quality to each skill than does the frictionless assignment. This is because the possibility of failing to get matched in the frictional economy reduces the expected marginal product of any machine quality and so, to equate the latter to the marginal cost of the machine, the machine quality assigned to each skill must be lower. I will leave other properties of the efficient assignment to later discussions and now turn to the decentralization of the efficient assignment.

4. MARKET ASSIGNMENT WITH FRICTIONS

To decentralize the efficient assignment described above, consider the following markets. Firms can choose any machine quality to pair with a skill. When contemplating a machine quality \(k\) for a skill \(s\), a firm expects to face a market tightness, \(B(k, s)\), which is determined by a zero-profit condition for entry. That is, the firm anticipates that there are many other firms that can make the same pairing and so such pairing cannot make a positive expect net profit. This tightness schedule is specified later for every pair \((k, s)\). Taking the tightness schedule as given, firms select machine qualities \(\phi(s)\) for each skill \(s\) and choose a wage \(W(k, s)\) for \(k \in \phi(s)\). If a firm chooses machine \(k\) for skill \(s\), the firm commits to hiring only skill \(s\) workers for the machine and to offering the wage \(W(k, s)\). Then all firms simultaneously announce the skill and wages to which they commit their machines. Observing these, workers decide which firm to apply to, possibly with mixed strategies. If a firm receives applicants of the skill that the firm committed to, it randomly chooses one of them to produce. Otherwise, the firm remains unmatched. Note that, in contrast to the frictionless world, the market tightness schedule rather than the wage schedule is the one taken as given by firms; wages are endogenously chosen by firms here.
The equilibrium assignment problem can be solved backward. First, given firms' commitments, workers' application decisions can be analyzed. Second, anticipating workers' application decisions, firms' choices of machine qualities and wages must be optimal given the market tightness schedule. Finally, the market tightness is such that the expected net profit of each pair \((k, s)\) is zero (ex ante).

4.1. Wages for Skill \(s\) Workers

Given firms' commitments, skill \(s\) workers will apply only to the machines \(\phi(s)\), since firms that provide other machines will not consider skill \(s\) workers at all. Similarly, only skill \(s\) workers will apply to machines \(\phi(s)\). I can then isolate skill \(s\) workers and machines \(\phi(s)\) from other workers and firms. Note that I do not restrict \(\phi(s)\) to be a singleton a priori. Let \(M(k, s)\) be the number of machines \(k \in \phi(s)\) chosen for skill \(s\) and let \(W(k, s)\) be the wage that such firms announce for skill \(s\) workers. I will first analyze skill \(s\) workers' application decisions and then characterize the wage decisions by firms that commit machines \(\phi(s)\) to these workers.

The end products of this strategic exercise will be endogenous wages and endogenous matching rates that are functions of the market tightness. Although this relationship between matching rates and the market tightness is a standard one in the wage-posting framework, explicitly carrying out the strategic analysis is necessary for clarifying the meaning of the market tightness for the current environment. In previous wage-posting models, agents on each side of the market are homogeneous and so (under symmetry) the market tightness is simply the ratio of the numbers of agents on the two sides of the market. As a short cut, some authors (e.g. [1, 24]) have assumed a relationship between matching rates and the tightness and justified it by appealing to [17]. With heterogeneous agents, however, different machine qualities may in principle share the same skill \(s\) workers, in which case the tightness for a machine depends on how workers apply to each machine quality in the set \(\phi(s)\) and hence it is a strategic outcome. Only after the strategies are specified can I later show that \(\phi(s)\) is indeed a singleton.

Throughout this paper, I am interested only in the equilibrium that is symmetric in the sense that all workers of the same skill use the same strategy and all firms that choose the same machine quality for the same skill use the same strategy. Of course, firms can use different strategies if they wish to target different machine qualities to the same skill or target the same machine quality to different skills. Similarly, workers of the same skill can use different strategies when applying to different machine qualities in the set \(\phi(s)\).
In this subsection, I suppress the index $s$ in all variables and functions, referring to a skill $s$ worker as a worker and to a machine $k$ in the group $\phi(s)$ as a machine $k$. Let $p(k)$ be the probability with which a worker applies to a firm with machine $k$ (and a wage $W(k)$). If a firm gets only one worker, the worker is rewarded the job. If the firm gets more than one worker, each worker is selected with equal probability. In either case, production begins immediately after the match and output is divided between the worker and the firm according to the posted wage. If a firm fails to recruit any worker of the desired skill, output is zero.

Since workers observe all posted wages and then choose which firm to apply, firms can directly influence workers' application strategies through the posted wages. To be precise, consider a deviation by a single firm with machine $k$ to a wage $W^d(k)$, while all $M(k) - 1$ other firms with machines $k$ continue to post the wage $W(k)$ and all $M(k')$ firms with a different machine $k' \in \phi$ continue to post the wage $W(k')$. If $W^d(k) > W(k)$, the deviator can attract more workers than does a non-deviator who offers the same machine. However, not all workers go to the deviator with probability one; if they did so, each would be selected with a very small probability (in the order $1/n$) and the workers could improve the expected wage by applying to a non-deviator. Let $p^d(k)$ be the probability with which each worker applies to the deviator.

To find how $p^d(k)$ depends on $W^d(k)$, let me compute the expected wage of an arbitrary applicant to the deviator, say worker $a$, where the expected wage is the actual wage scaled by the probability that the worker gets the job. When worker $a$ applies to the deviator, there might be $j$ other workers applying to the same firm, which occurs with probability $C_{n-1}^j [p^d(k)]^j \left[1 - p^d(k)\right]^{n-1-j}$ where $C_{n-1}^j = n!/[j!(n-j)!]$. In this case worker $a$ is chosen by the firm with probability $1/(j+1)$. Since $k$ can be any integer from 0 to $n-1$, worker $a$ gets the job from the deviator with the following probability:

$$
\sum_{j=0}^{n-1} \frac{1}{j+1} C_{n-1}^j [p^d(k)]^j \left[1 - p^d(k)\right]^{n-1-j} = \frac{1 - \left[1 - p^d(k)\right]^n}{np^d(k)}.
$$

If worker $a$ gets the job from the deviator, his ex post gain (wage) is $W^d(k)$ and so the expected wage is $W^d(k)$ multiplied by the above probability.

---

9 The qualitative results will be similar if each worker observes only two independently drawn wages, but the exercise is more cumbersome (see [1]). Similarly, one can allow firms to post the reserve wage rather than the actual wage and then hold an auction after receiving two or more applications. With this setup the actual wage equals the reserve wage if the firm receives only one application and equals zero if the firm receives at least two applications. The reserve wage serves a role very much like the actual wage in the current framework but there is a dispersion in actual wages (see [12]). Such a dispersion complicates the analysis without contributing much to the main issues here.
A worker must be indifferent between the deviation wage $W_d(k)$ and the equilibrium wage $W(k)$. That is, the expected wage worker $a$ gets from the deviator and the non-deviator must be the same. Let $EW$ be the expected wage that worker $a$ gets in equilibrium. (See a later discussion on $EW$ and note that $EW$ is not indexed by $k$.) Then

$$1 - \left[ 1 - p^d(k) \right]^n \frac{1}{np^d(k)} W_d(k) = EW. \quad (4.1)$$

This equation implicitly defines $p^d(k)$ as a function of $W_d(k)$, which can be shown to be a continuous and increasing function. Therefore, by posting a higher wage the deviator can obtain a higher expected number of workers. Since this function is continuous, workers respond to a marginal increase in the offer by only a marginal increase in the application probability.

The deviator chooses $W_d$ to maximize the expected profit, taking the dependence of $p^d(k)$ on $W_d(k)$ into account but taking other firms’ wages as given. Since the probability with which the deviator has at least one worker is $1 - \left[ 1 - p^d(k) \right]^n$, the deviator’s problem is to solve:

$$\max_{W_d(k)} \left\{ \left[ 1 - (1 - p^d(k))^n \right] \left[ F(k) - W_d(k) \right] \right\}. \quad (4.1)$$

For $W(k)$ to be an equilibrium wage, the deviation cannot be profitable and so the solution to the above problem must be $W_d(k) = W(k)$, which implies $p^d(k) = p(k)$. Evaluating the first-order condition of the above problem at $W_d(k) = W(k)$ yields

$$p(k) = 1 - \left( \frac{EW}{F(k)} \right)^{1/(n-1)}. \quad (4.2)$$

An important qualification need be made for the above analysis. I have implicitly assumed that a worker’s expected wage ($EW$) does not respond to the single firm’s deviation. Strictly speaking, this is not true in the finite economy: By posting a higher wage the deviator attracts workers away from other firms, reduces the congestion that workers face in other firms and hence increases workers’ expected wage from applying to those firms. However, the above characterization is valid in the limit where there are infinitely many workers and firms of each type (see [5, 7]). This limit economy will be my focus.

---

10 More precisely, when $n$ and $M$ are both large with a finite ratio $n/M$, the effect of a single firm’s deviation on the probability that each worker applies to the deviator, $p^d$, is of order $1/n$, but the effect on the probability that each worker applies to a non-deviator, $p=(1-p^d)/(M-1)$, is of order $1/n^2$. Thus, the deviation has a non-negligible effect on the probability that a worker obtains a job from the deviator, $[1 - (1 - p^d)^n](np^d)$, but a negligible effect on the probability that a worker obtains a job from a competing firm, $[1 - (1 - p)^n](np)$.\footnote{More precisely, when $n$ and $M$ are both large with a finite ratio $n/M$, the effect of a single firm’s deviation on the probability that each worker applies to the deviator, $p^d$, is of order $1/n$, but the effect on the probability that each worker applies to a non-deviator, $p=(1-p^d)/(M-1)$, is of order $1/n^2$. Thus, the deviation has a non-negligible effect on the probability that a worker obtains a job from the deviator, $[1 - (1 - p^d)^n](np^d)$, but a negligible effect on the probability that a worker obtains a job from a competing firm, $[1 - (1 - p)^n](np)$.}
In the limit \( n \), \( M(k) \to \infty \) (but \( n/M(k) \) is bounded below and above), define the tightness for each machine with quality \( k \) as \( B(k) \equiv \lim np(k) \). Then \( (1 - p(k))^n \to e^{-B(k)} \). The matching probability for a firm with machine \( k \) is \( 1 - (1 - p(k))^n \to 1 - e^{-B(k)} \) and a worker’s probability of matching with machine \( k \) is

\[
\frac{1 - (1 - p(k))^n}{np(k)} \to \frac{1 - e^{-B(k)}}{B(k)}
\]

These matching probabilities are the ones used in Section 3 in the planner’s problem. Also, the wage share of output, denoted \( A(k) \), worker’s expected wage, \( EW \), and firms’ expected profit (before deducting the machine cost), denoted \( EP(k) \), can be obtained from (4.1) and (4.2) as:

\[
A(k) = \frac{W(k)}{F(k)} = \frac{B(k)}{e^{B(k)} - 1} . \quad (4.3)
\]

\[
EW = e^{-B(k)}F(k); \quad EP(k) = [1 - (1 + B(k)) e^{-B(k)}] F(k). \quad (4.4)
\]

The wage share is a decreasing function of \( B \), i.e., workers get a smaller share of the output if they all apply to a firm with a higher probability.

### 4.2. Market Tightness, Assignment, and Equilibrium

The market tightness schedule \( \{B(k, s)\}_{s \in S, k \in K} \) is such that the expected net profit from each pair \((k, s)\) is zero, provided that output from the pair is at least as high as the cost of the machine. If output from a pair \((k, s)\) is less than the machine cost, no firm will adopt \( k \) for \( s \) and so \( B(k, s) = \infty \). Since the expected net profit from the pair \((k, s)\) is \( EP(k, s) - C(k) \), where \( EP \) is given by (4.4), the market tightness obeys:

\[
\begin{aligned}
\begin{cases}
\left[ 1 - \left( 1 + B(k, s) \right) \right] e^{-B(k, s)} = \frac{C(k)}{F(k, s)}, & \text{if } C(k) \leq F(k, s) \\
B(k, s) = \infty, & \text{otherwise.}
\end{cases}
\end{aligned}
\quad (4.5)
\]

It is worth repeating that this specifies a tightness for every pair \((k, s)\), not just for the pairs observed in equilibrium. For pairs \((k', s)\), where \( k' \notin \phi(s) \), (4.5) is a restriction on beliefs on the tightness off the equilibrium path. Similar to the restriction on the wage schedule in the frictionless assignment, this restriction can be rationalized by firms’ entry. That is, when contemplating the choice of quality \( k' \notin \phi(s) \) for skill \( s \), a firm expects that
other firms can do the same and leave each firm with a non-positive expected net profit.

Taking the schedule \{B(k, s)\}_{s \in S, k \in K}, as given, each firm chooses a machine quality to maximize the expected wage for each skill \(s\). That is, for each skill \(s\) firms solve:

\[
(P) \quad \max_{(B, k)} \left\{ e^{-B}F(k, s); \quad 1 - (1 + B) e^{-B} = \frac{C(k)}{F(k, s)} \right\}.
\]

**Definition 4.1.** An equilibrium consists of an assignment \(\phi\), a wage function \(W\) and a tightness schedule \(B\) such that the following conditions hold:

(i) For each pair \((k, s)\), the tightness \(B(k, s)\) satisfies (4.5);

(ii) Given the tightness schedule, it is optimal for a firm that offers a quality \(k \in \phi(s)\) to skill \(s\) to post the wage \(W(k, s)\) if other firms that offer \(k\) post \(W(k, s)\);

(iii) Given the tightness schedule, it is ex ante optimal for firms to choose machine qualities \(\phi(s)\) and commit to hiring only skill \(s\) workers for such machines.

Condition (i) gives the market tightness for equilibrium pairs as well as for pairs off the equilibrium path. Condition (ii) requires wages to satisfy (4.3). Condition (iii) requires the assignment \(\phi(s)\) to solve the dual problem to (\(P\)). That is, it requires each firm to choose machine qualities to maximize the expected net profit, subject to the constraint that workers get at least the equilibrium expected wage.

Since (\(P\)) is identical to (\(P^o\)), the following proposition is evident from Proposition 3.6:

**Proposition 4.2.** The market assignment is identical to the efficient one and hence \(\phi(s)\) is a singleton for each \(s\).

**Remark 1.** Since \(\phi(s)\) is a singleton, \(p(k, s) = 1/M(k, s)\) and so \(B(k, s) = n(s)/M(k, s)\). I can denote \(m(s) = M(\phi(s), s), b(s) = B(\phi(s), s)\) and \(w(s) = W(\phi(s), s)\).

Only one machine quality is chosen for a skill in equilibrium because, with different machines qualities, workers and firms cannot be both made indifferent. If workers are indifferent between different machines, then a higher quality machine must be more crowded with workers than a lower quality machine (see (4.2)). But this means that each higher quality machine gets a match more likely than does a lower quality machine. Since a higher quality machine is also more productive with the same worker, the
expected net profit from a higher quality machine is higher in this case. This cannot be consistent with firms’ rational choices of machines.\footnote{Later I will also show that each machine quality is paired with only one skill generically (Section 5). That is, different skills do not share the same machine quality.}

The equilibrium assignment is depicted by Fig. 1. The notation $IND(k)$ means that a worker (of skill $s$) is indifferent between combinations $(k, B)$ along the curve $B = IND(k)$; the notation $ZNP(k)$ means that combinations $(k, B)$ along the curve $B = ZNP(k)$ all generate zero expected net profit for firms in the market with skill $s$ workers.

4.3. Explanation for Efficiency

An important reason why the market assignment is efficient is that equilibrium wages are tied endogenously to the market tightness in the way described by (4.3). This condition is a well-known requirement for efficiency in a frictional matching environment, established first by Hosios [10]. In the current context, the Hosios condition takes the following form:

$$
 \frac{\partial \ln [m(1 - e^{-\frac{n}{m}})]}{\partial \ln n} = \frac{B}{e^B - 1} = A. \quad (4.6)
$$

That is, the elasticity of the number of matches with respect to the number of workers is equal to the wage share. As explained in [10], this condition is necessary for efficiency because it ensures each side of the market to be rewarded with the side’s contribution to the match.

It is not well understood why a wage posting framework can meet the Hosios requirement. To explain why, note that the requirement can be expressed in the following form, which can be verified by comparing (4.4) and (3.5):

**Corollary 4.3.** The expected wage in equilibrium for skill $s$ workers, denoted $E_w(s)$, is equal to the social marginal value of such workers. That is,

$$
 E_w(s) = \lambda(s) \equiv \frac{dL}{dh(s)}, \quad (4.7)
$$

where $L$ is the maximized value of the social welfare defined in (3.1).

The equality between the equilibrium expected wage and the social marginal value of workers is a natural outcome of the equilibrium mechanism employed here. Since firms commit to wages before workers compete, the expected wage a firm is willing to pay does not exceed the expected output.
loss from the firm's failure to match, i.e., the amount $e^{-BF}$. Neither will a firm pay less than this amount as the expected wage, since competition among firms will drive up the wage otherwise. Thus, the expected wage in equilibrium must be exactly equal to the expected loss in output resulted from a firm's match failure. Such an expected loss in output is precisely what an additional worker can save, i.e., the social marginal value of the workers.

To reiterate the explanation, consider an extreme case where a skill $s$ and a machine $k$ can produce output which is infinitely larger than the machine cost. Since there is not much to lose for a firm and the potential output is great when the machine is matched, there will be many firms offering machine $k$ and competing for the workers. The large entry ensures $B \to 0$ and pushes the expected wage up arbitrarily close to the level of output, i.e., the social marginal value of workers in this case. In general, as the market gets tighter, i.e., as $B$ increases, the worker's expected wage as a share of output decreases and the firm's share increases.

The above discussion reveals two elements that are necessary for any equilibrium allocation to meet the Hosios condition. The first is that the side of the market that creates matches is given the “rights” to decide how to split the match surplus. In the current setting, firms are the ones that organize matches: They not only incur machine costs but also bear entirely the market externality. The cost of congestion created by firms' entry, $Be^{-BF}$, is deducted from firms' profit, as shown in (4.4). By giving firms the power to choose wages, the market mechanism aligns the decision rights properly with incentives. The second element for the Hosios condition is full-fledged competition on the side of the market that organizes matches, by that I mean that agents on the other side of the market can knowingly choose between the organizers.

The Hosios condition is only a necessary condition for efficiency but not a sufficient one; efficiency also requires that the social marginal value of each skill be maximized by the choice of a machine quality and that the social marginal value of a machine be equal to the machine cost. Both are achieved in the market economy through firms' competition and their commitment of machines to the desired skill. As firms compete, no firm is expected to make a positive net profit and only those firms that select a machine quality to maximize the expected wage for a skill and commit this machine quality to the targeted skill will expect to break even.

To summarize, the market mechanism described in this paper achieves efficiency by (i) properly allocating the decision rights on surplus division to agents who actively organize the market, (ii) allowing for full-fledged competition among these agents, and (iii) featuring commitments by these agents to the surplus division method and to the particular types of agents who they choose to match with. None of these elements can be eliminated without destroying efficiency, as illustrated below in turn.
Improper allocation of the decision rights. An example for this case is the random search literature pioneered by [9, 16, 18], where wages are determined ex post by Nash bargaining. Since firms actively organize the market but the decision rights are chosen arbitrarily through the Nash bargaining rule, firms do not get compensated properly. Efficiency fails unless the bargaining weights are exogenously set to satisfy (4.3), as shown by [10].

Limited competition. If only an arbitrarily fixed number of firms are allowed to compete, the expected wage will not be equal to the social marginal value of workers and so efficiency fails. A subtler example of limited competition is the well-known search literature surveyed by [13]. In contrast to the random matching literature discussed above, this literature allows firms to choose wages before workers make their search decisions and allows for free-entry by firms. However, in most models in this literature, workers know only the distribution of wage offers and discover a firm’s offer only after a costly search. Since workers do not know two particular firms’ offers before deciding which one to go for, particular wage offers do not have the ex ante allocative role in those models. For this reason, competition among firms is limited and the Hosios condition is unlikely to be met.

Lack of commitment. Suppose that firms post and commit to wages and workers observe all posted wages before application, but firms cannot commit the chosen machine, say $\phi(s)$, to the particular skill $s$. When such a firm has already incurred the machine cost and then unexpectedly receives an application from a worker $s'$, where $s'$ is close to but higher than $s$, the firm has every incentive to hire worker $s'$ instead of worker $s$ since the surplus with worker $s'$ is higher than with worker $s$. Knowing that firms with machines $\phi(s)$ cannot commit their machines to skill $s$ workers and that applying to firms with machine $\phi(s)$ will ensure employment, a skill $s'$ worker will surely apply to those firms. This destroys the equilibrium constructed in this paper. In such an environment without commitment, a skill can be matched with a range of machine qualities. Each firm tries to choose high-quality machines to attract high-skill workers but, when all firms choose high-quality machines, each firm’s chance does not improve and so high-quality machines are under-utilized. The sum of expected net output is not maximized. The paper [23] analyzes an example of such a non-commitment economy with two types of workers and two types of machines.\footnote{The segmentation of skills resulted from firms’ commitment of different machine qualities to different skills improves efficiency by ensuring that firms maximize each skill’s expected wage with the least expensive machine. This has some similarity to the result in [8] that a social norm that ranks agents and separates them into different classes can improve efficiency. However, potential inefficiency in that paper is intertemporal and arises from over-saving that does not appear in the current environment.}
The above elements can be compared with those identified by Acemoglu and Shimer [1], who examine firms’ choices of an investment level to match with a group of homogeneous workers. They identify two elements that are necessary for the Hosios condition to hold—firms’ ability to direct workers’ search and their ability to commit to the posted wages. For a heterogeneous labor force, the ability to commit to the posted wages is not enough; firms must also commit to the particular skill for which the machine quality is chosen. Moreover, directed search can be viewed as an outcome of the proper allocation of the decision rights and wage-posting is a special surplus division method chosen ex ante. Other ex ante chosen mechanisms, such as auctions, can do equally well in implementing efficiency (see footnote 8).

It is debatable whether firms can commit a particular machine quality to a particular skill, given the possibility of a match failure and the existence of ex post surplus. In fact, the same criticism can be voiced against the assumption that firms in a frictional matching environment can commit to wages or, more generally, to any allocation mechanism that is chosen ex ante (such as auctions). Nevertheless, commitment does occur in reality. For example, the hiring procedure can be sufficiently rigid to induce firms to hire only workers qualified for the advertised positions, and parental influence can be sufficiently strong to induce the offsprings to seek particular types of partners for marriage.

5. PROPERTIES OF THE FRICTIONAL ASSIGNMENT

5.1. Assignment and Tightness

Since the efficient assignment and the equilibrium assignment are the same, I will refer to them as frictional assignment. An important feature of the frictional assignment is that each skill level is associated with a market tightness as well as a machine quality. Obviously, the numbers of machines and workers need not be the same. Although a high matching rate and a high-quality machine are two ways to compensate a high skill, the two ways are not always used simultaneously and so the assignment is not always positive, as summarized by the following proposition: 13

PROPOSITION 5.1. The assignment is positive, i.e., \( \phi_s(s) > 0 \), if and only if

\[
\frac{FF_{kS}}{F_k F_s} > \frac{CF_k(F_k - C_k)}{C_k(FC_k - CF_k)}. \tag{5.1}
\]

13 The proof is omitted; it is straightforward but tedious algebra involving (3.7) and (3.8).
A higher skill has a higher matching rate, i.e., $b_s(s) < 0$, if and only if

$$\frac{FF_{ss}}{F_s F_s} < \frac{CF_k C_{sk} - C_k F_{sk}}{C_k F_k (FC_k - CF_k)^{-1}}.$$  \hspace{1cm} (5.2)

Thus, $b_s \geq 0$ implies $\phi > 0$ and $\phi \leq 0$ implies $b_s < 0$. Under (v) in Assumption 1, there is a non-empty parameter region in which both $\phi > 0$ and $b_s < 0$.

Remark 2. Except when (5.1) holds with equality, each machine quality is assigned to only one skill. With this qualification, I will ignore the case where (5.1) holds with equality.

This proposition can be illustrated in Fig. 1 by increasing the skill level from $s$ to $s'$. When $s$ increases to $s'$, the zero-net-profit curve $ZNP(k)$ shifts down. The new tangency point in Fig. 1 can be either on the left or on the right side of the original tangency point and so $\phi(s')$ can be either smaller or greater than $\phi(s)$. Similarly, the new tangency point can be either above or below the original tangency point and so a higher skill is not necessarily associated with a less tight market or a higher matching probability. Nevertheless, if the new solution is at least as high as the original one along the vertical axis, it must be on the right side of the original one. That is, for $s' > s$, $b(s') \geq b(s)$ implies $\phi(s') > \phi(s)$ and so $\phi(s') \leq \phi(s)$ implies $b(s') < b(s)$.

Example 5.2. Non-positive assignment: $C(k) = C_0 k^\gamma$ with $\gamma > 1$ and $F(k, s)$ is the CES type, $F = F_0[(1 - \gamma) s^\sigma + \gamma k^\rho]^{1/\rho}$ with $\rho \neq 0$. Let $C_0 = 0.2$, $\gamma = 3$, $F_0 = 1$, $\sigma = 0.35$ and $\rho = 0.8$. Then a higher skill has a lower assignment, $\phi(3) = 0.25 < 0.254 = \phi(2)$, but is compensated with a higher matching rate (i.e., a less tight market), $b(3) = 0.064 < 0.075 = b(2)$.

To explain why equilibrium/efficient assignment may sometimes be non-positive, consider the case where skills and machine qualities are barely complementary, i.e. $F_k \approx 0$. This is the case where the condition for a positive assignment, (5.1), fails. Since high skills and high quality machines are both productive, efficiency requires both to be highly utilized. Given the matching friction, this cannot be achieved if high skills and high quality machines are assigned to match with each other. The only way to achieve high utilization for both is to assign each high-skill worker with many low-quality machines and to assign each high-quality machine with many low-skill workers. With this non-positive assignment, expected net output from units involving high-skill workers is high because the good skill is highly utilized and because the corresponding under-utilized machines have low costs; expected net output from units involving high-quality machines is also high because the good machines are highly utilized and because the corresponding under-utilized workers have low skills. This is why the non-positive assignment maximizes the sum of expected net output in this case.
This explanation clarifies two things. First, if high skills are ever assigned with lower quality machines by the efficient assignment, they must be given higher matching rates than low skills. That is, $\phi_s \leq 0$ implies $b_s < 0$, as stated in Proposition 5.1. Equivalently, if high skills are ever given lower matching rates, then they must be assigned with higher quality machines, i.e., $b_s \geq 0$ implies $\phi_s > 0$. Second, the equilibrium/efficient assignment is positive if skills and machine qualities are sufficiently complementary to each other in production. In this case, although assigning high skills and high-quality machines to each other still reduces the utilization rate for at least one of them, the corresponding loss in output is outweighed by the increase in output brought about by the complementarity between the two sides.

Exactly how complementary should the two factors be to make a positive assignment efficient depends on the factors’ marginal productivity and the marginal cost of machine qualities. Other things being equal, increasing the marginal productivity of either factor or reducing the marginal cost of machine quality relative to its productivity increases the threshold of the complementarity that is required for a positive assignment. This is intuitive. If the marginal productivity of a factor, say the skill, is higher, then leaving the skill under-utilized is more costly and so the complementarity must be higher to justify such cost created by a positive assignment. Similarly, if the marginal cost of machine qualities relative to the marginal productivity ($C_k/F_k$) is lower, increasing the machine quality and making it more highly utilized may increase net output, in which case a low utilization rate of such machines created by a positive assignment can be justified only by a stronger complementarity between the two factors.

When skills and machine qualities are moderately complementary, a higher skill is assigned to both a higher machine quality and a higher matching rate. If the two factors are extremely complementary to each other but skills by themselves are not very productive at the margin, the efficient assignment requires high skills to have a lower matching rate than low skills. That is, (5.2) fails when $F_{ks}$ is sufficiently high and $F_s$ is sufficiently low. This is intuitive. When the two factors are very complementary, it is efficient to match high-quality machines with high skills. Since the marginal productivity of skills is not high, the most important efficiency consideration is to ensure a high utilization rate for high-quality machines and so sufficiently many high skill workers are assigned to match with each high-quality machine, producing a low matching rate for each high-skill worker.

5.2. Wages and Skill Premium

I now turn to equilibrium wages and skill premium. Direct computation yields:
Proposition 5.3. \( E_{w_s}(s) > 0 \) for all \( s \), regardless of the signs of \( \phi_s \) and \( b_s \). If \( \phi_s \geq 0 \) then \( w_s > 0 \). That is, a higher skill is rewarded a higher wage if the assignment is positive. Moreover, \( w_s < F_s(\phi(s), s) \) if and only if \( b_s(s) < 0 \).

The fact that the worker’s indifference curve \( IND(k) \) in Fig. 1 moves toward southeast when \( s \) increases shows that a higher skill gets a higher expected wage, regardless of the signs of \( \phi_s \) and \( b_s \). A close inspection of the expression for the expected wage reveals that the skill level can affect the expected wage in three ways. An increase in \( s \) (i) affects the machine quality assigned to it; (ii) increases output directly; and (iii) attracts more firms to the market and hence reduces the workers’ congestion. The effects of (ii) and (iii) on the expected wage are unambiguously positive. The effect of (i) vanishes when the machine quality is chosen optimally. The output increased by a better machine is exactly canceled by the increased congestion that the better machine creates for workers (since, for any given skill, the higher cost of the better machine make fewer firms choose it).

Proposition 5.3 also states that there is a positive skill premium in wages if the assignment is positive. This result is not obvious ex ante. As stated in Proposition 5.1, a positive assignment might be accompanied by an increasing matching rate for high skills. Since what matters to workers’ decisions is not the actual wage but rather the expected wage, the outcome \( w_s < 0 \) can be consistent with \( \phi_s > 0 \), a priori, if \( b(s) \) is sufficiently decreasing. The proposition shows that this does not happen in equilibrium.

The marginal skill premium, measured by \( w_s \), is less than the marginal product of skill if and only if the matching rate increases with skill. This is simple to explain. When the matching rate increases with skill, a worker with a higher skill is compensated with a higher matching probability and so wage need not increase by as much as the contribution of the incremental skill to output. In fact, when the compensation through an increased matching probability is sufficient, the wage can even fall with skill.

The analysis indicates that actual wages are not a good measure of the skill premium. The true measure should be the difference in expected wages between skills, which reflects the difference in matching rates as well as in actual wages. Since matching rates are likely to be an increasing function of skills in reality, the wage differential under-estimates the inequality between skills.

5.3. The Cases of Constant Tightness

The possibility of a non-positive assignment is associated with a non-constant tightness, as discussed in the last subsection. It is interesting to know when \( b(s) \) is constant over \( S \). The following proposition lists some features of this special case (see Appendix B for a proof).
Proposition 5.4. Suppose \( b(s) = \text{constant} \). The assignment is then positive. If \( F(k, s) \) is linearly homogeneous in \((k, s)\), there exist positive constants \((\delta_1, \delta_2)\) such that \( \delta_1 < \delta_2 < 1 \) such that \( \phi(s) \) is implicitly given by the following equation:

\[
s = s_L \exp \left[ \frac{\phi(s)}{s_L} \left( \frac{\delta_1}{\delta_2} f(y) - y \right)^{1} \right], \tag{5.3}
\]

where \( f(k/s) = F_k/F_s \). If the production function \( F \) is the CES type, \( F = F_0\left[ \delta k^\rho + (1 - \alpha) s^\alpha \right]^{1/\rho} \), then the assignment is

\[
\phi(s) = \left[ \frac{\phi(s_L)}{s_L} \right]^\rho + \frac{(1 - \alpha) \delta_1}{\alpha(\delta_2 - \delta_1)} \left( s^\alpha - s_L^\alpha \right)^{1/\rho}, \tag{5.4}
\]

and the cost function must have the following form:

\[
C(k) = \delta_1 \cdot F_0 \left[ s \delta_2 + (1 - \alpha) s^\rho_L \right]^{1/\rho} \frac{\alpha(\delta_2 - \delta_1)}{\delta_1} \left[ \phi(s_L) \right]^\rho. \tag{5.5}
\]

In this case the assignment is concave if and only if \( C_{kk} > 0 \). Conversely, if \( C(k) \) is given by (5.5) and \( F(k, s) \) has the corresponding CES form, then \( b(s) = \text{constant} \).

Example 5.5. \( C(k) = C_0 k^\gamma \) and \( F(k, s) = F_0 s^{\alpha(1 - \gamma)} \), \( \alpha \in (0, 1) \).

This is a special case described in Proposition 5.4, with \( \rho = 0 \). In fact, taking the limit \( \rho \to 0 \) on (5.5) shows that \( C(k) = C_0 k^{\alpha(1 - \gamma)} \) for some \( C_0 > 0 \). Therefore, \( \alpha(\delta_2 - \delta_1) = \gamma \) and the unique assignment in (5.4) becomes

\[
\phi(s) = \phi(s_L) \left( \frac{s}{s_L} \right)^{(1 - \alpha)(\gamma - \alpha)},
\]

The assignment is concave if and only if \( \gamma > 1 \), an implication of Proposition 5.4. Since \( b_s = 0 \), Proposition 5.3 implies that the marginal wage \( w_s \) is equal to the marginal product of skill.

Example 5.6. \( C(k) = C_0 k^\rho \) and \( F \) is the CES form with \( \rho < 1 \).

This is another special case of Proposition 5.4, with \( C_{kk} = 0 \). Using (5.5) and setting \( C_k = 0 \) yields a restriction on \( \phi(s_L) \). Substituting such \( \phi(s_L) \) into (5.4) yields \( \phi(s) = \phi_0 s \) for some constant \( \phi_0 > 0 \). The assignment is positive and linear. This example is interesting because Jovanovic [11]
shows that, in a frictionless assignment, non-degenerate distributions of skills and machine qualities are consistent with positive long-run growth in per-capita income only when the cost of machine is linear in quality, at least at the aggregate level. This example indicates that a similar result can be established when the assignment is frictional.

6. CONCLUSION

This paper has examined the efficient assignment in a world with matching frictions and constructed a decentralization mechanism. The frictional assignment assigns each skill with a market tightness as well as a machine quality. The addition of the market tightness as an allocative device implies that the efficient/equilibrium assignment does not always assign high skills to match with high-quality machines even when the two are complementary in production—sufficient complementarity is required for a positive assignment. The decentralization mechanism features free entry by firms, a surplus division method and a targeted skill chosen by firms ex ante, and commitments to those decisions.

As indicated in the introduction, the current framework is useful not only for the labor market but also for other large, two-sided matching markets. Even if one restricts the attention to the labor market, the analysis sheds light on a number of issues. First, the true measure of the skill premium should be the difference in expected wages between skills. Given the reality that high-skill workers have a higher matching rate than low-skill workers, wage differentials underestimate the skill premium. Second, a worker's wage depends on market characteristics such as the market tightness, in addition to the worker's characteristics such as skill and the firm's characteristics such as machine quality and capital intensity. Since skilled workers are more likely to find a job than unskilled workers, an estimation of the earning function should pay attention to market characteristics. Third, a technological progress may affect the skill premium in a novel way: It may change the matching rates for different skills differently and so change wage inequality.

To examine these issues in a satisfactory fashion, the current model needs to be extended to a dynamic setting to allow agents to re-match over time. In [22] I carry out this extension and calibrate the model; the results will be collected separately in a sequel. Also, the current analysis has ignored other important aspects of the labor market, such as multi-dimensional skills, investments on both sides of the market, match-specific productivity, private information and/or uncertainty in productivity. Characterizing the efficient allocation and its decentralization mechanism with these complex elements will be the challenging tasks for future research.
APPENDIX

A. Proof of Proposition 3.3

First, let me reformulate the problem \((P^o)\). Denote \(\Theta(x) = 1 - (1 + x)e^{-x}\). Since \(\Theta(x)\) is strictly increasing for all \(x > 0\), its inverse exists which is denoted \(\theta(\cdot)\). Then, (3.3) can be rewritten as \(B(k, s) = \Theta(C(k), F(k, s))\) and the problem \((P^o)\) can be rewritten as

\[
\max_{k \in \mathbb{R}} A(k, s) = F(k, s) e^{-\theta(C(k), F(k, s))}.
\]

Next, let me find the upper bound and the lower bound on the solutions to \((P^o)\). For the upper bound, notice that \(\theta' = e^\theta/\theta = [1 - e^{-\theta} - C(k, F(k, s))]^{-1}\) and calculate

\[
A_k(k, s) = [F_k(1 - e^{-\theta}) - C_k].
\]

Any solution to \((P^o)\) must satisfy \(C_k < F_k\); otherwise \(A_k(k, s) < 0\) and so \(A(k, s)\) can be increased by reducing \(k\). Since the frictionless assignment \(\phi^*(s)\) satisfies \(C_k = F_k\) and since \(C_k - F_k\) is an increasing function of \(k\), the requirement \(C_k < F_k\) is equivalent to \(k < \phi^*(s)\) for all \(k \in \phi^*(s)\).

The lower bound on the solutions to \((P^o)\) is \(k_{\min}(s)\), where \(k_{\min}(s)\) is defined by (3.6). Since the function \(F(k, s) C_k(k) - C(k) F_k(k, s)\) is an increasing function of \(k\) under Assumption 1 and is zero-valued at \(k_{\min}(s)\), it suffices to show that this function has positive values at all solutions to \((P^o)\). To show this, suppose that there is \(k_1 \in \phi^*(s)\) that yields a non-positive value for this function. Then,

\[
A_k(k_1, s) = \left[ F_k(k_1, s)(1 - (1 + \theta) e^{-\theta}) - C_k(k_1) \right]/\theta
\]

where the equality in the middle step follows from (3.3) and the inequality is supposed. In this case, \(k_1\) cannot possibly solve \((P^o)\) since increasing the machine quality slightly above \(k_1\) increases \(A(k, s)\). A contradiction.

The above arguments have also shown that \(A_k(k, s) > 0\) for \(k = k_{\min}(s)\) and \(A_k(k, s) < 0\) for \(k = \phi^*(s)\). These properties imply that \((P^o)\) has at least one maximum in \((k_{\min}(s), \phi^*(s))\) and all such maxima satisfy the first order condition \(A_k(k, s) = 0\). To show that \((P^o)\) has a unique solution, it suffices to show that \(A_k(k, s) < 0\) for \(k \in (k_{\min}(s), \phi^*(s))\) whenever \(A_k(k, s) = 0\). (If \(A(k, s)\) has a second maximum, it must have a minimum between the two maxima.) This is guaranteed by (v) in Assumption 1. Q.E.D.
B. Proof of Proposition 5.4

When $b$ is a constant, Proposition 5.1 immediately implies $\phi_s(s) > 0$. Also, (3.7) and (3.8) imply that $C_k/F_k = \delta_2$ and $C/F = \delta_1$ are constants, with $\delta_1 < \delta_2 < 1$. Totally differentiating the equation $C/F = \delta_1$ with respect to $s$, substituting $C_k$ by $\delta_2 F_k$ and writing $F_r/F_k$ as $f(k/s)$, I obtain:

$$\phi_s(s) = \frac{\delta_1}{\delta_2 - \delta_1} \cdot f\left(\frac{\phi(s)}{s}\right). \quad (B.1)$$

Making a transformation $z(s) = \phi(s)/s$ and substituting $\phi$, I have:

$$\frac{ds}{s} = \left(\frac{\delta_1}{\delta_2 - \delta_1} f(z) - z \right)^{-1}.$$

Integrating from $s_L$ to $s$ yields the solution (5.3) in the proposition.

If $F$ is the CES type, then $f(y) = \left(\frac{L+y}{y}\right)^{-\rho}$ and integrating (B.1) gives (5.4). Substituting $s = \phi^{-1}(k)$ into the function $F$ and using $C = \delta_1 F$, one recovers the cost function (5.5). With $\rho < 1$, the cost function is strictly convex if and only if

$$\left[\phi(s_L)\right]^\rho > \frac{(1 - \rho)}{\alpha(\delta_2 - \delta_1)} s_L^{\delta_2},$$

which is also necessary and sufficient for $\phi_s(s) < 0$.

Conversely, if $C(k)$ is given by (5.5) and $F(k, s)$ has the corresponding CES form, then substituting these functions into (3.7) and (3.8) shows $b(s) = \text{constant.}$ Q.E.D.

REFERENCES

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