Materials appearing in Appendix D

D.1. Derivations for (3.2) through (3.13)

Start with (PH), the maximization problem of the representative household of country 1. We first derive the optimality conditions (3.2) through (3.8) for $i = 1$, and then derive the remaining conditions. These conditions for $i = 2$ can be derived similarly from the maximization problem of the representative household of country 2, which we omit.

Use (2.4) and (2.5) to solve for $x_{1k} = (q_{1k})^\sigma/\Omega_{1k}$ and $x_{1k}^f = (q_{1k}^f)^\sigma/\Omega_{2k}$. Substituting these solutions into (PH) and using the multipliers λ_{ik} and λ_{ik}^f introduced in section 3.2, we can write the Lagrangian of (PH) as follows:

$$\begin{aligned}\mathcal{L} \equiv\ & A\left[n_1\left(\tfrac{T_{11}^1}{N_1}q_{11}+\tfrac{T_{12}^1}{N_1}q_{11}^f\right)+(1-n_1)\left(\tfrac{T_{11}^2}{1-N_1}q_{12}+\tfrac{T_{12}^2}{1-N_1}q_{12}^f\right)\right]\\ & -\left[T_{11}^1\left(Q_{11}\right)^\sigma+T_{21}^1\left(Q_{21}^f\right)^\sigma\right]-\left[T_{11}^2\left(Q_{12}\right)^\sigma+T_{21}^2\left(Q_{22}^f\right)^\sigma\right]\\ & +\lambda_{11}n_1\tfrac{T_{11}^1}{N_1}\left[\tfrac{m_{11}-f_{11}}{n_1}-\tfrac{(q_{11})^\sigma}{\Omega_{11}}\right]+\lambda_{11}^f n_1\tfrac{T_{12}^1}{N_1}\left[\tfrac{m_{11}-f_{11}}{n_1}-\tfrac{\left(q_{11}^f\right)^\sigma}{\Omega_{21}}\right]\\ & +\lambda_{12}(1-n_1)\tfrac{T_{11}^2}{1-N_1}\left[\tfrac{m_{12}+ef_{11}}{1-n_1}-\tfrac{(q_{12})^\sigma}{\Omega_{12}}\right]+\lambda_{12}^f(1-n_1)\tfrac{T_{12}^2}{1-N_1}\left[\tfrac{m_{12}+ef_{11}}{1-n_1}-\tfrac{\left(q_{12}^f\right)^\sigma}{\Omega_{22}}\right]\\ & +\beta v\left(m_{11,+1},m_{12,+1}\right)\end{aligned}$$

where $m_{11,+1}$ and $m_{12,+1}$ are given below:

$$\begin{aligned}\gamma_1 m_{11,+1} =\ & (m_{11}-f_{11})-n_1\left[\tfrac{T_{11}^1}{N_1}\tfrac{(q_{11})^\sigma}{\Omega_{11}}+\tfrac{T_{12}^1}{N_1}\tfrac{\left(q_{11}^f\right)^\sigma}{\Omega_{21}}\right]\\ & +\left(T_{11}^1X_{11}+T_{21}^1X_{21}^f\right)+(\gamma_1-1);\end{aligned}$$

$$\begin{aligned}\gamma_2 m_{12,+1} =\ & (m_{12}+ef_{11})+\left(T_{11}^2X_{12}+T_{21}^2X_{22}^f\right)\\ & -(1-n_1)\left[\tfrac{T_{11}^2}{1-N_1}\tfrac{(q_{12})^\sigma}{\Omega_{12}}+\tfrac{T_{12}^2}{1-N_1}\tfrac{\left(q_{12}^f\right)^\sigma}{\Omega_{22}}\right].\end{aligned}$$

Recall the definition that $\omega_{1k} = (\beta/\gamma_k)v_k(m_{11,+1}, m_{12,+1})$. For $k = 1, 2$, it is easy to derive the first-order conditions for q_{1k} and q_{1k}^f as:

$$A = (\omega_{1k}+\lambda_{1k})\frac{\sigma(q_{1k})^{\sigma-1}}{\Omega_{1k}}, \quad A = (\omega_{1k}+\lambda_{1k}^f)\frac{\sigma(q_{1k}^f)^{\sigma-1}}{\Omega_{2k}}$$

These are the results (3.2) and (3.3) for $i = 1$.

The first-order condition for n_1 yields:

$$\begin{aligned}0 =\ & A\left[\tfrac{T_{11}^1}{N_1}q_{11}+\tfrac{T_{12}^1}{N_1}q_{11}^f-\left(\tfrac{T_{11}^2}{1-N_1}q_{12}+\tfrac{T_{12}^2}{1-N_1}q_{12}^f\right)\right]\\ & -\tfrac{T_{11}^1}{N_1}(\omega_{11}+\lambda_{11})\tfrac{(q_{11})^\sigma}{\Omega_{11}}-\tfrac{T_{12}^1}{N_1}\left(\omega_{11}+\lambda_{11}^f\right)\tfrac{(q_{11}^f)^\sigma}{\Omega_{21}}\\ & +\tfrac{T_{11}^2}{1-N_1}\left(\omega_{12}+\lambda_{12}\right)\tfrac{(q_{12})^\sigma}{\Omega_{12}}-\tfrac{T_{12}^2}{1-N_1}\left(\omega_{12}+\lambda_{12}^f\right)\tfrac{(q_{12}^f)^\sigma}{\Omega_{22}}\end{aligned}$$

Using the conditions for optimal q to substitute $(\omega + \lambda)$, we have the following results:

$$(\omega_{1k} + \lambda_{1k})\frac{(q_{1k})^\sigma}{\Omega_{1k}} = \frac{A}{\sigma} q_{1k}, \quad \left(\omega_{1k} + \lambda^f_{1k}\right)\frac{(q^f_{1k})^\sigma}{\Omega_{2k}} = \frac{A}{\sigma} q^f_{1k}, \quad k = 1, 2.$$

Substituting these results into the above condition for optimal n_1, we obtain (3.4) for $i = 1$.

It is not difficult to verify that the first-order condition for f_{11} is (3.5). Similarly, the first-order condition for f_{22} from a country 2 household's maximization problem is (3.6). Moreover, the derivative of the Lagrangian above with respect to m_{11} is $\left(\omega_{11} + \frac{T^1_{11}}{N_1}\lambda_{11} + \frac{T^1_{12}}{N_1}\lambda^f_{11}\right)$. By the envelope condition, this derivative is equal to $\frac{\partial}{\partial m_{11}} v(m_{11}, m_{12})$, which is equal to $(\gamma_1/\beta)\omega_{11,-1}$ by the definition of ω. This equality is (3.7) for $i = 1$. Similarly, the envelope condition for m_{12} is (3.8) for $i = 1$.

We now derive the conditions in section 3.3. To derive (3.9), set $i = 1$ in (3.7) and (3.8). Substituting the results into (3.5) and moving the time index forward by one period, we get: $\gamma_1\omega_{11} = e_{+1}\gamma_2\omega_{12}$. Here we have used the fact that the money growth rates are constant over time. Similarly, substituting the envelope conditions into (3.6) yields: $\gamma_1\omega_{21} = e_{+1}\gamma_2\omega_{22}$. Now, (3.9) becomes evident under the stationarity of e. Re-arranging (3.9) yields (3.10).

To obtain (3.11), suppose that all the trading constraints in (2.6) and (2.7) bind, which is the case when $\gamma_1, \gamma_2 > \beta$. Substituting x_{1k} and x^f_{1k} from (2.4) and (2.5), the constraints imply:

$$\frac{(q_{11})^\sigma}{\Omega_{11}} = \frac{m_{11} - f_{11}}{n_1} = \frac{(q^f_{11})^\sigma}{\Omega_{21}}$$

$$\frac{(q_{12})^\sigma}{\Omega_{12}} = \frac{m_{12} + ef_{11}}{1 - n_1} = \frac{(q^f_{12})^\sigma}{\Omega_{22}}$$

Using (3.10), the first line above yields $q_{11}/q^f_{11} = \theta^{1/\sigma}$, and the second line yields $q_{12}/q^f_{12} = \theta^{1/\sigma}$. This is one part of (3.11). The remaining part of (3.11) comes similarly from the trading constraints on the buyers of a country 2 household.

We derive (3.12) from (3.7). To do so, impose the symmetry requirements, $\Omega_{ik} = \omega_{ik}$ and $\Omega_{i'k} = \omega_{i'k}$. Substituting $\omega_{2k} = \omega_{1k}/\theta$ from (3.10) and $q^f_{1k} = q_{1k}\theta^{-1/\sigma}$ from (3.11), we can rewrite (3.2) and (3.3) (for $i = 1$) as:

$$\lambda_{1k} = \omega_{1k}\left[\frac{A}{\sigma(q_{1k})^{\sigma-1}} - 1\right], \quad \lambda^f_{1k} = \omega_{1k}\left[\frac{A\theta^{-1/\sigma}}{\sigma(q_{1k})^{\sigma-1}} - 1\right]$$

Substituting these λ'es into (3.12) and imposing stationarity on ω_{11}, we have:

$$\frac{\gamma_1}{\beta} - 1 = \frac{T^1_{11}}{N_1}\left[\frac{A}{\sigma(q_{11})^{\sigma-1}} - 1\right] + \frac{T^1_{12}}{N_1}\left[\frac{A\theta^{-1/\sigma}}{\sigma(q_{11})^{\sigma-1}} - 1\right]$$

Using the assumed form of the matching function and the definition $\mu(N) = L(N)/N$, we get: $T^1_{11}/N_1 = \mu(N_1)s$ and $T^1_{12}/N_1 = \mu(N_1)(1-s)$. Substituting these results into the above equation and invoking the symmetry requirement, $N_1 = n_1$, we obtain (3.12) for $i = 1$.

The equation (3.13) for $i = 1$ can be obtained similarly from (3.8). We can also apply the same procedure to the optimality conditions of a country 2 household's maximization problem to obtain (3.12) and (3.13) for $i = 2$.

D.2. Indeterminacy of the Equilibrium under Symmetric Matching

Consider an alternative matching function that preserves concavity but implies symmetric matching. It is convenient to specify the matching probabilities directly. Let μ_{ik} be the probability with which a buyer of country i gets a trade match in area k. Let μ_{ik} be strictly decreasing in the number of buyers of country i in area k, N_{ik}. Then, the matching function is strictly concave in N. In contrast to the model in the paper, let us assume $\mu_{1k} = \mu_{2k}$ for $k = 1, 2$, so that matching is symmetric between the two countries' buyers. We show that the allocation of buyers to the two areas and the nominal exchange rate are indeterminate in the equilibrium.

To begin, note that μ_{ik} specified above is the probability with which a buyer of country i gets a trade match in area k with either a domestic seller or a foreign seller. To calculate the probability of a match with a specific country's sellers, consider area $k = i \in \{1, 2\}$. In area i, a fraction s of the sellers are from country i and a fraction $(1-s)$ from country i'. Thus, a buyer of country i in area i has a trade match with a country i seller with probability $s\mu_{ii}$ and with a country i' seller with probability $(1-s)\mu_{ii}$. Similarly, a buyer of country i in area i' has a trade match with a country i seller with probability $(1-s)\mu_{ii'}$ and with a country i' seller.

With the above matching rates, we can analyze a country i household's optimal choice of n_i. The optimal choice leads to the following condition similar to (3.4):

$$\mu_{ii}\left[sq_{ii} + (1-s)q^f_{ii}\right] = \mu_{ii'}\left[(1-s)q_{ii'} + sq^f_{ii'}\right].$$

The derivation for (3.11), presented in the previous section, is still valid. Using (3.11) to substitute q^f_{ii} and $q^f_{ii'}$ in the above equation, we obtain:

$$\frac{q_{ii'}}{q_{ii}} = \frac{\mu_{ii}}{\mu_{ii'}}\left[\frac{s + (1-s)\theta^{(i-i')/\sigma}}{1 - s + s\theta^{(i-i')/\sigma}}\right].$$

Similar to (3.12) and (3.13), the envelope conditions for m_{ii} and $m_{ii'}$ lead to:

$$1 + \frac{\gamma_i/\beta - 1}{\mu_{ii}} = \left[s + (1-s)\theta^{(i-i')/\sigma}\right]\frac{A}{\sigma q_{ii}^{\sigma-1}}$$

$$1+\frac{\gamma_{i'}/\beta-1}{\mu_{ii'}}=\left[1-s+s\theta^{(i-i')/\sigma}\right]\frac{A}{\sigma q_{ii'}^{\sigma-1}}$$

Dividing these equations and substituting $q_{ii'}/q_{ii}$ from the above, we have:

$$\frac{s+(1-s)\theta^{(i-i')/\sigma}}{1-s+s\theta^{(i-i')/\sigma}}=\left(\frac{\mu_{ii'}}{\mu_{ii}}\right)^{\frac{\sigma-1}{\sigma}}\left[\frac{1+\frac{\gamma_i/\beta-1}{\mu_{ii}}}{1+\frac{\gamma_{i'}/\beta-1}{\mu_{ii'}}}\right].$$

This equation holds for $i=1,2$. When matching is asymmetric for the two countries' buyers, the equation for $i=1$ is distinct from the equation for $i=2$, and hence they provide a restriction on n_1 and n_2. When matching is symmetric, however, $\mu_{1i}=\mu_{2i}$ for $i=1,2$. In this case, the above equation for $i=1$ is identical to the equation for $i=2$. Thus, the equilibrium does not pin down n_1 and n_2 uniquely. As a result, the nominal exchange rate is indeterminate.

D.3. A Walrasian Model with Cash-in-advance Constraints

We modify Helpman's (1981) model of cash-in-advance by introducing elastic labor supply which as in our model. With this modification, we show that there is no incentive for a country to deviate from the Friedman rule, and hence no need for monetary coordination. This contrast with our results shows that the deviations from the LOP are important for our results.

In Helpman's model, each household receives endowment of goods. To modify the model, assume that goods are produced with labor and the cost of production is the disutility of labor, as in our model. That is, producing q units of output incurs disutility q^σ, where $\sigma>1$. Maintain the cash-in-advance constraint in Helpman's model; i.e., all purchases of the goods produced in a country i must be made with currency i, where $i=1,2$. Let Q_i be the quantity of goods sold by a household of country i, which is now the household's choice. Because the goods markets are Walrasian and all goods appear symmetrically in a household's utility function, all goods produced by country i are sold for the same price. Let p_i be such a price. Most of the other variables have the same meaning as in our model.

A household in country i chooses $(q_{ii},q_{ii'},Q_i,f_{ii})$ and $(m_{ii,+1},m_{ii',+1})$ to solve the following maximization problem:

$$v(m_{ii},m_{ii'})=\max\left\{A\left(q_{ii}+q_{ii'}\right)-Q_i^\sigma+\beta v(m_{ii,+1},m_{ii',+1})\right\}$$

subject to:

$$p_iq_{ii}\le m_{ii}-f_{ii}$$

$$p_{i'} q_{ii'} \leq m_{ii'} + e^{i'-i} f_{ii}$$

$$\gamma_i m_{ii,+1} = m_{ii} - f_{ii} + p_i (Q_i - q_{ii}) + (\gamma_i - 1)$$

$$\gamma_{i'} m_{ii',+1} = m_{ii'} + e^{i'-i} f_{ii} - p_{i'} q_{ii'}$$

Here $e^{i'-i} f_{ii}$ is the amount of foreign currency obtained by the household in the currency market.

The first two constraints above are cash-in-advance constraints. Let λ_{ii} and $\lambda_{ii'}$ be the Lagrangian multipliers of these constraints. Then, the optimal choices of q_{ii} and $q_{ii'}$ yield:

$$\lambda_{ii} = \frac{A}{p_i} - \omega_{ii},\ \lambda_{ii'} = \frac{A}{p_{i'}} - \omega_{ii'}$$

The optimal choice of f_{ii} yields: $\omega_{ii} + \lambda_{ii} = e^{i'-i}(\omega_{ii'} + \lambda_{ii'})$. Substituting the λ'es from the above, we have the following purchasing power parity:

$$p_{i'} = p_i e^{i'-i}. \tag{D.1}$$

The optimal choice of Q_i yields:

$$\omega_{ii} p_i = \sigma Q_i^{\sigma-1} \tag{D.2}$$

Deriving the envelope conditions for m_{ii} and $m_{ii'}$, and substituting the λ'es, we get:

$$\omega_{ii} = \frac{\beta A}{\gamma_i p_{i,+1}},\ \omega_{ii'} = \frac{\beta A}{\gamma_{i'} p_{i',+1}}.$$

In a stationary equilibrium, $\omega_{ii,+1} = \omega_{ii}$ and $\omega_{ii',+1} = \omega_{ii'}$ for $i = 1, 2$. Then the (normalized) prices p_i and $p_{i'}$ are also stationary. The above equations for the ω'es imply $p_i \omega_{ii} = \beta A/\gamma_i$ and $p_{i'} \omega_{ii'} = \beta A/\gamma_{i'}$. Write these results equivalently as

$$p_k \omega_{ik} = \beta A/\gamma_k, \text{ all } i, k \in \{1, 2\}.$$

That is, ω_{ik} is independent of the country index i. Hence, $\omega_{11}/\omega_{21} = \omega_{12}/\omega_{22} = \theta = 1$. There is no differential between the two countries in the absolute valuation of a currency. Thus, the main incentive for inflation emphasized in our model is absent here.

To show formally that there is no incentive to inflate, we complete the characterization of the equilibrium. The market-clearing condition for each country i's goods is:

$$q_{ii} + q_{i'i} = Q_i \tag{D.3}$$

The market-clearing condition for currency exchange is $f_{22} = ef_{11}$. Total holdings of a currency add up to one: $m_{ii} + m_{i'i} = 1$. Moreover, because the cash in advance constraints bind, a household spends all of its foreign currency on foreign goods. As a result, the household holds no foreign currency at the beginning of a period, i.e., $m_{12} = m_{21} = 0$. (For the purchases of foreign goods, the household obtains the foreign currency in the currency market.) Thus, $m_{11} = m_{22} = 1$.

Focus on the equilibrium where all the cash-in-advance constraints bind. This is guaranteed if and only if $p_i\omega_{ii} < A$ and $p_{i'}\omega_{ii'} < A$ for $i = 1, 2$. In turn, these conditions are satisfied in the stationary equilibrium if and only if $\gamma_1, \gamma_2 > \beta$. The binding constraints imply:

$$q_{ii} = \frac{m_{ii} - f_{ii}}{p_i},\ q_{i'i} = \frac{m_{i'i} + f_{ii}}{p_i}$$

Here we used the fact that $f_{i'i} = f_{ii}$. Adding up these two equations, we have $q_{ii} + q_{i'i} = 1/p_i$. Combining this result with (D.3), we have $1/p_i = Q_i$. Substituting the ω'es into (D.2), we obtain Q_i. Let us write the result as follows:

$$\frac{1}{p_i} = Q_i = \left(\frac{\beta A}{\sigma\gamma_i}\right)^{\frac{1}{\sigma-1}}. \tag{D.4}$$

Then, (D.1) yields:

$$e = \frac{p_2}{p_1} = \left(\frac{\gamma_2}{\gamma_1}\right)^{\frac{1}{\sigma-1}}. \tag{D.5}$$

We compute the levels of consumption and utility of a household of country i. The household's consumption of domestic goods is $q_{ii} = (1 - f_{ii})/p_i$, because $m_{ii} = 1$. The household's consumption of foreign goods is

$$q_{ii'} = \frac{m_{ii'} + e^{i'-i}f_{ii}}{p_{i'}} = \frac{e^{i'-i}f_{ii}}{p_{i'}} = \frac{f_{ii}}{p_i}.$$

The second equality follows from $m_{ii'} = 0$, and the third equality from (D.1). Because the household's total consumption is $c_{ii} = q_{ii} + q_{ii'}$, then

$$c_i = \frac{1 - f_{ii}}{p_i} + \frac{f_{ii}}{p_i} = \frac{1}{p_i} = Q_i = \left(\frac{\beta A}{\sigma\gamma_i}\right)^{\frac{1}{\sigma-1}}.$$

Note that both the levels of consumption and production depend only on the country's domestic money growth rate. Thus, the household's utility depends only on the country's domestic money growth. Denote the household's steady state utility as $W_i(\gamma_i) \equiv (1 - \beta)v(m_{ii}, m_{ii'})$. Then,

$$W_i(\gamma_i) = Ac_i - Q_i^\sigma = AQ_i - Q_i^\sigma.$$

With the expression for Q_i in (D.4), it is easy to show that $W_i'(\gamma_i) < 0$ for all $\gamma_i > \beta$. Thus, country i's optimal choice of money growth is $\gamma_i = \beta$. That is, each country has no incentive to deviate from the Friedman rule in the regime of policy competition. The Nash equilibrium of the policy game has $\gamma_1 = \gamma_2 = \beta$. Therefore, there is no need for monetary coordination.