1 A quick and dirty guide to derivatives for intermediate micro

Consider a function of one variable, x,

$$y = f(x)$$

Often we want to find the value of x which maximizes the value of y. Let this value of x be $\widehat{x}$.

Recall that the derivative of a function at point x is the slope of the function at that point. The derivative of $f(x)$ is denoted by

$$\frac{dy}{dx}$$

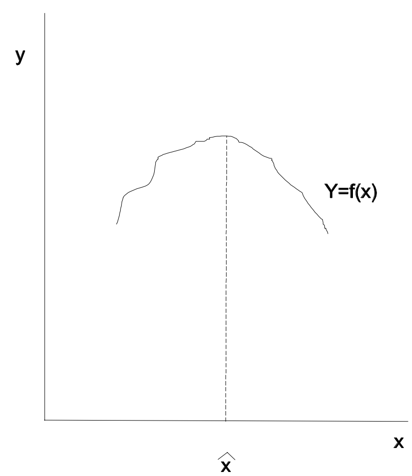
or

$$f'(x)$$

or

$$f_x$$

From the figure below, it is easy to see that the maximum value of y occurs when the slope (or derivative) is zero, at $x = \widehat{x}$.



Unfortunately the minimum value of y also occurs when the derivative is zero. So in general, you have to check that you have found a maximum rather than a minimum. One way to check is to look at derivative of $\frac{dy}{dx}$ with respect to x. This is known as the second derivative of y with respect to x. You have found a maximum if the second derivative evaluated at $\widehat{x}$ is negative.

1.1 Derivatives of a quadratic function

Let

$$y = a + bx + cx^2$$

where a, b, c are numbers.
Then

$$\frac{dy}{dx} = b + 2cx$$

The maximum of this function occurs when the derivative is equal to zero. Thus at the maximum,

$$\begin{aligned} \frac{dy}{dx}|_{\widehat{x}} &= 0 \\ b + 2c\widehat{x} &= 0 \\ \widehat{x} &= -\frac{b}{2c} \end{aligned}$$

You have found a maximum if the second derivative is negative. The second derivative of y is

$$\begin{aligned} \frac{d(\frac{dy}{dx})}{dx} &= \frac{d(b + 2cx)}{dx} \\ &= 2c \end{aligned}$$

So you have found a maximum if $2c < 0$ or $c < 0$.
If $c > 0$, you have found a minimum.

2 Marginal benefit = marginal cost

Often economic objectives are stated as benefits minus cost (net benefit or profit). The objective of the agent is to choose an action such that it maximizes profit. Let the agent's choice of action be x. Let the benefit function be

$$B(x)$$

We will assume that $B(x)$ is increasing in x. That is, the slope or the marginal benefit of x, $B_x = MB(x) > 0$.

Let the cost function be

$$C(x)$$

We will assume that $C(x)$ is decreasing in x. That is, the slope or the marginal cost of x, $C_x = MC(x) < 0$.

Then profit due to action x is

$$\pi(x) = B(x) - C(x)$$

The derivative of $\pi(x)$ is

$$\begin{aligned}\frac{d\pi}{dx} &= \frac{dB}{dx} - \frac{dC}{dx} \\ &= MB(x) - MC(x)\end{aligned}$$

To find the action, $\widehat{x}$, which gives the maximum profit, we want to set the derivative of $\pi(x)$ to zero:

$$\begin{aligned}\frac{d\pi(x)}{dx}|_{\widehat{x}} &= 0 \\ \frac{dB}{dx}|_{\widehat{x}} - \frac{dC}{dx}|_{\widehat{x}} &= 0 \\ \frac{dB}{dx}|_{\widehat{x}} &= \frac{dC}{dx}|_{\widehat{x}} \\ MB(\widehat{x}) &= MC(\widehat{x})\end{aligned}$$

So we have found the standard formula of equating marginal benefit to marginal cost for the profit maximizing action.

2.1 Quadratic example

Let

$$\begin{aligned}B(x) &= b_0 + b_1 x + b_2 x^2 \\ C(x) &= c_0 + c_1 x + c_2 x^2\end{aligned}$$

Then

$$\begin{aligned}\pi(x) &= b_0 + b_1 x + b_2 x^2 - (c_0 + c_1 x + c_2 x^2) \\ MB(x) &= b_1 + 2b_2 x \\ MC(x) &= c_1 + 2c_2 x \\ \frac{d\pi}{dx} &= b_1 + 2b_2 x - (c_1 + 2c_2 x) \\ \frac{d(\frac{d\pi}{dx})}{dx} &= 2b_2 - 2c_2\end{aligned}$$

At the maximum,

$$\begin{aligned}MB(\widehat{x}) &= MC(\widehat{x}) \\ b_1 + 2b_2\widehat{x} &= c_1 + 2c_2\widehat{x} \\ \widehat{x} &= \frac{b_1 - c_1}{2(c_2 - b_2)}\end{aligned}$$

We also need $\frac{d(\frac{d\pi}{dx})}{dx} < 0$ for a maximum which means that we need $b_2 - c_2 < 0$ or $c_2 > b_2$. If $b_2 < 0$ and $c_2 > 0$, then $b_2 - c_2 < 0$.

3 Two choices

What if the function we want to maximize is a function of two variables x and z?

Let

$$y = f(x, y)$$

Let $\widehat{x}$ and $\widehat{z}$ be the values of x and z respectively which maximizes y. Then we know that $\widehat{x}$ and $\widehat{z}$ must satisfy:

$$\frac{\partial y}{\partial x}|_{\widehat{x},\widehat{z}} = 0$$
$$\frac{\partial y}{\partial z}|_{\widehat{x},\widehat{z}} = 0$$

where $\frac{\partial y}{\partial x}$ is the partial derivative of y with respect to x and $\frac{\partial y}{\partial y}$ is the partial derivative of y with respect to z.

3.1 Calculating partial derivatives

You calculate the partial derivative of y with respect to x by taking the derivative of y with respect to x, treating z as a constant. Similarly, you calculate the partial derivative of y with respect to z by taking the derivative of y with respect to z, treating x as a constant.

E.g.

$$y = ax + bz + cxz + ex^2 + hz^2$$

Then

$$\frac{\partial y}{\partial x} = a + cz + 2ex$$
$$\frac{\partial y}{\partial z} = b + cx + 2hz$$

To find the maximum, you want to set the two partial derivatives to zero which means

$$a + c\widehat{z} + 2e\widehat{x} = 0$$
$$b + c\widehat{x} + 2h\widehat{z} = 0$$

You can solve the above simultaneous equations for $\widehat{z}$ and $\widehat{x}$.

We will not check to see if you have found a maximum or not in this case (Its beyond this class).

4 Cobb Douglas case

Let the benefit function $B(x,z)$ be

$$B(x,z) = ax^{\alpha}z^{\beta}$$

where $a > 0$ and $0 \geq \alpha, \beta > 0$. Then

$$\begin{aligned}\frac{\partial B}{\partial x} &= a\alpha x^{\alpha-1}z^{\beta}\\ \frac{\partial B}{\partial z} &= a\beta x^{\alpha}z^{\beta-1}\end{aligned}$$

Let the cost function $C(x,z)$ be

$$C(x,z) = c_1 x + c_2 z$$

where $c_1, c_2 > 0$.
Let profits be

$$\begin{aligned}\pi(x,z) &= B(x,z) - C(x,z)\\ &= ax^{\alpha}z^{\beta} - (c_1 x + c_2 z)\end{aligned}$$

The objective of the agent is to maximize π with respect to x and z.
The partial derivatives of π are

$$\begin{aligned}\frac{\partial \pi}{\partial x} &= \frac{\partial B}{\partial x} - \frac{\partial C}{\partial x} = a\alpha x^{\alpha-1}z^{\beta} - c_1\\ \frac{\partial \pi}{\partial z} &= \frac{\partial B}{\partial z} - \frac{\partial C}{\partial z} = a\beta x^{\alpha}z^{\beta-1} - c_2\end{aligned}$$

At $\widehat{x}$ and $\widehat{z}$,

$$\begin{aligned}\frac{\partial \pi}{\partial x}|_{\widehat{x},\widehat{z}} &= 0\\ \frac{\partial \pi}{\partial z}|_{\widehat{x},\widehat{z}} &= 0\end{aligned}$$

which implies

$$\begin{aligned}\frac{\partial B}{\partial x}|_{\widehat{x},\widehat{z}} &= \frac{\partial C}{\partial x}|_{\widehat{x},\widehat{z}}\\ \frac{\partial B}{\partial z}|_{\widehat{x},\widehat{z}} &= \frac{\partial C}{\partial z}|_{\widehat{x},\widehat{z}}\end{aligned}$$

The above implies that we want to equate the marginal benefit equal to marginal cost for every choice variable.

The above is also

$$a\alpha\widehat{x}^{\alpha-1}\widehat{z}^{\beta} = c_1$$
$$a\beta\widehat{x}^{\alpha}\widehat{z}^{\beta-1} = c_2$$

We can solve the above two equations to get $\widehat{x}$ and $\widehat{z}$. (Hint: Multiply through the top equation by α and then divide the new top equation by the bottom equation).