Innovation, Growth, and Welfare-Improving Cycles*

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This paper establishes necessary and sufficient conditions for the existence of stationary cycles in an economy comprising independent investing firms. The economy is not subject to aggregate uncertainty, investors have no direct complementarities, and all agents act independently. The cycle arises due to general equilibrium contemporaneous complementarities between investors devoting resources to innovation which yields temporary profits. With numerical examples we show that there are multiple cyclical equilibria that differ in the cycle length. Welfare and the long-run growth rate can be increased from an equilibrium where innovations occur rapidly to one with longer cycles; however, there exists a finite cycle length that maximizes welfare. Journal of Economic Literature Classification Numbers: E32, L16, O31, O41. © 1999 Academic Press

1. INTRODUCTION

With increasing research in endogenous growth it is becoming clear that economic cycles cannot be separated simply from long-run growth. Forces that drive long-run growth such as innovation may also create cycles. Moreover, the long-run growth path and associated cycles are sensitive to

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This Schumpeterian view can be supported if there is either direct technological complementarity or strategic complementarity among innovators. In the current paper we focus on the welfare side of the Schumpeterian view and ask: can innovation cycles improve welfare? This question is interesting because previous studies of economies that experience cyclical growth seem to provide an unambiguously negative answer. For example, in Shleifer’s [14] model of implementation cycles, Gale’s [6] model of dynamic coordination games, and Cheng and Dinopoulos’ [4] neo-Schumpeterian model of cyclical growth, delays and cycles have no social welfare gain and welfare would be maximized if innovators could coordinate their actions to eliminate cycles altogether.

This paper constructs a model in which aggregate level cycles in productivity growth arise endogenously. The economy is composed of a large number of independent industries in which productivity increases due to purposeful innovative activity undertaken through time by firms. Such activity yields for firms a short term advantage over rivals which translates into supernormal profits that are eventually competed away by rivals’ imitation. Firms attempt to obtain their advantage over rivals when the discounted value of profits is highest relative to the costs of developing the innovation. It is shown that the value of profits relative to costs for any one firm is increasing in the aggregate level of profit in the economy and is thus complementary with the innovation decisions of firms in other sectors. Importantly, the contemporaneous complementarities here are entirely pecuniary in nature, that is, they do not arise for ad hoc reasons of technological complementarity, exogenous aggregate shocks or collusion among innovators, but instead through the general equilibrium interaction of investing firms.

Innovators are rewarded for successful innovation through temporary profit at the industry level. In periods when profits across sectors are high, factors of production and consumers receive relatively little of the benefit to innovation since much is accruing to successful firms as rents. Thus, in the interval between innovation and imitation, factors are cheap in comparison with aggregate income which is relatively large. Since a large part of the costs of innovation are payments to and opportunity costs of factors of production, innovators would like to incur the costs of their innovation when factors are cheap but obtain their brief span of leadership when aggregate income, and hence sales, are high. This implies that they should match, in the timing of their innovation and implementation decisions, the timing of innovators in other sectors, since by doing so they incur costs

before factor prices rise through competition but reap profits when income has grown through innovation. There thus endogenously emerges a tendency for innovating firms to cluster together in innovation activity, the aggregate effect of which is an \((N+1)\)-period cycle in output, where \(N\) is the length of innovation.\(^2\)

Multiple cyclical equilibria exist as there can be more than one value for \(N\) that is consistent with rational expectations. The shortest cycle \((N = 1)\) does not necessarily generate the fastest output growth or the highest welfare. With suitable parameter values, increasing the innovation length or the cycle length from the shortest one to a longer one increases output growth and welfare. Agents in an economy that is capable of sustaining a smooth growth path may prefer a more volatile alternative growth path. However, there is a limit to the positive welfare effect of a long cycle.

The model developed here is closely related to that developed by Shleifer [14] where the focus was similarly on pecuniary externalities in generating cycles. Given the models' similarities, we pause here to explicitly spell out the substantive differences. Shleifer's analysis pre-dates the modern literature on endogenous growth through innovation\(^3\) and therefore necessarily treats the innovation process in an ad hoc fashion. Innovations arrive to firms smoothly, exogenously, without costs, and in a pre-determined and highly specialized order across sectors. Innovators simply decide when to implement and do not invest in creating innovations. Here we model firms' intertemporal innovation decisions, as in this growth literature, but extend this literature by explicitly considering innovation in a multi-period framework.

As in Shell [13], Grossman and Helpman [7], and Aghion and Howitt [1] we take the view that innovation should properly be considered as an activity taking place through time: firms devote current resources in hope of future rewards. However, as well as deciding on the total amount of resources to devote to innovation, firms must also decide on the timing of arrival and the intertemporal allocation of those resources. As innovations generate productivity increases, and as productivity increases are central to any real cycle, we feel it important to explicitly model the process of innovation and not just implementation. In fact, we shall impose restrictions on parameter values so that delay in implementation is never optimal for any innovator.

The focus on innovation rather than implementation cycles generates marked differences in welfare conclusions. Shleifer's implementation cycles do not improve welfare without the existence of fixed costs to implementation,\(^2\) A similar mechanism of inter-industry pecuniary connections is explored in Baland and Francois [2], though there the focus is growth traps and underdevelopment.\(^3\) Roemer [11], Aghion and Howitt [1], Grossman and Helpman [7], Segestrom, Anant and Dinopolous [12], though the contemporaneous work of Judd [10] provides a more thorough analysis of innovation.
and even with fixed costs, shorter cycles will always be preferred over longer ones. In our framework of innovation cycles welfare can rise with cycle length, even though the innovation technology is smooth and implementation is costless. This is because longer cycles allow longer gestation periods in research and improve the likelihood of successful innovation, which induces a form of self-enforcing multiplier connection between innovating firms. This together with the intertemporal innovation externalities that are essential for sustaining growth, may create a social preference for longer cycles.

The contribution of this paper is thus in generating cycles which arise endogenously from innovating firms' maximizing behavior, and in showing that longer cycles can improve welfare. The aim is to explicitly link the processes causing growth to those causing cycles. To emphasize the general equilibrium effects at the heart of our analysis, we therefore abstract from considerations which have been shown to generate cycles in other similar models, even though those may be complementary to the mechanism analyzed here.

The cycle developed here is entirely different from that which occurs in Aghion and Howitt [1] even though their model is similarly based on growth through technological innovation. In their model, a representative innovating firm undertakes the economy's aggregate level of research. The current firm's incentive to innovate is linked to future innovation levels because future firms will eventually displace them from their leading position with new innovations. High expected future levels of research lower current innovation incentives since profit as industry leader is expected to

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4 A further distinction arises between the fixed costs model of Shleifer's paper, Appendix B, and our framework. His costs of implementation have to be non-convex, i.e. a fixed cost, $F$, which is a lump sum. However, in our analysis, one is able to invest in innovation smoothly and the production function of innovations is weakly concave. Instead of a non-convexity, our result relies on diminishing marginal returns in production being offset by the increase in income with a higher cycle, so that firms will invest more in a higher cycle.

5 A different approach to the beneficial impact of cycles is analyzed in Caballero and Hammour [3]. Their paper looks at the benefits of reallocating resources in a downturn, where opportunity costs are lower, so that cycles can allow more efficient allocations while minimizing adjustment costs. However, they show that specificity in production and incompleteness in contracting between workers and firms can scuttle this potentially beneficial effect of downturns and argue that observed patterns of employment creation and destruction in recession suggest a pessimistic view of their role.

6 Cheng and Dinopoulos [4] are the most directly related. They similarly deal with a Neo-Schumpeterian process of growth through innovation but establish cycles by introducing a distinction between breakthroughs and improvements to capture the notion of technological opportunities eventually diminishing after a major innovation, see also Jovanovic and Rob [9]. Gale [6] characterizes equilibria in a class of games in which exogenously posited complementarities between investors can lead to welfare reducing delay in implementation and cycles. An important departure from his approach is our endogenous modelling of the magnitude of investment which can lead cycles to be welfare improving.
be short lived and currently low levels of innovation vindicate previous decisions to invest heavily in research, since successful innovators in the immediate past are unlikely to be displaced by the presently low levels of activity. Thus an expectations driven cycle in innovation and hence output levels develops. The representative firm assumption focuses their analysis on innovations within a sector, or product line, and rules out analysis of interaction between sectors, which are the key to the cycle developed here. Similarly, Deneckere and Judd [5] analyze cycles in an endogenous innovation framework. In addition to the significant structural differences in the innovative process and the prominent role of obsolescence in their framework, the mechanism leading to cycles is fundamentally distinct. In their framework, incentives to innovate have a non-linear dependence on current levels of innovation. This non-linearity can give rise to cyclical dynamics; high innovation and therefore increased variety induces low innovation and a decline in variety which in turn raises incentives for future innovation. Thus, as in Aghion and Howitt [1], it is the interdependence between distinct generations of innovators which gives rise to cycles. The present paper, in contrast, examines complementarities between multiple contemporaneous innovators who are located in non directly related sectors and is, in this sense, more closely related to the direction taken by Shleifer [14].

The paper proceeds as follows. Section 2 develops a simple model in which we specify the intertemporal process of innovation, rewards to innovators, imitation lengths and solve for general equilibrium prices. Section 3 establishes conditions for the existence of a cycle. Section 4 examines the qualitative features of the cycle and the relationship between cycle length and welfare. Section 5 concludes and the appendices supply necessary proofs.

2. THE MODEL

2.1. Goods and Technologies

The basic structure of the economy resembles a standard quality-ladder model examined by Grossman and Helpman [7], except for the innovation technology. Time is discrete, denoted by $t$, and agents live forever. The economy comprises a competitive final good sector and a unit interval of intermediate good sectors. There is a continuum of agents with a unit measure and each agent has one unit of primary input, labor, which can be allocated to two activities, innovating new technologies and producing intermediate goods. The wage rate for both activities is $w$ in terms of final goods. Innovation and imitation take place in the intermediate good sector. The innovation decisions are the novel part of this paper which will be
described in detail later. In contrast, the modelling of imitation is kept to a minimum. As in Shleifer [14] we assume that an innovation is imitated costlessly after one period.

The available production technologies in intermediate sector \( i \in [0, 1] \) at time \( t \) are represented by the marginal productivity of labor delivered by the technologies, which depend on the successive quality innovations that the sector has had. Normalize the productivity of labor delivered by the most preliminary technology in all intermediate sectors to one. Each successive innovation raises the productivity by a factor \( \gamma_{i} > 1 \) in sector \( i \).

To simplify, let \( \gamma_{i} = \gamma \) for all \( i \). If \( k_{i} \) is the number of successive innovations having occurred in sector \( i \), the output in sector \( i \) is

\[
x_{it} = \gamma^{k_{i}} l_{it},
\]

where \( l_{it} \) is the amount of labor used in production. Competition between intermediate producers and constant returns to scale in each sector ensure that the most recent technology is used in each sector each period.

Intermediate goods are used to produce the economy's unique final good according to a constant-returns-to-scale technology. Output of the final good, denoted \( y \), is as follows:

\[
y_{t} = \exp \left( \int_{0}^{1} \ln x_{it} \, di \right). \tag{2.1}
\]

The unit price of the final good is one in all periods. As in similar models, for simplicity, production of the final good does not require labor inputs. For simplicity, we also assume that the final good is perishable and so all final goods are consumed each period.\(^7\) Given the price \( p_{i} \) for each input \( i \), the profit maximization problem of a final good producer is

\[
\max_{x} \left[ \exp \left( \int_{0}^{1} \ln x_{it} \, di \right) - \int_{0}^{1} p_{i} x_{it} \, di \right],
\]

which yields the following demand for each input \( i \):

\[
x_{it}^{d} = \frac{y_{t}}{p_{i}}. \tag{2.2}
\]

Notice that the demand for one sector's input is independent of the price in other sectors so that there is no direct complementarily between innovations on the demand side.

\(^7\)One can allow for intertemporal consumption decisions but the gist of the exercise will remain the same.
2.2. Profits in Each Intermediate Sector

Assume all sectors are symmetric and ignore, for the moment, the process generating innovation and imitation. With the wage rate $w$, the profit of a producer of intermediate good $i$ is $p_i x_i - w_i P_i$. There are two cases to consider.

The first case is when the most recent innovation in sector $i$ (with productivity $\gamma^k_i$) has not yet been imitated. In this case, the sector is monopolized by the innovator who uses limit pricing to squeeze out the competitors (incumbents). That is, the innovator's marginal cost of labor, $w_i/\gamma^k_i$, is lower than the incumbents' cost, $w_i/\gamma^k_i - 1$, and he can set the price below the incumbents' marginal cost by an arbitrarily small amount $\varepsilon > 0$. In the limit $\varepsilon \to 0$, the price is $p_i = w_i/\gamma^k_i - 1$ and the innovator is the only producer in that sector. When the market for good $i$ clears, the innovator's profit is

$$\pi_a = p_i x_i - w_i \frac{x_i}{\gamma^k_i} = \left(1 - \frac{1}{\gamma^k_i}\right) p_i x_i = \left(\frac{\gamma - 1}{\gamma^k_i}\right) Y_i, \quad (2.3)$$

where we have used (2.2). Notice that each successful innovator's profit increases with aggregate income—a feature that underlies the central result of this paper.

The second case is when the most recent innovation has already been imitated. In this case, there can be more than one producer in the sector and Bertrand competition yields zero profit for each producer. The price for good $i$ is thus $p_i = w_i/\gamma^k_i$ and the sector's output again equals the demand given by (2.2).

2.3. Labor Market Clearing

Labor is used in production and innovation. The demand for labor in production by each intermediate sector $i$ can be deduced from the production technology and the demand for the sector's goods, (2.2):

$$l_i = \frac{x_i}{\gamma^k_i} = \frac{Y_i}{\gamma^k_i P_i}.$$

If the leading technology in the sector has not yet been imitated in sector $i$, the demand for labor is $l_i = Y_i/(\gamma w_i)$; if imitation of the leading technology has already occurred in the sector, the demand for labor is $l_i = Y_i/w_i$. Monopolistic (pre-imitation) sectors use less labor since they raise price and reduce output by limit pricing at competitors' marginal costs.

The extreme Bertrand assumption used here is not critical. All that is required for the persistence of the paper's qualitative results is that profit of incumbents in sectors with a unique firm having a technological lead are lower than those in sectors where competitors have equivalent technologies.
If \( \xi_t \) denotes the measure of monopolistic sectors in period \( t \), then total labor used in producing intermediate goods is

\[
L^p_t = \xi_t \frac{y_t}{w_t} + (1 - \xi_t) \frac{y_t}{w_t}.
\]

Let \( l_i \) be the amount of labor used in innovation in sector \( i \) and \( L_t \) be the sum of such labor inputs in all sectors. The labor market clearing condition is:

\[
1 = L^p_t + L_t = \xi_t \frac{y_t}{w_t} + (1 - \xi_t) \frac{y_t}{w_t} + \int_0^1 l_i \, dl,
\]

which yields

\[
\frac{y_t}{w_t} = (1 - L_t) \frac{y_t}{y - (y - 1) \xi_t}. \tag{2.4}
\]

The important feature of (2.4) is that the ratio of aggregate income to wages is increasing in \( \xi_t \), so that income is high relative to wages when there is a large proportion of sectors that are monopolistic. This is so because monopolistic sectors use new technologies that yield profit in addition to wages. Since \( \xi_t \), in equilibrium, is also equal to the fraction of innovators who have just succeeded, the positive dependence of demand on \( \xi_t \) represents a pecuniary externality among innovators. That is, if there are more firms trying to bring about new technologies at the same date, aggregate income will be higher when the innovators implement the new technology and so the profit for any other innovator at that target date rises.

It is worth repeating that this externality works through the market clearing condition, without the need for complementarity or spillover assumed in other models of innovation.

Innovators' decisions are also inter-dependent through the aggregate labor input into production of the final good. If more innovators target the same date for the innovation process, more labor will be released from innovation to the production of the final good when the innovators implement their new technologies. That is, \( 1 - L \) will be higher in the period of implementation if more innovators target the same date. This also contributes to a higher income-wage ratio.

2.4. The Innovation Activity

Innovators build on the previous generation of technology. To simplify the discussion, we continue to suppress the sector index \( i \). At time \( t + 1 \), a
research firm targets a date, \( t + n \), at the end of which the innovation process terminates.\(^{10}\) We call this innovation process an \( n \)-period innovation. During the innovation process the firm chooses a labor input profile \( \{l_{t+i}\}^n_{i=1} \), which generates project-specific human capital

\[
h^n_{t+n} = h(l_{t+1}, l_{t+2}, \ldots, l_{t+n}, n),
\]

where the superscript of \( h \) indicates the start and the subscript indicates the end of the project. This project-specific human capital determines the probability of success of the innovation at the end of the innovation process. In particular, the probability of success at the end of period \( t + n \) is an increasing function of the human capital:

\[
\phi(h^n_{t+n}).
\]

The probability of success at any time before \( t + n \) is 0. The project-specific human capital depreciates completely after the outcome of the innovation is realized, successful or not. If an innovation project is successful, the innovation is ready for use (implementation) in period \( t + n + 1 \). In the period immediately following implementation the new technology is imitated by other firms without cost.

The physical environment that we have in mind is an innovation process where ideas and blueprints must be tried in order to determine whether they work, i.e., there is project specific uncertainty which is resolved only by undertaking the project. If the blueprints or prototype are not ready (before the scheduled project length \( n \)), they cannot be usefully checked. If the project is found to be unsuccessful, the firm must try a different project in which previous efforts do not enhance the chance of success of the new project.\(^{11}\)

\[^{10}\] We do not explicitly detail analysis of the firm’s source of financing since this has already been thoroughly analyzed in Grossman and Helpman [7], and has no interesting effects on the cycle. For simplicity, it can be assumed that the firm finances its factor payments by selling equity shares in its future expected profit stream to risk averse investors. Thus, under this interpretation, all individuals in the economy will hold a diversified portfolio of equity in the economy’s innovation firms and receive non-stochastic, though time varying, income flows. Firms will thus also pursue profit maximizing investments without risk aversion.

\[^{11}\] Why does an innovator choose an \( n \)-period innovation project rather than following a sequential strategy each period? There are a number of possible underlying reasons. Firstly, it may simply be a realistic feature of the innovative process that multi-period projects cannot be equivalently broken down into multiple single period actions. Also, with sufficiently high fixed costs in setting up an innovation, once firms commit to a time frame, it may not be optimal to sequentially reconsider the investment in the light of project specific information each period. Alternatively, there may be insufficient project specific information to warrant reconsideration of the project until the innovation is complete. For example, information about the success of new consumer products is not available until the design is created and tested on consumers.
To simplify the analysis, for now we assume $\phi$ is linear, so that $\phi(h) = \phi h$, and the probability of success equals $\phi h$ if $h \leq 1/\phi$ and 1 for any level of $h$ higher. Moreover, the project-specific human capital $h$ has a CES type functional form:

$$h_{t+n}^{\rho+1} = \left(n^{-\phi(1+\theta)} \sum_{s=1}^{n} p_s^{(1+\theta)}\right)^{1/\theta+1}, \quad -1 < \theta < \infty,$$

with $1 + \theta (> 0)$ denoting the elasticity of substitution between labor inputs at different times. The following assumptions on $h$ are imposed:

$$\frac{\theta}{1 + \theta} \leq \rho < \theta. \quad (2.6)$$

These restrictions are motivated by the following considerations. Consider innovation processes that have the same fixed amount of labor in each period, $l$. For such innovation process with a length $n$, the human capital at the end of the project is

$$h_n = l \cdot n^{(\theta - \rho)(1 + \theta)}.$$
delay, it is desirable to assume that such human capital does not increase with \( n \) while \( l \) is fixed. This is satisfied if and only if \( \rho \geq \theta/(1 + \theta) \).

An innovator’s maximization problem is:

\[
\max_{(l,n)} \left\{ \beta^{n+1} \pi_{t+n+1} \phi \cdot h(l_{t+1}, \ldots, l_{t+n}; n) - \sum_{s=1}^{n} \beta^{s} w_{t+s} l_{t+s} \right\}.
\]

Since the function \( h \) has a CES type, the maximization problem can be divided into two stages. In the first stage, one solves for the profile of labor inputs that minimizes the labor cost under any given \( h \):

\[
\min \sum_{s=1}^{n} \beta^{s-1} w_{t+s} l_{t+s}
\]

s.t. \( \left( n - \rho \theta \sum_{s=1}^{n} \beta^{s} (1 + \theta)^{s} \right) \geq h. \)

Denote the wage profile by \( W = (w_{t+1}, \ldots, w_{t+n}) \). The minimization problem yields:

\[
\begin{align*}
I_{t+1} &= I_{t+1}(h, W) \equiv h \cdot \rho^{\theta (1 + \theta)^{n}} \left[ \sum_{s=1}^{n} \left( \beta^{1-s} \frac{W_{t+s+1}}{W_{t+s}} \right)^{(1 + \theta)^{s}} \right]^{-1/(1 + \theta)^{n}}, \quad (2.7) \\
I_{t+s} &= I_{t+s}(h, W) \equiv I_{t+s}(h, W) \cdot \left( \beta^{1-s} \frac{W_{t+s+1}}{W_{t+s}} \right)^{(1 + \theta)}. \quad (2.8)
\end{align*}
\]

These solutions for \( l \) yield the following unit cost of human capital \( h \), expressed in terms of labor in period \( t+n+1 \):

\[
G(n, W) \equiv \frac{1}{\rho^{\theta n(1 + \theta)^{n}}} \left[ \sum_{s=1}^{n} \left( \beta^{1-s} \frac{W_{t+s+1}}{W_{t+s}} \right)^{(1 + \theta)^{s}} \right]^{-1/(1 + \theta)^{n}}.
\]

In the second stage of the innovator’s problem, the firm solves for \((h, n)\). Substituting (2.7) and (2.8) into the objective function yields the following expected profit for the innovator:

\[
\beta^{n} h \left[ \pi_{t+n+1} \phi - w_{t+1} G(n, W) \right] \quad (2.9)
\]
Choosing $h$ to maximize (2.9) yields:

$$h = \begin{cases} 
0 & \text{if } \frac{\pi_{t+n+1}}{w_{t+1}} < G(n, W)/\phi \\
\frac{1}{\phi} & \text{if } \frac{\pi_{t+n+1}}{w_{t+1}} = G(n, W)/\phi \\
\frac{1}{\phi} & \text{otherwise.} 
\end{cases} \quad (2.10)$$

The linearity of $\phi$ (as in Grossman and Helpman [7]) leads to the solution’s sensitive dependence on the unit cost of human capital, $G$. The innovator inputs a positive amount of labor into the innovation only when the expected rate of return to human capital, $\frac{\pi_{t+n+1}}{w_{t+1}}$, is greater than or equal to the unit cost of human capital, $G(n, W)$.

Given the solution for $l$, we will refer to the pair $(h, n)$ as the innovator’s strategy. Since the conditions in (2.10) depend on the project length $n$, (2.10) also implicitly characterizes the choice of $n$ which we examine in the next section. Note from (2.3) and (2.4) that the profit associated with a successful innovation, $\pi_{t+n+1}$, depends on the innovation strategies of other firms through $\pi_{t+n+1}$. Thus, the above result implicitly defines each individual innovator’s innovation strategy as a best response correspondence to aggregate innovation strategies. Let $(H, N)$ denote other innovators’ choices of $h$ and $n$ respectively. (2.10) may be written as $h = h(H, N)$.

3. EQUILIBRIUM

Suppose now that all other innovators choose $N$ as the length of innovation and implement successful innovations immediately next period, the conditions under which this holds will be established subsequently. The time sequence of events through such an $(N + 1)$-cycle is depicted in Fig. 1. Let $t + 1$ be a generic date where the last innovation is imitated and where the new cycle starts, with $L_{t+1} > 0$ aggregate labor input in innovation. The new innovation process continues until the end of $t + N$ where innovative effort finally ceases and project successes or failures are realized. In period $t + N + 1$ successful innovators implement. Knowledge of the new inventions diffuses at the end of that period so that, at the start of period $t + N + 2$ imitators compete away profits. Diffusion of knowledge also allows investment in the next generation of technology to commence so that $L_{t+N+2} > 0$ and the cycle starts anew. Over the stationary cycle
$L_{t+1+N+2} = L_{t+1}$ and these efforts bear fruit in period $t + 2N + 1$ for implementation in $t + 2N + 2$, so that the cycle repeats itself every $N+1$ periods. We now establish the conditions under which such an $(N+1)$-cycle exists.

If all innovators choose $N$ as the length of innovation then $H = h(H, N)$, where the correspondence $h(H, N)$ is implicitly defined by (2.10). Under the linear specification of the $\phi$ function used here the fixed points of the correspondence $h(\cdot, N)$ are 0, $1/\phi$ and an interior value $H^* \in (0, 1/\phi)$. The fixed point $H = 0$ is not interesting as it implies that there is no innovation. The fixed point $H = 1/\phi$ is an equilibrium in which innovation activity is so high that inventions are successful with probability one, i.e., $\phi H = 1$. We ignore this equilibrium because it implies certainty in innovation success with sufficiently large labor input, which is unrealistic and is simply an artifact of the simplifying linearity assumption. Furthermore, modifying the $\phi$ function to rule out the possibility of certainty in innovation success eliminates $\phi H = 1$ as an equilibrium. In what follows we focus on the interior solution for $H$, which will be denoted $H(N)$.

We refer to the pair $(H(N), N)$ as the aggregate innovation strategy, where $H(N)$ is given by $\pi_{t+1+N+1}/w_{t+1} = g(N)/\phi$. To find $H(N)$, note first that if all innovators choose $N$ as the length of innovation then

$$w_{t+1+N} = w_{t+1}$$

for all $s \in \{1, 2, \ldots, N+1\}$.

With all innovators following the same strategy described by $\{L_{t+1}, \ldots, L_{t+1+N}; N\}$, the fraction of monopolistic intermediate sectors in the period of implementation is equal to the success probability of innovation, which is $\pi_{t+1+N+1} = \phi H$. Later we will also show that no firm has the incentive to innovate in the period of implementation so that $L_{t+1+N+1} = L_{t+1} = 0$. Then (2.3) and (2.4) imply

$$\frac{\pi_{t+1+N+1}}{w_{t+1+N+1}} = \frac{\pi_{t+1+N+1}}{w_{t+1}} = \frac{\gamma - 1}{\gamma - (\gamma - 1) \phi H}.$$ (3.2)

Also, given the flat wage profile, the unit cost of human capital for $n = 1, 2, \ldots, N+1$ becomes

$$g(n) \equiv G(n, W)|_{W = \{w_{t+1}^1\}_{t=1}} \equiv n^{\rho(1+\rho)/\rho} \left( \frac{B - 1}{B^{1-n}} \right)^{1/\theta}.$$ (3.3)
Therefore, $H(N)$ is given by

$$H(N) = \frac{\gamma}{\phi(\gamma - 1)} \frac{1}{g(N)}.$$  \hspace{1cm} (3.4)

For $0 < H < 1/\phi$ it is necessary that

$$\frac{g(N)}{\gamma - 1} < \frac{\gamma \cdot g(N)}{\gamma - 1}.$$  \hspace{1cm} (3.5)

In Appendix A the following lemma is demonstrated:

**Lemma 3.1.** Under (2.6), the function $g(n)$ defined above is an increasing function of $n$ so that, for $1 \leq n \leq N + 1$, $G(n, W)$ is also increasing in $n$. Moreover, $H(N)$ is increasing and concave in $N$.

The properties of $H(N)$ will be important for the later discussion on welfare. For now, let us denote aggregate labor input into innovation at time $t + s$ by $L_{t+s}$. With the wage profile in (3.1), the dependence of $(l_{t+1}, ..., l_{t+N})$ on $W$ vanishes and so, for $n \leq N + 1$, the aggregate forms of (2.7) and (2.8) generate:

$$L_{t+1} = H \cdot R^{e(1+\theta)\theta} \left( \frac{\beta - \theta - 1}{\beta - \theta - 1} \right)^{(1+\theta)\theta},$$  \hspace{1cm} (3.6)

$$L_{t+s} = L_{t+1} \beta^{(1-s)(1+\theta)} \quad s = 1, 2, ..., N.$$  \hspace{1cm} (3.7)

Aggregate labor inputs into innovation rise and aggregate labor inputs into production of the final good fall over time as the innovation process continues. Since the fraction of monopolistic intermediate sectors and the wage rate remain constant between $s = 1$ and $N - 1$, output of the final good falls (see (2.4)), which implies that the profit of a successful innovation falls. The following lemma summarizes these results and provides the condition for $L_{t+s} < 1$ (see Appendix B for a proof).

**Lemma 3.2.** For all $1 \leq s \leq N - 1$, the aggregate labor input into innovation, $L_{t+s}$, and the profit, $\pi_{t+s}$, satisfy $L_{t+s+1} > L_{t+s}$ and $\pi_{t+s+1} < \pi_{t+s}$. Moreover, (3.5) implies $L_{t+s} < 1$ for all $s = 1, 2, ..., N$.

The rising labor input into innovation and the falling innovation profit is reversed in period $t + N + 1$ when the new innovation is implemented. (3.2) shows that the profit/wage ratio is higher in $t + N + 1$ than in period $t + 1$ and hence higher than the profit/wage ratio in any other periods $t + 2$ through $t + N$. This is because not only labor inputs into production of
the final goods increase in $t + N + 1$ ($1 - L_{t+N+1} = 1$) but also because implementing new innovation increases output (the presence of $\phi H$).

We now find a necessary and sufficient condition for $(H, N)$ and the implied labor input profile to form the described cyclical equilibrium. This is done by checking that no individual innovator wants to choose $n \neq N$ and that no innovator wants to delay implementation of a successful innovation. First, we rule out $n < N$ by the following proposition:

**Proposition 3.3.** Given that all other innovators use $(h, n) = (H, N)$, no individual innovator has incentive to choose $n < N$ if and only if

$$\frac{1}{\beta \phi} > \frac{\gamma - 1}{\gamma} (1 - L_{t+2}). \tag{3.8}$$

**Proof.** We first demonstrate that, if an innovator ever wants to deviate to $n < N$, the best deviation is $n = 1$. That is, the choice $n = 1$ is better than any other choice $2 \leq n \leq N - 1$. Lemma 3.1 states that $g(n)$ rises with $n$ for all $n \leq N + 1$. Lemma 3.2 says that the profit of a successful innovation $\pi_{t+n+1}$ falls with $n$ for all $n < N - 1$. Since the wage rate remains at $w_{t+1}$ for periods $t + 1$ through $t + N + 1$, for any $2 \leq n \leq N - 1$ we have

$$\frac{\pi_{t+n+1}}{w_{t+1}} g(n) < \frac{\pi_{t+n+1}}{w_{t+1}} < \frac{g(1)}{w_{t+1}} \phi, \frac{\pi_{t+2}}{w_{t+1}} < \frac{g(1)}{w_{t+1}} \phi.$$

Therefore, a choice $n = 1$ is indeed better than any choice $n \in \{2, 3, ..., N - 1\}$.

We now show that the choice $n = 1$ is dominated by the equilibrium strategy $n = N$ if and only if (3.8) holds, and so (3.8) implies that all choices $n < N$ are dominated by the equilibrium strategy. Consider an individual innovator who chooses $n = 1$. If he is successful, implementing the new technology in $t + 2$ generates a profit $\pi_{t+2}$. When $N \geq 2$, the fraction of monopolistic sectors is 0 in period $t + 2$ since the innovator’s sector is the only one in a continuum that might have succeeded in the innovation by then. In this case, $y_{t+2}/w_{t+2} = 1 - L_{t+2}$ by (2.4) and the expected rate of return to the innovator is

$$\frac{\phi \pi_{t+2}}{w_{t+1}} = \phi \cdot (1 - L_{t+2}) \frac{\gamma - 1}{\gamma}.$$

(3.8) is necessary and sufficient for this rate of return to be less than the corresponding unit cost of human capital, $g(1) = 1/\beta$. By (2.10), the innovator will not choose $n = 1$ if and only if (3.8) holds.
We now consider the possible deviation to \( n > N \). Intuitively, even when there is positive discounting, incentives for delay may arise from the potential increase in aggregate income from period \( t + N + 1 \) to \( t + N + 2 \), which in turn arises from the diffusion of technology due to imitation in period \( t + N + 2 \). By innovating till period \( t + N + 1 \) rather than \( t + N \), the innovator reaps her one period of profit when aggregate income is potentially higher and may thus obtain greater profit. However, the incentive to delay is mitigated by the aggregate allocation of productive resources to new research, \( \Lambda_{t + N + 2} \). Also, the innovator will not wish to delay for too long since wages rise with the influx of imitators, as we now establish.

To illustrate, let us first calculate aggregate income, wage rate and profit in period \( t + N + 2 \). The result is stated in the following lemma (see Appendix C for a proof):

**Lemma 3.4.** In an \((N + 1)\)-cycle stationary equilibrium income, wage rate and profit of implementing a successful innovation in period \( t + N + 2 \) are

\[
\begin{align*}
    y_{t + N + 2} &= \gamma^{\phi H} y_{t + 1}, \\
    w_{t + N + 2} &= \gamma^{\phi H} w_{t + 1}, \\
    \pi_{t + N + 2} &= \gamma^{\phi H} \pi_{t + 1}.
\end{align*}
\]

(3.9)

This lemma states intuitively that, since a fraction \( \phi H \) of the sectors have upgraded the technology by \( \gamma \) in period \( t + N + 2 \) relative to period \( t + 1 \), aggregate income and the wage rate are both higher in \( t + N + 2 \) than in \( t + 1 \) by a factor \( \gamma^{\phi H} \). If an individual innovator implements a new technology in \( t + N + 2 \), he is able to capture a higher demand generated by the higher aggregate income. The profit is thus higher in \( t + N + 2 \) than in \( t + 1 \) by a factor \( \gamma^{\phi H} \). It is also possible that per period profit to a successful innovation may rise in period \( t + N + 2 \) relative to period \( t + N + 1 \), as stated in the following lemma (see Appendix D for a proof):

**Lemma 3.5.** Suppose all innovators choose \( N \) as the length of innovation, and \( \Lambda_{t + N + 2} = 0 \), then \( \pi_{t + N + 2} > \pi_{t + N + 1} \).

However, in an \((N + 1)\)-cycle, \( \Lambda_{t + N + 2} > 0 \), and delaying innovation also delays the stream of profits. These disincentives can be strong enough to prevent delay in innovation.

**Proposition 3.6.** Given that all innovators succeeded at time \( t + N \) and implement the new innovation at \( t + N + 1 \), an individual innovator will choose not to delay innovation past \( t + N \) if and only if

\[
L_{t + 1} > 1 - \frac{\gamma^{1 - \phi H}}{\gamma - \phi} \frac{g(N + 1)}{\phi},
\]

(3.10)
Proof. Lemma 3.2 implies that in a stationary cycle, for all \( 1 \leq s \leq N - 1 \), \( \pi_{t+N+2+s} > \pi_{t+N+2+s} \). Also from (3.9), \( w_{t+N+2} > w_{t+N+1} \) and wage rates are the same in \( t+N+2 \) through \( t+2N+2 \). Thus, if an innovator ever chooses to delay innovation past \( t+N \), delaying innovation to \( t+N+1 \) and implementing at \( t+N+2 \) is better than delaying to any other period beyond \( t+N+1 \). Thus it suffices to verify that the innovator does not delay innovation to \( t+N+1 \). To do so, we demonstrate that (3.10) is equivalent to

\[
\frac{\pi_{t+N+2}}{w_{t+1}} < \frac{G(N+1,w)}{\phi}.
\]

(3.11)

Then (2.10) implies that the strategy \((h,N+1)\) is sub-optimal for the innovator if (3.10) holds. Using Lemma 3.4 and the fact \( L_{t+N+2} = L_{t+1} \) in a stationary cycle we have

\[
\frac{\pi_{t+N+2}}{w_{t+1}} = \left( \frac{\gamma - 1}{\gamma} \right) (1 - L_{t+1}) r^H.
\]

And, since \( w_{t+N+1} = w_{t+1} \), we have \( G(N+1,w) = g(N+1) \) from equation (3.3). Substituting these into (3.11) yields (3.10).

Note the distinction with Shleifer's [14] incentives for delay that arises from the optimal time path of innovations in our framework. In Shleifer's analysis, the only factor ruling out delay to obtain higher nominal profits is discounting. In contrast, here, the increase in nominal profits is mitigated by the outflow of resources to innovation. Lemma 3.2 implies a strictly increasing path for aggregate labor input into innovation through the downturn. Thus, upon implementation, resources devoted to production are at a maximum, since fewer resources are being invested in the next innovation. If a firm were to delay innovation one period, it could, in the extreme, face smaller aggregate non-discounted income (if the effect of increased labor in research outweighed the improvement in allocation after imitation).

Finally we examine an innovator's incentive to delay implementation. That is, under what conditions does an innovator who succeeds at the end of period \( t+N \) choose to implement the innovation at \( t+N+1 \) rather than a later date? The incentive to delay implementation of an already successful innovation at \( t+N+1 \) is similar to the incentive to delay innovation examined above. However, the two types of delays have subtle differences. At \( t+N+1 \) when an innovator already has a successful innovation in hand, the implementation decision depends only on the relative profit between implementing immediately and later—past labor costs are sunk and out of consideration. On the other hand, at \( t+1 \) when an innovator
embarks on an innovation process, the outcome of innovation is unknown and the decision on whether to innovate until \( t + N \) or \( t + N + 1 \) depends on the labor costs planned for all the periods. Ruling out one type of delay does not automatically rule out the other.

**Proposition 3.7.** If all other innovators choose \((h, n) = (H, N)\) and implement successful innovations at \( t + N + 1\), a successful innovator does not choose to delay implementation if and only if

\[
L_{t+1} > 1 - \frac{1}{\beta} \left[ H^\gamma (1 - \phi H) + \phi H^{\gamma H - 1} \right]^{-1}. \tag{3.12}
\]

**Proof.** As in the case of delaying innovation, if an innovator ever wants to delay implementation, delaying to \( t + N + 2 \) is better than delaying to any other periods. Thus we only have to rule out the possibility of delaying implementation to \( t + N + 2 \), which is ensured if and only if \( \beta \pi_{t + N + 2} < \pi_{t + N + 1}. \) The condition can be rewritten as

\[
\beta \frac{\pi_{t + N + 2} + 2}{w_{t+2} + 1} \left( \frac{\gamma - 1}{\gamma} \right) (1 - L_{t+1}) H^{\gamma H} < \frac{\gamma - 1}{\gamma - (\gamma - 1)} \phi H = \frac{\pi_{t + N + 1}}{w_{t+1}}.
\]

Rearranging the inequality yields (3.12). \( \square \)

We summarize the existence conditions implied from the above analysis.

**Corollary 3.8.** The necessary and sufficient conditions for the existence of a symmetric \((N+1)\)-cycle steady state are: (3.5), (3.8), (3.10) and (3.12), where \( L_{t+1} \) and \( H(N) \) are given by (3.6) and (3.4).

In the next section we will give numerical examples to show that there are equilibria with \( N \geq 2 \) and compare welfare across equilibria with different cycle lengths. Before doing so however, it is worth contrasting these equilibria with implementation cycles. Firstly note that explicitly modelling innovation rules out an arrival process as assumed by Shleifer, where innovations arrive in a uniformly dispersed fashion through the cycle. This follows directly from Lemma (3.2), where it is seen that if a firm desires implementation in period \( t + 1 \) it is optimal for it to input resources to innovation along a monotonically increasing trajectory up until period \( t \). Similarly, there cannot be delay in implementation. A firm delaying implementation could always do better by delaying innovation and implementing immediately. However, in a slightly more general setting than the one here,
implementation could again arise. For instance, if there were some uncertainty as to the arrival date of innovations, in an $N$ cycle, firms would aim to innovate after $N$ periods, but those receiving innovations before or after this time may have incentive to delay until the main body of economy wide innovations are implemented.\footnote{We thank an anonymous referee for pointing out this possibility.}

4. WELFARE

4.1. The Welfare Measure

Using an equally weighted utilitarian welfare concept, welfare can be measured by aggregate income which comprises profit from the production of intermediate goods and wage income. Because all incomes are spent in the same period, aggregate income must be equal to aggregate consumption of the final good which in turn must be equal to output of the final good in every period.\footnote{To see this identity between income and output $y_t$, consider period $t+s$ in the innovation process that started at $t+1$, $s \in \{1, 2, \ldots, N\}$. Since no new innovation is implemented in $t+s$ the profit is zero and the welfare measure is $(1-L_{t+s})w_{t+s} \cdot 0 = y_{t+s}$.} Equilibria with different lengths of cycles can be compared using the discounted present value of aggregate income. Denote such present value in an $(N+1)$-period stationary cycle by $Y(N)$. Let the economy start at $t=1$ with a given set $\phi H$ of innovators and wage rate $w_1=1$. We can express income in each period within the $j$th cycle in terms of $y_{(j+1)(N+1)}$ and then express the latter in terms of $y_1$. The procedure leads to the following lemma (see Appendix E for a proof).

**Lemma 4.1.** The weighted income levels within the $j$th cycle in an $(N+1)$-period cyclical equilibrium are

\begin{align*}
y_{(j+1)(N+1)+s} &= (1-L_j) \gamma^j \phi H, \quad s = 1, 2, \ldots, N, \quad (4.1) \\
y_{(j+1)(N+1)} &= \frac{\gamma^{1+j} \phi H}{\gamma - (\gamma - 1) \phi H}. \quad (4.2)
\end{align*}

In period $t+1$, a fraction $\phi H$ of intermediate good producers implement new innovations, each generating profit $\pi_{t+1}$. The welfare measure is again $y_{t+1}$, as shown below:

\[\phi H(\pi_{t+1} + w_{t+1}) + (1 - \phi H)w_{t+1} = \phi H \frac{\gamma - (\gamma - 1) \phi H}{\gamma} y_{t+1} = y_{t+1}.\]
for all integers, \( j \). The discounted present value of income in such an equilibrium is

\[
Y(N) = \frac{(1 - \beta^{N+1})/(1 - \beta)}{1 - \beta^{N+1}H}\gamma^{\beta H}.
\] (4.3)

One way to understand the measure \( Y(N) \) is as follows. The sum of discounted output within each cycle \( j \), discounted to the first period of the cycle, is \((1 - \beta^{N+1})/(1 - \beta)\) times the output in the first period of the cycle. Since output grows at rate \( \gamma^{\beta H} \) between two adjacent cycles, the effective discount factor between cycles is \( \beta^{N+1}H \). Using this to discount output between cycles yields (4.3). Clearly, for (4.3) to be useful, we need \( \beta^{N+1}H < 1 \).

The welfare measure reveals two ways through which the innovation length \( N \) affects overall welfare. One is negative, represented by the appearance of \( \beta^{N+1} \). For any given level of the project-specific human capital, an increase in the project length makes new technologies take a longer time to come by and so reduces welfare. The second is positive, represented by the appearance of the project-specific human capital, \( H \). Recall from Lemma 3.1 that \( H \) is an increasing function of the innovation length \( N \). An increase in the innovation length yields a larger prize—more sectors will be successful in innovation and so aggregate output will jump up by a larger amount. The potential for welfare improvement is partly due to feedback effects between innovators. An increase in innovation in one sector, by increasing expected income, induces increased innovation elsewhere. Since these positive pecuniary externalities are not taken into account in decentralized equilibria, longer cycles, which induce higher investment can act to mitigate this externality.

The dependence of long-run growth on the cycle length turns on the trade-off between smaller but frequent jumps in output versus jumps that are more dispersed but larger in size. Welfare increases with the cycle length if and only if the project-specific human capital increases sufficiently with the innovation length:

\[
H'(N) > \frac{1 - \gamma^{\beta H}}{\phi(1 - \beta^{N+1})} \left( \frac{\ln \beta}{\ln \gamma} \right).
\]

Thus welfare may increase from an equilibrium with a very short innovation process to another equilibrium with a longer innovation. It should also be clear that too long an innovation process does not yield the maximum level of welfare either. While the marginal cost of discounting increases with the length of innovation, the marginal contribution of a longer innovation process to welfare falls because the project-specific human capital is a concave
function of the project length (see Lemma 3.1). Therefore, the length of the innovation process that maximizes welfare is finite.

There are numerous sources of inefficiency in equilibrium that we pause to consider more carefully. Firstly there is the already recognized allocative inefficiency caused by the successful $\phi H$ monopolists, see Shleifer [14, p. 1178]. This disappears after imitation in one period. Secondly there are those inefficiencies that arise due to the Schumpeterian growth process. That is, innovators provide positive externalities to future innovators by increasing the quality step from which future innovations depart while receiving returns to their innovation for a finite time only. The usual business stealing effect present in this literature as discussed in Aghion and Howitt [1], is absent however since monopoly profits are destroyed by imitators before the next innovation arrives. Finally, pecuniary externalities tend the economy towards an inefficiently low level of innovation. If, ceteris paribus, a firm increases its investment in innovation and hence its probability of successfully innovating, it raises expected aggregate income in the upturn and thus expected profits of all other investing firms. It is this positive externality which is not taken into account by investing firms, that is central to the possibility of welfare improving with cycle length, a case of which we now exemplify.

4.2. A Numerical Example

Here we numerically compare different $N + 1$ cycles satisfying the conditions in Corollary 1. We can use (4.1) and (4.2) to plot the graph of $y_t$ and compute the present value of output, $Y(N)$ for all cases. The following parameterization provides a good example of the forces affecting the path of output over different cycle lengths. We consider an economy with $\beta = 0.95^2$, $\gamma = 2$, $\theta = 0.1$, $\rho = 0.092$, and $\phi = 2$. The value of $\beta$ gives the interpretation of a period here as roughly two years. The choices of $\theta$ and $\rho$ satisfy (2.6).

This economy is capable of sustaining equilibrium cycles with $N = 1$ through 8. The top panel of Fig. 2 presents the plot of output levels for two such equilibria from periods 1 to 30, $N = 1$ (graph $Y_1$) and $N = 2$ (graph $Y_2$). This panel illustrates the feature that output growth is increased by switching from the equilibrium with the smallest cycle ($N = 1$) to a slightly longer cycle ($N = 2$). Although in the few periods at the beginning of growth the level of output corresponding to the $N = 2$ cycle is quite close to and sometimes below the output level corresponding to the $N = 1$ cycle, it soon overtakes its counterpart. This feature accords well with our earlier intuition. The smallest cycle provides only weak incentives for innovation since all others are investing a small amount and, in the period of implementation, the low level of aggregate research success implies wages exhaust a large part of aggregate income. In the $N = 2$ cycle, however,
larger scale implementation in the aggregate every third period implies a relatively larger jump in aggregate income relative to factor costs that provides innovators with relatively large incentives to innovate simultaneously. Similarly, the long run trend of output in the $N=3$ cycle exceeds that of the $N=2$ cycle, which eventually lifts the output path above that of the $N=2$ cycle. The increased growth rate from $N=1$ to $N=2$ and from $N=2$ to $N=3$ contrasts with Cheng and Dinopoulos [4] in which there is a

FIG. 2. Output levels of different cycle lengths.
TABLE 1
Cyclical Equilibria

<table>
<thead>
<tr>
<th>Innovation length (N)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>μ(N)(%)*</td>
<td>7.0</td>
<td>9.7</td>
<td>9.8</td>
<td>9.5</td>
<td>9.0</td>
<td>8.6</td>
<td>8.2</td>
<td>7.8</td>
</tr>
<tr>
<td>Y(N)</td>
<td>28.14</td>
<td>89.94</td>
<td>97.11</td>
<td>69.30</td>
<td>50.57</td>
<td>39.56</td>
<td>32.68</td>
<td>28.07</td>
</tr>
</tbody>
</table>

negative correlation between the duration of each cycle and the long run growth trend. The same contrast can be made with Shleifer [14] and Gale [6].

However, as argued before, too long an innovation cycle is not good either. The bottom panel of Fig. 2 supports this argument by plotting output paths for the \( N = 3 \) cycle and the longest cycle (\( N = 8 \)). Although implementation of each successful innovation in the longest cycle raises output by much more than that in the \( N = 3 \) cycle, such success comes so infrequently that the output path for the longest cycle stays below that of the \( N = 3 \) in most periods.

The non-monotonic relationship between the cycle length and economic performance is summarized in Table 1, where \( μ(N) \) is the average output growth rate per period defined by \( μ(N) = \frac{γ^N R(N)}{(N+1)} - 1 \). When \( N \) increases from 1 to 2, there is a large increase in the growth rate and welfare. The subsequent increase from \( N = 2 \) to \( N = 3 \) is much smaller. After \( N = 3 \), the growth rate and welfare begins to fall with the innovation length. Therefore the \( N = 3 \) cycle maximizes the average growth rate and welfare among all equilibria.16

The relationship between factor costs and output over the cycle is also of interest. The immediate output surge from implementation has no positive impact on wage payments since all benefits accrue as monopoly rents to the successful innovators. However, after imitation, competition destroys rents and in the process diffuses benefits to factors of production so that wages rise after a lag. Output may also increase with imitation since the distortion of monopoly pricing in the subset of successful industries is removed by the influx of imitators (an example is when \( ϕ = 2.3 \) and \( N = 4 \)).

The top panel of Fig. 3 charts the movements of the real wage rate relative to output over the cycle. The labor share of income is counter-cyclical, as shown in the bottom panel of Fig. 3. Positive deviations in the labor share from the long-run level precedes the output surge.

16 It is a coincidence in this example that the growth-maximizing and welfare-maximizing cycle length are the same. There exist examples where the two differ.
The existence of multiple equilibria makes comparative statics difficult. Nevertheless we can examine how the number of equilibria depends on parameters. The number of equilibria increases with the magnitude of each quality ladder, γ. This is because a longer innovation length can be supported by a larger improvement in productivity when the innovation is successful. Also, the number of equilibria falls when ρ increases. This is because for the same innovation length and same labor input profile a larger ρ implies lower project-specific human capital.
Equilibria also depend on the productivity of human capital, measured by $\phi$. For example, when $\phi$ increases from 2 to 2.3 in the above example, the $N = 1$ cycle is no longer an interior equilibrium—the productivity of human capital in this case is high enough to support $H = 1/\phi$ and $N = 1$ as an equilibrium. Among the interior equilibria, now equilibrium innovation length can be $N = 2$ through 9. The welfare and growth implications of such a change are, in general, not clear. A tendency to increase growth occurs because, for any given $H$, higher productivity of human capital increases the rate of return to its accumulation, but, in equilibrium, aggregate human capital must fall in order to restore equality between rates of return and unit costs of human capital, so that, in net, growth effects are ambiguous.

5. CONCLUSIONS

In this paper we have constructed a model where innovation cycles underly long-run growth. An implication is that there can be multiple stationary equilibria with different cycle lengths. The growth rate and welfare are non-monotonically related to the cycle length. Both can be increased from an equilibrium with a very short cycle to a longer cycle, but extending the cycle for very long also reduces the long-run growth rate and welfare. Thus, amongst equilibria of different cycle length, there exists a medium cycle length which maximizes welfare (and/or growth). This welfare-maximizing cycle length depends positively on the productivity of innovation. The potential welfare-improving role of longer cycles arises here simply because, with a longer innovation process, the innovator is able to build up higher project-specific human capital that makes innovation more likely to succeed.

The general interpretation of our result is that when the return to an individual’s innovation depends on both the length of gestation and aggregate demand, there is a tendency for innovation activities to cluster along the time dimension, which can be welfare improving. In the current setup, innovators can perfectly time their innovation length if they find it desirable. In reality, projects may experience some unexpected delay but the tendency for innovation activity to cluster remains.

Note that we have not formally ruled out all other possible steady states and therefore do not unequivocally claim that cycles will increase welfare. The results established in the previous section are a direct comparison of welfare levels within qualitatively similar, but time varying, cycles only. Though we have not been able to find other stable steady states yielding higher welfare than the $N$ cycles examined, we have also not been able to formally rule out their existence.\footnote{We are grateful to an anonymous referee for alerting us to the possible existence of such qualitatively different equilibria.}
This paper highlights a form of pecuniary externality between contemporaneous innovators that can generate both productivity cycles and growth endogenously. One may interpret such an endogenous mechanism as an explanation for the exogenously assumed productivity “shocks” in most real business cycle models. In addition, incorporating the particular mechanism in real business cycle models will enhance the match between the models’ predictions and data. For example, the pecuniary externality generates large volatility of profit over the cycles, as observed in the data.

The existence of multiple equilibria and the possibility of growth-enhancing cycles seem to accord with Schumpeter’s verbal discussions on innovation and cycles. We believe that these two central results are valid for more general environments. In contrast, too much emphasis on the exact form of the cycle developed here would be inappropriate. The example presented here is merely illustrative since there is no possibility of estimating the precise form of the \( \phi \) or \( H \) functions at this preliminary stage. Its purpose is to depict the counteracting forces which contribute to the net effect of cycle length on output, and hence welfare, over time. One can experiment with different functional forms for \( \phi \) (e.g., we have also examined \( \phi(H) = 1 - e^{-\phi H} \)) and show that the qualitative nature of the cycle is not exclusive to the linear specification used here.

One can also experiment with different assumptions on the imitation process. In the current model, new innovation is assumed to be imitated immediately with a one period lag. With this assumption, rents associated with new innovation are competed away one period after implementation. The result is that the labor share of income drops only for one period where new innovation is implemented, after which it bounces up immediately (see the bottom panel of Fig. 3). If new technologies take a longer time to diffuse, one would see a smoother pattern. The labor share would rise gradually as imitation takes place, accompanied by falling output as a new round of innovation draws labor away from production of the final goods.

APPENDIX

A. Proof of Lemma 3.1

Taking the derivative of \( \ln g(n) \) yields

\[
\frac{g'(n)}{g(n)} = \frac{\rho(1 + \theta)}{\theta^2} + \frac{\beta^\rho N}{1 - \beta^\rho N} \ln \beta.
\]
Since $\rho(1 + \theta) \geq \theta$ by (2.6), we have

$$\frac{g'(n)}{g(n)} > \frac{1}{N\theta} + \frac{\beta^{\text{m}}}{1 - \beta^{\text{m}}} \ln \beta = \frac{1 - \beta^{\text{m}} + \beta^{\text{m}} \ln(\beta^{\text{m}})}{N(1 - \beta^{\text{m}})}.$$  

The denominator is always positive regardless of the sign of $\theta$. The numerator is also positive because the function $1 - x + x \ln x$ is convex and attains a global minimum (zero) at $x = 1$.

Since $H'(N) = g'/g^2$, the property $g' > 0$ implies $H'(N) > 0$. To show $H''(N) < 0$, compute:

$$gH''(N) = \frac{gg'' - 2(g')^2}{g^3} = \left(\frac{g'}{g}\right)' - \left(\frac{g'}{g}\right)^2 = -\frac{\rho(1 + \theta)}{N^2\theta^2} + \frac{\theta(\ln \beta)^2 x}{(1-x)^2} - \left(\frac{\rho(1 + \theta)}{N\theta^2} + \frac{x}{1-x} \ln \beta\right)^2,$$

where $x = \beta^{\text{m}}$. The property $g' > 0$ and the assumption $\rho(1 + \theta) > \theta$ in (2.6) imply

$$gH'' < -\frac{1}{\theta N^2} \left[ \frac{\theta(\ln \beta)^2 x}{(1-x)^2} - \left(\frac{1}{N\theta} - \frac{x}{1-x} \ln \beta\right)^2 \right]$$

$$= \frac{-1}{\theta^2 N^2(1-x)^2} \left[ \theta((1-x)^2 - x(\ln x)^2) + (x \ln x + 1 - x)^2 \right].$$

It can be verified that the function $(1-x)^2 - x(\ln x)^2$ achieves the global minimum zero at $x = 1$. Then $\theta > -1$ implies

$$\theta((1-x)^2 - x(\ln x)^2) + (x \ln x + 1 - x)^2$$

$$> x(\ln x)^2 - (1-x)^2 + (x \ln x + 1 - x)^2$$

$$= x(x+1) \cdot (\ln x) \left[ \ln x - \frac{2(x-1)}{x+1} \right].$$

The function $\ln x - 2(x-1)/(x+1)$ is an increasing function which is positive if and only if $x > 1$. Also, the function $\ln x$ is positive if and only if $x > 1$. The product of the two is thus positive for all $x$ (except for $x = 1$ at which the value is zero), implying $H'' < 0$.  

Since $\beta \in (0, 1)$, it is apparent that $L_{t+s} > L_{t+s+1}$ for $s = 1, 2, ..., N$. For $1 \leq s \leq N-1$, $w_{t+s+1} = \pi_{t+s+1}$ and so

$$\pi_{t+s+1} = w_{t+s+1} \left( \frac{\gamma - 1}{\gamma} \right) (1 - L_{t+s+1}) = w_{t+s} \left( \frac{\gamma - 1}{\gamma} \right) (1 - L_{t+s+1})$$

$$< w_{t+s} \left( \frac{\gamma - 1}{\gamma} \right) (1 - L_{t+s}) = \pi_{t+s}.$$

The inequality follows from the first result, $L_{t+s+1} > L_{t+s}$.

Since $L_{t+s+1} > L_{t+s}$ for $s = 1, ..., N-1$, $L_{t+s} < 1$ for all $s = 1, ..., N$ if and only if $L_{t+N} < 1$, i.e., if and only if $L_{t+1} < \beta^{(N-1)(1+\theta)}$, which can be rewritten as follows using (3.6):

$$H < \beta^{(N-1)(1+\theta)Np(1+\theta)q} \left( \frac{\beta - 1}{\beta^{\phi} - 1} \right)^{(1+\theta)/\theta} \cdot \frac{\rho N}{\partial \phi g(N)}.$$ 

Substituting (3.4) yields:

$$\left( \frac{\rho N}{\partial \phi g(N)} \right)^{(1+\theta)} \frac{\gamma}{\phi(\gamma - 1)} - \frac{1}{g(N)} \text{ i.e.}$$

$$\phi > \frac{\gamma}{\gamma - 1} g(N) \cdot \left[ 1 + \left( \frac{\rho N}{\partial \phi} \right)^{(1+\theta)} \left[ g(N) \right]^{-\theta} \right]^{-1}.$$ 

Since $\theta/\rho > 0$, the right-hand side of the inequality is less than $\gamma g(N)/(\gamma - 1)$, which in turn is less than $\phi$ under (3.5). \(\square\)

## C. Proof of Lemma 3.4

In period $t+N+2$, all other new innovations are imitated and so aggregate profit equals zero. Thus total income equals the wage bill, $y_{t+N+2} = w_{t+N+2} (1 - L_{t+N+2})$. (The costs of labor inputs into innovation are opportunity costs of the innovator—if the innovator does not take on the project he can earn the competitive wage in the economy. These costs are paid off by the potential profit at the end of the project and counted then into aggregate income through the profit. They are not counted explicitly into aggregate income in any other periods.) Final goods’ market equilibrium implies

$$w_{t+N+2} (1 - L_{t+N+2}) = g \left[ \ln x_{t+N+2} \right] - \theta \left[ \ln (x_{t+N+2}) \right] + \phi \left[ \ln (x_{t+N+2}) \right] \right] \cdot \frac{\rho N}{\partial \phi g(N)}.$$  

$$\int$$
where, with a slight abuse of notation, \( \gamma_{t+N+2}^{i} \) denotes the quality ladder at \( t+N+2 \) in sector \( i \). Since a fraction \( \phi H \) of intermediate sectors are improved relative to period \( t+1 \) while the rest are not, we have

\[
\int_{0}^{1} \ln(\gamma_{t+N+2}^{i}) \, di = \phi H \int_{0}^{1} \ln(\gamma_{t+1}^{i}) \, di + (1-\phi H) \int_{0}^{1} \ln \gamma_{t}^{i} \, di
\]

Thus

\[
y_{t+N+2} = w_{t+N+2}(1-L_{t+N+2}) = \gamma^{\phi H} \frac{L^{p}_{t+N+2}}{L^{p}_{t+1}} e^{\phi H \ln(\gamma_{t+1}^{i})} \int_{0}^{1} \ln \gamma_{t+1}^{i} \, di
\]

Since \( \gamma_{t+1} = 0 \) (when \( N \geq 2 \)), (2.4) implies \( y_{t+1} = (1-L_{t+1}) w_{t+1} \). In an \((N+1)\)-period stationary equilibrium, periods \( t+N+2 \) and \( t+1 \) are both the first periods of innovation cycles and so \( L_{t+N+2} = L_{t+1} \). Then \( y_{t+N+2} = \gamma^{\phi H} y_{t+1} \). The expressions for \( \pi_{t+N+2} \) and \( w_{t+N+2} \) can be obtained similarly.

D. Proof of Lemma 3.5

If all others implement at \( t+N+1 \), imitation occurs at \( t+N+2 \), thus \( \pi_{t+N+2} = 1 \), so that (2.4) and (2.3) imply

\[
\pi_{t+N+2} = \left( \frac{\gamma-1}{\gamma} \right) y_{t+N+2} = \left( \frac{\gamma-1}{\gamma} \right) (1-L_{t+N+2}) w_{t+N+2}
\]

Since

\[
\pi_{t+N+1} = w_{t+1} \frac{\gamma-1}{\gamma + (1-\gamma) \phi H}
\]

then

\[
\frac{\pi_{t+N+2}-\pi_{t+N+1}}{w_{t+1}} = \left( \frac{\gamma-1}{\gamma} \right) (1-L_{t+N+2}) \gamma^{\phi H} - \frac{\gamma-1}{\gamma + (1-\gamma) \phi H}.
\]
If $L_{t+N+2} = 0$, then $\pi_{t+N+2} > \pi_{t+N+1}$ if and only if

$$\gamma^{\phi H} > \frac{\gamma}{\gamma + (1 - \gamma) \phi H},$$

i.e., $1 < \gamma^{\phi H}(1 - \phi H) + \phi H \gamma^{\phi H - 1}$.

The function $\gamma^{\phi H}(1 - \phi H) + \phi H \gamma^{\phi H - 1}$ equals 1 when $\phi H = 0$ or 1, and it can be verified that its derivative is positive for $0 < \phi H < 1$, so that $\pi_{t+N+2} > \pi_{t+N+1}$ when $L_{t+N+2} = 0$.

**E. Proof of Lemma 4.1**

Start with the first innovation cycle ($j = 0$). Since $w_1 = 1$, then for $1 \leq s \leq N$,

$$y_s = \frac{1 - L_s}{1 - L_1}, y_1 = (1 - L_1) w_1 = 1 - L_1;$$

$$y_{N+1} = \frac{\gamma}{\gamma - (\gamma - 1) \phi H(N)} y_1 = \gamma = \frac{\gamma}{\gamma - (\gamma - 1) \phi H(N)} y_{N+1},$$

where the first equality for $y_{N+1}$ follows from (2.4). In the first period of the second cycle ($j = 1$), the welfare level is

$$y_{N+2} = \gamma^{\phi H} y_1 = \gamma^{\phi H}(1 - L_1).$$

One can then calculate $y_{N+3}$ through $y_{2(N+1)}$. In general, for $j = 0, 1, 2, \ldots$ and $s \in \{1, 2, \ldots, N\}$, $L_{(N+1)s} = L_s$ and we have (4.1) and (4.2). The movement of wages over the cycle can be obtained directly from (3.9) and (3.1).

To calculate the discounted value of the entire output stream over the infinite horizon, we first calculate the present value of income within a cycle interval. For a cycle interval, $j(N+1) + 1$ to $(j+1)(N+1)$, the present value of output discounted to period $(N+1) + 1$ is

$$Y_j = \beta^N y_{j(N+1) + N+1} + \sum_{s=1}^{N} \beta^{s-1} y_{j(N+1) + s}.$$

Substituting from (4.1) and (3.7) yields

$$Y_j = \gamma^{\phi H} \left\{ \frac{\beta^N}{\gamma - (\gamma - 1) \phi H} + \sum_{s=1}^{N} \beta^{s-1}(1 - L_s) \right\}$$

$$= \gamma^{\phi H} \left\{ \frac{\beta^N}{\gamma - (\gamma - 1) \phi H} + \frac{1 - \beta^N}{1 - \beta} - L_1 N \left[ \frac{\beta^{N-1} g(N)}{N} \right]^{-\theta} \right\}.$$
Substituting for $L_1$ from (3.6) yields

$$y_j = \gamma_j \phi^{-1} \left\{ \beta_j \frac{1 - \beta^{N_j}}{1 - \beta} - H \beta^{N_j-1} g(N) \right\}$$

$$= \gamma_j \phi^{-1} \left\{ \beta_j \frac{1 - \beta^{N_j}}{1 - \beta} - \frac{H \beta^{N_j-1} g(N)}{\phi^{-1}} \right\}.$$

Substituting for $g(N)$ from (3.4) and re-arranging yields

$$y_j = \gamma_j \phi^{-1} \left\{ \beta_j \frac{1 - \beta^{N_j}}{1 - \beta} - \frac{H \beta^{N_j-1} g(N)}{\phi^{-1}} \right\}.$$

The present value of weighted income, $Y(N)$, is the discounted present value of the stream \{ $Y_j$ \}$_{j=0}^\infty$, where the discount factor is $\beta^{N+1}$ (since each cycle including the implementation period lasts for $N+1$ periods). Then

$$Y(N) = \sum_{j=0}^\infty \beta^{N_j+1} \left\{ \gamma_j \phi^{-1} \left\{ \frac{1 - \beta^{N_j}}{1 - \beta} - \frac{H \beta^{N_j-1} g(N)}{\phi^{-1}} \right\} \right\}.$$

This is bounded provided that output grows slower than discounting, i.e.,

$$\beta^{N+1} \gamma_j \phi^{-1} < 1.$$

**REFERENCES**