How does the marriage market clear? An empirical framework.

Aloysius Siow*
University of Toronto
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Abstract

The paper surveys the Choo Siow (CS) marriage matching model and its extensions. CS derives a behavioral marriage matching function (MMF). The collective model of intra-household allocations can be integrated into the CS framework. Spousal labor supplies respond to changing marriage market conditions. The hypothesis that spousal labor supplies vary to equilibrate the marriage market has overidentifying restrictions. The CS framework extends to a dynamic marriage matching environment. Empirically, this paper shows how the famine caused by the great leap forward in Sichuan affected the marital behavior of famine born cohorts. It also shows that marriage market tightness, the ratio of unmarried type $i$ men to unmarried type $j$ women, is a useful statistic for summarizing marriage market conditions. Marriage market conditions in contemporary United States primarily affect spousal labor force participation rather than hours of work.

Who marries whom and how does the marriage market clear are two classic questions in the study of marriage markets. There are few empirical studies on these questions because economists are just beginning to estimate bilateral matching models, and inter-spousal transfers in modern societies are difficult to observe.

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2Even when observed, Botticini and Siow 2003 cautions against interpreting dowries solely as transfers which clear the marriage market.
The objective of this paper is to survey and extend research by my coauthors and I on these two questions. Choo Siow (2006; hereafter CS) proposed and estimated a marriage matching function (MMF). A marriage matching function maps population supplies to who marries whom. The benefits of the CS MMF are: (1) It is non-parametric and easy to estimate. (2) It has behavioral foundations. Economists have proposed other behaviorally motivated marriage matching models. Most of these models impose strong apriori identifying restrictions on the model which are inappropriate in many empirical and normative contexts.

A theme that runs through this paper is that many structural parameters that economists use for welfare simulations in these contexts are not identified. Sections 1 and 2 will exposit marriage matching functions and the CS model respectively.

Brandt, Siow and Vogel (2008; hereafter BSV) uses the CS model to study the effects of the famine in China caused by the great leap forward on the marital behavior of the famine born cohorts. BSV shows that the famine born cohorts were “discriminated” against in the marriage market. Section 3 will provide some of that evidence here.

The CS MMF assumes transferable utilities (i.e. constant marginal utility of income) and that intra-household allocations do not affect total marital surpluses. We relax these restrictive assumptions in Choo, Seitz and Siow (2008a, 2008b; hereafter CSSa and CSSb) by integrating the collective model of intra-household allocation into the CS marriage matching framework. Developed by Chiappori and his collaborators, the collective model is an empirically tractable model of intra-household allocation where household members may have divergent interests (Chiappori 1988, 1992; Vermeulen 2002 has a survey). There is empirical evidence in favor of the collective model in many environments (Donni 2005 has a bibliography). CSSa nests a general collective model inside the CS marriage matching framework. We show existence of marriage market equilibrium. Thus the collective model, an influential empirical model on intra-household allocation, is shown to be consistent with marriage market equilibrium. This collective marriage matching model provides a solution to the long standing search for an empirical model of marriage matching and intra-household allocations. Section 4 exposits this collective marriage matching model using the collective model by Blundell, Chiappori and Meghir 2005.

An \(\{i,j,s\}\) marriage match has a type \(i\) husband, a type \(j\) wife and a type \(s\) living arrangement. In collective models, taking \(\{i,j,s\}\) and the utility weights of family members as given, a social planner determines intra-household allocations by maximizing the weighted sum of members’ utilities subject to an aggregate family budget constraint. In the collective marriage matching model, the utility weights of husbands relative to their wives, \(p_{ij,s}\), adjust to equilibrate the marriage market.

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4“Models that analyze bargaining within existing marriages can give only an incomplete picture of the determinants of the well-being of men and women. The marriage market is an important determinant of distribution between men and women. At a minimum, the marriage market determines who marries and who marries whom.” Lundberg and Pollak 1996.
is unobserved by the researcher. Collective models have developed different empirical strategies to relate observable intra-household allocations to empirical proxies for $p_{ijs}$. As noted by other researchers, marriage market conditions affect $p_{ijs}$. What is a useful empirical proxy for marriage market conditions? Based on convenience and ease of interpretation, researchers have primarily focused on the own sex ratio, the ratio of type $i$ men to type $j$ women. But as the own sex ratio changes, the sex ratios of spousal substitutes are also changing. Thus ignoring the availability of spousal substitutes makes it difficult to interpret the results of these studies.

I provide an alternative empirical proxy for marriage market tightness. In the CS class of models, a sufficient statistic for the marriage market condition type $i$ men and type $j$ women is marriage market tightness, the log ratio of the number of unmarried type $i$ men to the number of unmarried type $j$ women. $p_{ijs}$, bargaining power of the husband in the marriage is decreasing in marriage market tightness. Section 4.2 shows that the wife’s (husband’s) spousal labor supply is increasing (decreasing) as the bargaining power of the husband increases or as marriage market tightness decreases. Living arrangements $s$ does not appear in marriage market tightness which imposes further restriction on the data if we observe the ratio of the number of $\{i, j, s\}$ to $\{i, j, s'\}$ marriages. I will provide evidence on these issues.

While we and other researchers (E.g. Angrist 2002; Chiappori, Fortin and Lacroix 2002 (hereafter CFL); Grossbard-Shectman 1993) show that changes in marriage market conditions affect spousal labor supplies, it is not known if spousal labor supplies adjust sufficiently to clear the marriage market. To answer such a question, we need to estimate a structural model of spousal labor supplies and marriage matching. CSSb proposes and estimates such a model. Section 5.2 sketches how this is done.

Another restrictive feature of CS is that it is a static MMF whereas marital behavior is fundamentally dynamic. An individual may choose to remain unmarried today in order to marry in the future. Thus we need to develop dynamic MMF’s. Building on Choo Siow 2007, Section 6 presents an overlapping generations dynamic MMF where the number of entrants into the marriage market is time varying.

Finally, the empirical framework here assumes away the problem of unobserved heterogeneity. I currently do not know how to estimate equilibrium bilateral matching models with two sided unobserved heterogeneity.

1 Marriage matching functions

Building on the classic model of Leslie 1945, demographers use one-sex models to empirically model population growth. However, empirical male versus female one sex models of population growth lead to contradictory implications. These contradictions arise to a large part because one sex models, by construction, do not impose consistent mating and reproductive behavior across the two genders.

\footnote{This discussion builds on Pollak 1990, Pollard 1993,1997.}
In order to construct empirical two sex models of population growth, researchers have to first provide tractable empirical models of mating behavior, or who marries whom. Demographers construct MMFs to model who marries whom.

Let there be $I$ types of men and $J$ types of women. $M$ is a population vector where element $m_i$ is the number of eligible (single) type $i$ men. $F$ is a population vector where element $f_j$ is the number of eligible (single) type $j$ women. $\Pi$ is a vector of exogenous parameters. A marriage matching function (MMF) is an $I \times J$ matrix $\mu(M, F; \Pi)$ whose $\{i, j\}$ element is $\mu_{ij}$, the number of type $i$ men married to type $j$ women:

$$\mu_{ij} = \eta_{ij}(M, F; \Pi); \forall i, j$$

Let $\mu_{i0}$ and $\mu_{0j}$ be the numbers of unmarried type $i$ men and type $j$ women respectively.

The MMF must also satisfy the following accounting constraints:

$$\mu_{0j} + \sum_{i=1}^{I} \mu_{ij} = f_j \forall j$$

(2)

$$\mu_{i0} + \sum_{j=1}^{J} \mu_{ij} = m_i \forall i$$

(3)

$$\mu_{i0}, \mu_{0j}, \mu_{ij} \geq 0 \forall i, j$$

(4)

The accounting constraints are important because marriage rates for some types of individuals are close to 1. Thus MMFs which ignore accounting constraints will often result in predicted marriage rates above 1.

Although the objective is known for some time, it has been difficult to come up with empirically tractable and yet behaviorally attractive MMFs. The main difficulty is how to deal with alternative spousal choices while minimizing apriori identifying restrictions.

The current standard MMF is the harmonic mean MMF (E.g. Qian and Preston 1993; Schoen 1981):

$$\mu_{ij} = \left[ \frac{m_i f_i}{m_i + f_j} \right] \pi_{ij}$$

(5)

This MMF is attractive because it is non-parametric and easy to estimate.

However, it has three deficiencies. First, it ignores the problem of alternative spousal choices. Changes in $m_i'$ or $f_j'$ do not affect $\mu_{ij}$. Second, the accounting constraints, (2) to (4) are not imposed. As shown below, this problem leads to nonsensical predicted marital behavior in an actual application. Finally, it has no coherent behavioral foundation.

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6Pollak 1990a is an important contribution to this research agenda.

7... the frustrating search for a mathematically tractable and sociologically realistic “marriage function”. (Preston and Richards 1975)
2 Choo Siow

CS provides a framework to derive empirically tractable and behaviorally consistent MMFs.\textsuperscript{8}

Start by considering each marital match between two different types of individuals as a distinct marriage market. With $I$ types of men and $J$ types of women, there are $I \times J$ submarriage markets.

In an $\{i, j\}$ marriage, a systematic marriage surplus which depends on the type of the match, $\Pi_{ij}$, is generated. Let $\bar{\tau}_{ij}$ be the share of the surplus that is obtained by the wife. Each type $j$ wife also gets an idiosyncratic payoff which depends on her specific identify, and the type of spouse that he is and not his specific identity. Her idiosyncratic payoff also does not depend on $\bar{\tau}_{ij}$.

In an $\{i, j\}$ marriage, $\Pi_{ij} - \bar{\tau}_{ij}$ is the share of marital surplus that is obtained by the husband. Each type $i$ husband also gets an idiosyncratic payoff that is specific to him, and the type of spouse that she is and not her specific identity. His idiosyncratic payoff also does not depend on $\Pi_{ij} - \bar{\tau}_{ij}$.

The above assumptions imply that every type $i$ male regards every type $j$ female as perfect spousal substitutes and vice versa.

Each individual also gets a systematic payoff from remaining unmarried which depends on their type as well as an idiosyncratic payoff which depends on their specific identity.

Given their payoffs, both systematic and idiosyncratic, from every potential spousal choice including remaining unmarried, each individual will choose the spousal choice which maximizes their utility. Note that concern over different spousal choices are automatically built into the individual’s choice problem.

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Given $\bar{\tau}_{ij}$, we can solve each individual’s spousal choice problem. We can aggregate these individual decisions into demand and supply functions for spouses in every $\{i, j\}$ submarriage market.

Finally, we solve for the matrix of $\bar{\tau}_{ij}$ which will equilibrate demand with supply in every submarriage market simultaneously.

The equilibrium distribution of marriages is a function of population vectors and exogenous parameters which determine the systematic and idiosyncratic payoffs. Thus this equilibrium distribution of marriages is a MMF.

This framework for constructing MMF has three benefits. First, it is behaviorally coherent. Second, considerations of alternative spousal choices are built into the framework. Third, in equilibrium, all accounting constraints, (2) to (4) are automatically satisfied.

What remains is to find convenient functional forms to implement the above framework. An important assumption in CS is transferable utilities (constant marginal utilities of income). As will be shown below, this restriction is necessary for identification.

Let the utility of male $g$ of type $i$ who marries a female of type $j$ be:

$$v_{ijg} = \Pi_{ij} - \bar{\tau}_{ij} + \varepsilon_{ijg}$$

\textsuperscript{8}Dagsvik 2000 and 2001 developed a related model. See CS for a comparison.
As discussed above, $\Pi_{ij} - \bar{\tau}_{ij}$ is the systematic marital share of the husband. $\varepsilon_{ij}$ is his idiosyncratic payo\-off. Assume that $\varepsilon_{ij}$ is an i.i.d. type I extreme value random variable.\footnote{$\varepsilon \sim EV(0, 1)$, with the cumulative distribution given by $F(\varepsilon) = e^{-e^{-\varepsilon}}$. $E\varepsilon = e$ which is Euler’s constant $\approx 0.577$.}

If he chooses to remain unmarried, denoted by $j = 0$, his utility will be:

$$v_{i0g} = \Pi_{i0} + \varepsilon_{i0g}$$

(7)

where $\varepsilon_{i0g}$ is also an idiosyncratic payo\-ff which is another i.i.d. extreme value random variable.

This man $g$ can choose to marry one of $J$ types of spouses or not to marry. The utility from his optimal choice will satisfy:

$$v_{ig} = \max_j [v_{i0g}, \ldots, v_{ijg}, \ldots, v_{iJg}]$$

(8)

Let $\mu_{ij}$ be the number of type $i$ men who wants type $j$ spouses. When there are many type $i$ males, McFadden 1974 showed that type $i$’s quasi-demand for $j$ spouses satisfy:

$$\ln \frac{\mu_{ij}}{\mu_{i0}} = \Pi_{ij} - \bar{\tau}_{ij} - \Pi_{i0}$$

(9)

Turning to the marital choices of women, let the utility of female $k$ of type $j$ who marries a male of type $i$ be:

$$V_{ijk} = \bar{\tau}_{ij} + \epsilon_{ijk}$$

(10)

As discussed above, $\tau_{ij}$ is the systematic marital share of the wife. $\epsilon_{ijk}$ is her idiosyncratic payo\-ff. Assume that $\epsilon_{ijk}$ is an i.i.d. extreme value random variable.

If she chooses to remain unmarried, denoted by $i = 0$, her utility will be:

$$V_{0jk} = \Pi_{0j} + \epsilon_{0jk}$$

(11)

where $\epsilon_{0jk}$ is also an idiosyncratic payo\-ff which is another i.i.d. extreme value random variable.

This woman $k$ can choose to marry one of $I$ types of spouses or not to marry. The utility from her optimal choice will satisfy:

$$V_{jk} = \max_j [V_{0jk}, \ldots, V_{ijk}, \ldots, V_{Ijk}]$$

(12)

Let $\pi_{ij}$ be the number of type $j$ women who want type $i$ spouses. When there are many type $j$ females, type $j$’s quasi-supply for $i$ spouses satisfy:

$$\ln \frac{\pi_{ij}}{\pi_{0j}} = \bar{\tau}_{ij} - \Pi_{0j}$$

(13)
For every $I \times J$ submarriage market, let $\tau_{ij}$ be the female equilibrium share of marital surplus in the $\{i, j\}$ submarriage market which equilibrates the demand and supply of spouses in all submarkets simultaneously. In this case, the equilibrium number of $\{i, j\}$ marriages, $\mu_{ij}$, will satisfy:

$$\mu_{ij} = \mu_{0j} = \Pi_{ij} \forall i, j$$  \hspace{1cm} (14)

CSSa shows existence of marriage market equilibrium for a more general class of models.

Imposing marriage market clearing, (14), to the quasi-demand equation, (9), and to quasi-supply equation, (13), we get the male and female net gains equations respectively:

$$\ln \frac{\mu_{ij}}{\mu_{i0}} = \Pi_{ij} - \tau_{ij} - \Pi_{i0}$$  \hspace{1cm} (15)

$$\ln \frac{\mu_{ij}}{\mu_{0j}} = \tau_{ij} - \Pi_{0j}$$  \hspace{1cm} (16)

Net gains, $\ln \mu_{ij}/\mu_{i0}$ and $\ln \mu_{ij}/\mu_{0j}$, are observable by the researcher. However, observing the two net gains is not sufficient to identify the four unknowns, $\Pi_{ij}$, $\tau_{ij}$, $\Pi_{i0}$ and $\Pi_{0j}$. Moreover, $\tau_{ij}$ is endogenous. Thus net gains will change depending on changes in population supplies.

Add the two net gains equations to get the CS MMF:

$$\ln \frac{\mu_{ij}}{\mu_{0j}} = \frac{\Pi_{ij} - \Pi_{i0} - \Pi_{0j}}{2} = \pi_{ij} \forall i, j$$  \hspace{1cm} (17)

CS calls the left hand side of (17) the total gains to marriage. It is equal to ratio of the number of marriages to the geometric average of the unmarrieds. The right hand side, $\pi_{ij}$, is equal to the systematic marital surplus of an $\{i, j\}$ marriage minus their systematic surpluses from not marrying.

$\pi_{ij}$ is fixed. It does not depend on population supplies. The left hand side of (17) is observable. Thus we can estimate $\pi_{ij}$. The CS MMF is non-parametric. It will fit any cross section marriage distribution as long as there is no thin cell.

$\Pi_{ij}$, $\Pi_{i0}$ and $\Pi_{0j}$ are not separately identified. The discussion here assumes $\Pi_{ij}$ as a primitive. CS assumes that $\Pi_{ij}$ depends on the separate spousal contributions to marital surplus. These separate spousal contributions are not identified. There are $I \times J$ total gains and $I \times J$ unknown parameters, $\pi_{ij}$.

The CS MMF is just identified. The transferable utilities (constant marginal utilities of income) assumption is necessary for identification of the MMF with only marriage matching data. Without transferable utilities, parameters which determine the marginal utilities of income will also need to be identified and the MMF will be unidentified without additional restrictions. CSSa and CSSb extend the model beyond transferable utilities and brings in additional data, spousal labor supplies, to identify parameters.
Since not all structural parameters are identified, in what sense is CS MMF a marriage matching function? Let $\pi$ be the matrix of total gains parameters where a typical element is $\pi_{ij}$. CS shows that:

**Proposition 1** Consider an equilibrium distribution of marriages, $\mu$, with population vectors $M$ and $F$, and parameter matrix $\pi$. The equilibrium distribution of marriages varies uniquely for small variations in $M$, $F$ and $\pi$.

The above proposition says that, for small changes in population vectors and $\pi$, the CS MMF is a MMF as defined in section 1.

The question of global uniqueness remains open.

Although there are substitution effects in the CS MMF, their analytic properties are not fully understood.

Unlike Dagsvik’s MMF, the CS MMF is homogenous of degree one in population supplies. Botticini and Siow 2007 provides evidence from three different societies which shows that the constant returns to scale assumption for marriage markets is reasonable.

CS uses the MMF to study how the legalization of abortion in the United States affect the demand for marriage. Choo Siow 2006a uses it to study marriage and cohabitation in contemporary Canada. BSV uses it to study the impact of the famine caused by the great leap forward in China on marital behavior of the famine affected cohorts. I will use this application to show the difference between the harmonic mean MMF and CS.

Although not discussed in CS, another result is immediate. Subtract male net gains, (15), from female net gains, (16), to get:

$$\tau_{ij} = \ln \frac{\mu_{i0}}{\mu_{0j}} + \frac{\Pi_{ij} + \Pi_{0j} - \Pi_{i0}}{2}$$

(18)

$\ln \frac{\mu_{i0}}{\mu_{0j}}$ is the log ratio of the equilibrium number of unmarried type $i$ men to unmarried type $j$ women. I call $T_{ij} = \ln \frac{\mu_{i0}}{\mu_{0j}}$ marriage market tightness. $T_{ij}$ summarizes marriage market conditions for the \{i, j\} match.

Equation (18) implies:

**Proposition 2** Holding marital outputs, $\Pi_{ij}$, $\Pi_{i0}$ and $\Pi_{0j}$, constant, the equilibrium share of marital output which accrues to the wife, $\tau_{ij}$, is increasing in marriage market tightness, $T_{ij}$.

Proposition 2 is empirically useful. Holding marital outputs, $\Pi_{ij}$, $\Pi_{i0}$ and $\Pi_{0j}$, constant, variation in $T_{ij}$, marriage market tightness, can only be caused by variation in marital output of other spousal matches and or population supplies, $M$ and $F$. Marriage market tightness is observable. Holding $\Pi_{ij}$, $\Pi_{i0}$ and $\Pi_{0j}$, constant, variation in marriage market tightness summarizes variation in marriage market conditions for \{i, j\} marital matches. Thus if one has an empirical proxy for $\tau_{ij}$, one can estimate how this proxy varies with $T_{ij}$ controlling for $\Pi_{ij}$, $\Pi_{i0}$ and $\Pi_{0j}$. I will provide such an application below where I will justify and use spousal labor supplies as an empirical proxy for $\tau_{ij}$.

Finally,
Proposition 3. For every \( \{i, j\} \), the net gains equations, (15) and (16), imply (17) and (18) and vice versa.

3 Famine and marriage in China

An economic and social experiment, the great leap forward in 1959-1961, resulted in the largest 20th century famine in China. The famine drastically reduced the birth rates in those years thereby affecting the sex ratio for customary spousal age difference for the famine affected birth cohorts.

Sichuan province was hard hit by the famine which also disproportionately affected the countryside. Figure 1 shows the number of individuals in rural Sichuan by age in 1990. The figure shows that there were substantially fewer individuals between ages 29 to 31, who were born during the great leap, relative to adjacent ages. Figure 2 shows that the sex ratio (ratio of men to women) of the famine born cohorts did not substantially change relative to their adjacent age cohorts. However the sex ratios at the customary spousal age difference (males being three years older) for the famine affected cohorts were substantially affected. Famine born men and women faced a surplus of their customary aged spouses. On the other hand, pre famine born men and post famine born women faced a deficit of their customary aged spouses.

Figure 3 and 4 show the 1990 male and female marriage rates by age respectively. For both genders, the marriage rates of the famine affected cohorts were relatively small. Small marriage rate changes are consistent with what other researchers have found with other large exogenous demographic changes. This finding led previous researchers to argue that individuals are flexible in their spousal choice (E.g. Bergstrom and Lam 1994; Bhrolchain 2001; Esteve i Albert and Anna Cabré 2004).

Since famine born cohorts were small relative to adjacent aged birth cohorts, small changes in their marriage rates imply that they were “discriminated” against in the marriage market. In other words, marital surpluses with famine born spouses must have been low relative to other types of spouses. If their marital surpluses did not change, their marriage rates should have increased, particularly for men, because there was room to increase. As Georgens, et. al. 2005 has shown, famine born individuals have suffered adverse health consequences due to the famine. Thus their attractiveness as spouses were also likely to have suffered.

The observed marriage rates were due to a combination of population changes and changes in marital surpluses. To disentangle the two effects, we need to predict what the change in marital behavior would have been due to population changes alone. To this end, we estimated the harmonic mean and the CS MMFs using the 1982 census. The type of an individual is associated with his or her age. In 1982, the marital behavior of 29-31 year olds would not have been affected by the famine. Using the estimated 1982 marriage matching parame-

\(^{10}\) Porter 2007 has a related study. Also see Bergstrom and Lam 1994 and Francis 2007.

\(^{11}\) They were born at least 6 years before the famine began.
Figure 1: Sichuan number of individuals by age, 1990

Figure 2: Sichuan sex ratios female ages, 1990
Figure 3: Predicted and actual 1990 Sichuan male marriage rates

Figure 4: Predicted and actual 1990 Sichuan female marriage rates
ters, we predicted what the marriage rates in 1990 would have been using 1990 population vectors.

Figures 3 and 4 show the predicted male and female marriage rates from the two models respectively. For both genders, the predicted marriage rates from the harmonic mean MMF often exceed 1, a nonsensical result. These violations occurred because the harmonic mean MMF do not impose required accounting identities, substitution spousal effects are absent, and the changes in sex ratios of customary spousal age differences were large. Thus as previous researchers have observed, the standard MMF used by demographers is a poor empirical MMF.

On the other hand, the predicted marriage rates from the CS MMF behave sensibly. In Figures 3 and 4, the predicted marriage rates are above average for the famine born cohorts and below average for the adjacent aged birth cohorts. No accounting constraint is violated. Note that actual female marriage rates were over 0.95 for most ages. Even with large changes in sex ratios of the customary spousal age differences for the famine born cohorts, their predicted marriage rates remained below 1. The predicted female marriage rates for famine born cohorts were very similar to that for predicted adjacent aged cohorts. In other words, the CS MMF respects both the accounting constraints of MMFs and also captures the flexibility of individuals in their marital choices. These two attributes show the advantage of the CS MMF over the harmonic mean MMF.

In Figure 3, famine born males had lower marriage rates than predicted and pre famine born males had higher marriage rates than predicted. These two features are explained by the hypothesis that famine born males generated lower marital surpluses than pre famine born males. In other words, famine born males had lower total gains to marriage than their 1982 same aged peers.

Figure 4 shows that the discrepancies between predicted and actual female marriage rates were small. These small discrepancies obscure the changes in total gains to marriage for famine born females. The discrepancies between male predicted and actual marriage rates showed that total gains in 1990 changed substantially from 1982 estimated total gains. BSV showed that predicted marital matches in 1990 were different from actual marital matches. Thus the small discrepancies between predicted and actual female marriage rates in 1990 obscure estimated changes in total gains to marriage between 1982 and 1990 for famine affected birth cohorts.

BSV also estimated a CS MMF where the type of an individual was defined by their education and age. The predicted 1990 marriage rates were not significantly different from the simpler model where type only depended on age.

4 The collective marriage matching model

Economists have developed empirical models on intra-household allocations. A benchmark model is the collective model. The attractive feature of the collective model is that it is an empirically tractable model of intra-household allocation.
where intra-household bargaining power plays a central role in determining the allocation. An open question is whether the collective model can be embedded into an equilibrium marriage matching framework. CSSa and CSSb answer this in the affirmative. We show how the collective model can be embedded into the CS marriage matching framework. In addition, we also extend the CS model beyond transferable utilities.

Again, consider a society in which there are $I$ types of men and $J$ types of women, and population vectors $M$ and $F$. Although CSSa includes risk sharing between the spouses, it will be convenient for us to abstract from risk sharing here. In this case, the model is purely static. Individuals choose who to marry if they want to marry and also the type of marriage they want to engage in. An $\{i, j, s\}$ marriage is a marriage between a type $i$ man and a type $j$ woman in a $s$ type living arrangement. A living arrangement may depend on where the couple is cohabitating or formally married, whether the wife works or not, whether they have children or not, etc..

At the time of marriage, wages and non-labor income for each household are known. We normalize the prices of all consumption goods to one. Each household has to choose spousal labor supplies, public and private consumption. The rationale for including public good consumption within marriage is to capture resources allocated to children in the marriage.

Consider an $\{i, j, s\}$ marriage. Let $C_{ijs}$ ($c_{ijs}$) be the private consumption of the wife (husband), $K_{ijs}$ the household’s expenditures on public goods and $H_{ijs}$ ($h_{ijs}$) the wife’s (husband’s) labor supply. We normalize the total amount of time for each individual to 1. Utilities from private and public consumption, and labor supplies are described by:

$$U_{ijs}[C_{ijs}, 1 - H_{ijs}, K_{ijs}]$$

and

$$u_{ijs}[c_{ijs}, 1 - h_{ijs}, K_{ijs}]$$

respectively. The felicity functions, $U_{ijs}(\cdot)$ and $u_{ijs}(\cdot)$, depend on $i$ and $j$ to allow for differences in home production technologies across different types of marriages. If a woman chooses not to marry, then $i = 0$ and if a man chooses not to marry, $j = 0$.

Let $W_{ijs}$ be the wife’s wage, $w_{ijs}$ be the husband’s wage and $A_{ijs}$ be their joint non-labor or asset income. Here we are making a very strong assumption that all marriages of the same type, $\{i, j, s\}$, have the same wages and asset income. CSSa relaxes this strong assumption.

The hallmark of the collective model is that it assumes efficient allocation of intra-household resources. The efficient allocation can posed as a social planner solving:

$$\max_{\{C_{ijs}, c_{ijs}, H_{ijs}, h_{ijs}, K_{ijs}\}} U_{ijs}[C_{ijs}, 1 - H_{ijs}, K_{ijs}] + p_{ijs} u_{ijs}[c_{ijs}, 1 - h_{ijs}, K_{ijs}] \quad (P1)$$

subject to the family budget constraint

$$c_{ijs} + C_{ijs} + K_{ijs} \leq A_{ijs} + W_{ijs} H_{ijs} + w_{ijs} h_{ijs}$$

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In problem (P1), the planner chooses family consumption and labor supplies to maximize the weighted sum of the wife’s and the husband’s felicities subject to their family budget constraint.\(^{12}\) The weight allocated to the husband’s felicity is denoted \(\bar{p}_{ij}s, \bar{p}_{ij}s \in R^+\), where \(\bar{p}_{ij}s > 1\) implies the husband has more weight than the wife and vice versa. We will refer to \(\bar{p}_{ij}s\) as the husband’s power. The social planner takes \(\bar{p}_{ij}s\) as given when solving the intra-household allocation problem. The determination of \(\bar{p}_{ij}s\) itself occurs in the marriage market.

Let \(Q_{ij}(\bar{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s)\) and \(q_{ij}(\bar{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s)\) be the indirect felicity functions of the wife and the husband, respectively. It is straightforward to show that:

**Proposition 4** The changes in spousal utilities as the husband’s power, \(\bar{p}_{ij}s\), increases satisfy:

\[
\frac{\partial Q_{ij}(\bar{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s)}{\partial \bar{p}_{ij}s} = -\bar{p}_{ij}s \frac{\partial q_{ij}(\bar{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s)}{\partial \bar{p}_{ij}s} < 0 \tag{19}
\]

The wife’s utility falls and the husband’s utility increases as \(\bar{p}_{ij}s\) increases. Equation (19) traces the redistribution of spousal utilities as the husband’s power increases. We will now study how spousal labor supplies change as husband’s power changes. Denote \(H_{ij}(\bar{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s)\) and \(h_{ij}(\bar{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s)\) as the labor supplies for the wife and husband that result from solving (P1). In general, it is difficult to determine analytically how spousal labor supplies respond to changes in \(\bar{p}_{ij}s\). Building on Blundell, et. al., CSSa shows that if the public good is weakly separable from private consumption and leisure, and by restricting leisure (with suitably defined individual private income) and the public good to be normal goods for each spouse:

**Proposition 5** The wife’s labor supply is increasing in \(\bar{p}_{ij}s\), whereas the husband’s labor supply is decreasing in the husband’s power, \(\bar{p}_{ij}s\):

\[
\frac{\partial H_{ij}(\bar{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s)}{\partial \bar{p}_{ij}s} > 0 \tag{20}
\]
\[
\frac{\partial h_{ij}(\bar{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s)}{\partial \bar{p}_{ij}s} < 0 \tag{21}
\]

These propositions are expected.

### 4.1 Marriage decisions and outcomes

Following CS, agents decide whether to marry and who to marry and the type of living arrangement if they choose to marry. A woman can choose between \(I\) types of men and \(S\) type of living arrangements and also to remain unmarried (unattached). So she has \(I \times S + 1\) marital choice. Similarly, a man has \(J \times S + 1\)

\(^{12}\) Problem P1 is based on Blundell, et. al..
For a particular woman of type $j$, her indirect utility from an \{i, j, s\} marriage is:

$$V_{ij}s(\vec{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s, \epsilon_{ij}s) = Q_{ij}s(\vec{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s) + \Gamma_{ij}s + \epsilon_{ij}s$$ (22)

$\Gamma_{ij}s$ is every type $j$ woman’s invariant gain from marrying a type $i$ man in living arrangement $s$. It is independent of $\vec{p}_{ij}s$, $W_{ij}s$, $w_{ij}s$, $A_{ij}s$ and $\epsilon_{ij}s$. $\Gamma_{ij}s$ is used to fit the observed marriage matching distribution. $\epsilon_{ij}s$ is the particular woman’s idiosyncratic payoff which is different for every woman. $\epsilon_{ij}s$ is an i.i.d. extreme value random variable. If the woman remains unmarried, denote her indirect utility as $V_{0j}(W_{0j}, A_{0j}) = \Gamma_{0j} + Q_{0j}(W_{0j}, A_{0j}) + \epsilon_{0j}$ where $\epsilon_{0j}$ is an i.i.d. extreme value random variable.

Let her vector of idiosyncratic extreme value be $\epsilon_j$. The indirect utility from her optimal choice will satisfy:

$$V_j(\epsilon_j) = \max \{V_{0j}(p_{0j}, W_{0j}, A_{0j}, \epsilon_{0j}), .., V_{ij}s(\vec{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s, \epsilon_{ij}s), .., V_{ij}s(\vec{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s, \epsilon_{ij}s)\}$$

The indirect utility to a particular type $i$ man in an \{i, j, s\} marriage is:

$$v_{ij}(\vec{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s, \epsilon_{ij}s) = q_{ij}(\vec{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s) + \gamma_{ij}s + \epsilon_{ij}s$$ (24)

$\gamma_{ij}s$ is every type $i$ man’s invariant gain from marrying a type $j$ woman in living arrangement $s$. It is independent of $\vec{p}_{ij}s$, $W_{ij}s$, $w_{ij}s$, $A_{ij}s$ and $\epsilon_{ij}s$. $\gamma_{ij}s$ is used to fit the observed marriage matching distribution. $\epsilon_{ij}s$ is the particular woman’s idiosyncratic payoff which is different for every woman. $\epsilon_{ij}s$ is an i.i.d. extreme value random variable. If he remains unmarried, denote his indirect utility as $v_{i0}(w_{i0}, A_{i0}, \epsilon_{i0}) = \gamma_{i0}s + q_{i0}(w_{i0}, A_{i0}) + \epsilon_{i0}$ where $\epsilon_{i0}$ is an i.i.d. extreme value random variable.

The indirect utility from his optimal choice will satisfy:

$$v_i(\epsilon_i) = \max \{v_{i0}(p_{i0}, w_{i0}, A_{i0}, \epsilon_{i0}), .., v_{ij}s(\vec{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s, \epsilon_{ij}s), .., v_{ij}s(\vec{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s, \epsilon_{ij}s)\}$$ (25)

As in CS, if there are many woman of each type, the quasi demand of type $j$ women for \{i, j, s\} marriages satisfy:

$$\ln \bar{p}_{ij}s - \ln \bar{p}_{0j} = (\Gamma_{ij}s - \Gamma_{0j}) + Q_{ij}s(\bar{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s) - Q_{0j}(W_{0j}, A_{0j})$$ (26)

where $\bar{p}_{ij}s$ is the number of \{i, j, s\} marriages demanded by $j$ type females and $\bar{p}_{0j}$ is the number of type $j$ females who choose to remain unmarried.

Similarly, for every type of man $i$, the quasi demand of type $i$ men for \{i, j, s\} marriages is:

$$\ln \bar{w}_{ij}s - \ln \bar{w}_{i0} = (\gamma_{ij}s - \gamma_{i0}) + q_{ij}(\bar{p}_{ij}s, W_{ij}s, w_{ij}s, A_{ij}s) - q_{i0}(w_{i0}, A_{i0})$$ (27)

where $\bar{w}_{ij}s$ is the number of \{i, j, s\} marriages supplied by $j$ type males and $\bar{w}_{i0}$ is the number of type $i$ males who choose to remain unmarried.

When the marriage market clears, $\bar{p}_{ij}s = \bar{w}_{ij}s = \mu_{ij}s$ for all \{i, j, s\} sub-markets. In a more general setup, CSSa shows that:
Proposition 6 A market equilibrium exists. 

The sketch of the proof is as follows. Consider any \( \{i, j, s\} \) sub marriage market. Holding all other \( p_{i'j's'} \), \( i'j's' \neq ijs \), constant, as \( p_{ijs} \) increases, weakly more type \( i \) men will enter into \( \{i, j, s\} \) matches and weakly less of other types of matches. Weakly less type \( j \) women will be willing to go into \( \{i, j, s\} \) matches and weakly more into other types of matches. Thus the excess demand function for spouses in every \( \{i, j, s\} \) submarriage market satisfy the weak gross substitute condition. A standard result from general equilibrium theory says that a marriage market equilibrium exists.

Let \( p_{ijs} \) denote the equilibrium husband’s power. Imposing equilibrium, the male and female net gains equations respectively become:

\[
\ln \frac{\mu_{ijs}}{\mu_{i0}} = (\gamma_{ijs} - \gamma_{i0}) + q_{ijs}(p_{ijs}, W_{ijs}, w_{ijs}, A_{ijs}) - q_{i0}(w_{i0}, A_{i0}) \tag{28}
\]

\[
\ln \frac{\mu_{ijs}}{\mu_{0j}} = (\Gamma_{ijs} - \Gamma_{0j}) + Q_{ijs}(p_{ijs}, W_{ijs}, w_{ijs}, A_{ijs}) - Q_{0j}(W_{0j}, A_{0j}) \tag{29}
\]

We add the net gains equations to get:

\[
\ln \frac{\mu_{ijs}}{\sqrt{\mu_{i0}\mu_{0j}}} = \left(\frac{(\gamma_{ijs} - \gamma_{i0}) + q_{ijs}(p_{ijs}, W_{ijs}, w_{ijs}, A_{ijs}) - q_{i0}(w_{i0}, A_{i0})}{2}\right) + \left(\frac{(\Gamma_{ijs} - \Gamma_{0j}) + Q_{ijs}(p_{ijs}, W_{ijs}, w_{ijs}, A_{ijs}) - Q_{0j}(W_{0j}, A_{0j})}{2}\right) \tag{30}
\]

The left hand side of (30) is the total gains to marriage which is the same as in (17). The right hand side of (30) depends on \( p_{ijs} \), the husband’s power, wages and asset incomes. Note that \( p_{ijs} \) is an endogenous variable. It has to adjust to clear the marriage market. Thus in general, total gains is not equal to only exogenous variables as in CS.\(^{13}\)

4.2 Market tightness, discrete and continuous spousal labor supply decisions

Subtracting male net gain, (28), from female net gain, (29), to get:

\[
T_{ij} = \ln \frac{\mu_{i0}}{\mu_{0j}} = (\Gamma_{ijs} - \Gamma_{0j}) + Q_{ijs}(p_{ijs}, W_{ijs}, w_{ijs}, A_{ijs}) - Q_{0j}(W_{0j}, A_{0j})
\]

\[
- ((\gamma_{ijs} - \gamma_{i0}) + q_{ijs}(p_{ijs}, W_{ijs}, w_{ijs}, A_{ijs}) - q_{i0}(w_{i0}, A_{i0})) \tag{31}
\]

where \( T_{ij} \) is marriage market tightness. Equation (31) is a generalization of (18). Unlike proposition 2 which is silent on the determinants of marital output, equation (31) says that important determinants of marital outputs include

\(^{13}\)If husband’s power is approximately one, i.e. both spouses have equal utility weight in the planner’s problem (P1), then using proposition 4, one can show that the right hand side of (30) is exogenous.
spousal wages and asset incomes. It is important to emphasize that $T_{ijs}$ and $p_{ijs}$ are both endogenous variables and simultaneously determined. Thus equation (31) is not a statement about the causal effect of $T_{ijs}$ on $p_{ijs}$.

To obtain a causal statement, consider a change in an exogenous parameter, $x$. Let $Z_{ijs} = [W_{ijs}, w_{ijs}, A_{ijs} w_{i0}, A_{i0}, w_{0j}, A_{0j}]$. Using proposition 4 and (31)

$$\frac{\partial p_{ijs}}{\partial x} = \rho_{ijs} \frac{\partial \tilde{T}_{ijs}}{\partial x} + \rho_{ijs} \frac{\partial (Q_{ijs} - Q_{0j}) - (q_{ijs} - q_{0j})}{\partial Z_{ijs}} \frac{\partial Z_{ijs}}{\partial x} - \rho_{ijs} \frac{\partial T_{ijs}}{\partial x},$$

(32)

$$\tilde{T}_{ijs} = (\Gamma_{ijs} - \Gamma_{0j}) - (\gamma_{ijs} - \gamma_{0i}); \rho_{ijs} \equiv [(1 + p_{ijs}) \frac{\partial q_{ijs}}{\partial p_{ijs}}]^{-1} > 0$$

A change in $x$ induces three changes in the husband’s power. The first is the effect of a change in relative spousal invariant gains, $\tilde{T}_{ijs}$, on power. The second term is proportional to the change in the difference in expected spousal utilities due to a change in $Z_{ijs}$ caused by a change in $x$. The third term is proportional to the change in marriage market tightness. Since $\rho_{ijs} > 0$, when market tightness increases, the husband’s power is predicted to fall.

I can invert equation (31) to derive an expression for the husband’s power:

$$p_{ijs} = g_{ijs}(T_{ijs}, \tilde{T}_{ijs}, Z_{ijs})$$

(33)

The empirical implication of equation (33) is as follows. Holding $\tilde{T}_{ijs}$ and $Z_{ijs}$ constant, if $T_{ijs}$ stays the same, then $p_{ijs}$ stays the same and total gains in equation (30) stays the same. I now use equation (33) to derive empirical implications of our theory regarding the effect of marriage matching on spousal labor supplies. Recall that $H_{ijs}(p_{ijs}, W_{ijs}, w_{ijs}, A_{ijs})$ is the hours of work of a wife of type $j$ in an $\{i,j,s\}$ marriage. Using proposition 5, equation (32) and (33), the labor supply of a wife in an $\{i,j,s\}$ match is:

$$H_{ijs}(T_{ijs}, \tilde{T}_{ijs}, Z_{ijs})$$

(34)

$$\frac{\partial H_{ijs}}{\partial T_{ijs}} < 0$$

(35)

That is, holding invariant gains to marriage and $Z_{ijs}$ constant, the wife’s labor supply depends inversely on marriage market tightness. The exogenous variables that can change to affect tightness are the population vectors, $M$ and $F$ and determinants of marital surpluses from other marital matches that are uncorrelated with $\tilde{T}_{ijs}$ and $Z_{ijs}$.

Similarly, the labor supply of a husband in an $\{i,j,s\}$ match is:

$$h_{ijs}(T_{ijs}, \tilde{T}_{ijs}, Z_{ijs})$$

(36)

$$\frac{\partial h_{ijs}}{\partial T_{ijs}} > 0$$

(37)

Holding relative invariant gains to marriage, wages and asset incomes from $\{i,j,s\}$ and unmarried matches constant, the husband’s labor supply depends inversely on marriage market tightness.
Another use of market tightness is to consider two types of marital matches, \(\{i, j, s\}\) and \(\{i, j, s'\}\). For my application below, \(s\) are marriages in which a spouse works and \(s'\) are marriages in which a spouse does not participate in the marriage market. Using equation (31), market tightness is the same both kinds of marital matches which imply:

\[
\Gamma_{ij} + Q_{ij} (p_{ij}, W_{ij}, w_{ij}, A_{ij}) - \gamma_{ij} = q_{ij} (p_{ij}, W_{ij}, w_{ij}, A_{ij}) = \Gamma_{ij} + Q_{ij} (p_{ij}, W_{ij}, w_{ij}, A_{ij}) - \gamma_{ij} - q_{ij} (p_{ij}, W_{ij}, w_{ij}, A_{ij})
\]

(38)

Consider a change in population supplies, \(x\). Holding \(b_{ij}\) and \(Z_{ij}\) constant, using equation (38) and proposition 4,

\[
(1 + p_{ij}) \frac{\partial q_{ij}}{\partial p_{ij}} \frac{\partial p_{ij}}{\partial T_{ij}} = (1 + p_{ij}) \frac{\partial q_{ij}'}{\partial p_{ij}'} \frac{\partial p_{ij}'}{\partial T_{ij}}
\]

(39)

Equation (39) says that if the marginal utility of tightness is larger for living arrangement \(s\) than \(s'\), then \(p_{ij}\) is smaller than \(p_{ij}'\) and vice versa. So if one can compare marginal utilities of tightness between the two living arrangements, one can learn about their relative husband’s powers.

To this end, use the male net gains equations for marital matches, \(\{i, j, s\}\) and \(\{i, j, s'\}\) to get:

\[
\ln \frac{\mu_{ij}}{\mu_{ij}'} = \gamma_{ij} + q_{ij} (p_{ij}, W_{ij}, w_{ij}, A_{ij}) - \gamma_{ij} - q_{ij}' (p_{ij}', W_{ij}', w_{ij}', A_{ij}')
\]

(40)

Holding \(\Gamma_{ij}\) and \(Z_{ij}\) constant, equations (39) and (40) imply:

\[
\frac{\partial \ln \frac{\mu_{ij}}{\mu_{ij}'}}{\partial T_{ij}} = -\left(\frac{p_{ij} - p_{ij}'}{1 + p_{ij}}\right) \frac{\partial q_{ij}'}{\partial p_{ij}'} \frac{\partial p_{ij}'}{\partial T_{ij}}
\]

(41)

Since \(\frac{\partial p_{ij}'}{\partial T_{ij}} < 0\) and \(\frac{\partial q_{ij}'}{\partial p_{ij}'} > 0\), \(\frac{\partial q_{ij}'}{\partial p_{ij}'} > 0\). Then:

**Proposition 7** Holding \(\Gamma_{ij}\) and \(Z_{ij}\) constant,

\[
\text{sign}(\partial \ln \frac{\mu_{ij}}{\mu_{ij}'} / \partial T_{ij}) = \text{sign}(p_{ij} - p_{ij}')
\]

In words, holding \(\Gamma_{ij}\) and \(Z_{ij}\) constant, if the share of \(s\) living arrangements increases relative to that of \(s'\) living arrangements as tightness increases, the husband’s power in \(s\) living arrangements exceeds that in \(s'\) living arrangements.

I will apply the above proposition to interpret how the log odds of spouses participating in the labor market respond to variations in tightness.

Finally, there is the equivalent of proposition 3:

**Proposition 8** For every \(\{i, j, s\}\), the net gains equations, (28) and (29), imply (30) and (31), and vice versa.

As earlier, Proposition 8 implies that the net gains equations imply total gains and market tightness equations and vice versa.
5 Estimation and testing strategies

CS, Choo Siow 2006a and BSV estimated all the parameters of a MMF. The estimated MMF could be used to address counterfactual experiments for the entire marriage distribution as was shown with the Sichuan data presented earlier.

We have not yet attempted to estimate all the parameters of the collective marriage matching model, partly because we have not yet establish what is needed for identification, and partly because there are so many more parameters to estimate.

Although the collective marriage matching model is a general equilibrium model, it admits decentralized estimation and testing strategies as is usual for competitive equilibrium models. For any marital match, \{i, j, s\}, the husbands’ and wives’ labor supply functions are respectively:

\begin{align*}
H_{ijs}(p_{ijs}, W_{ijs}, w_{ijs}, A_{ijs}) & \Leftrightarrow h_{ijs}(p_{ijs}, W_{ijs}, w_{ijs}, A_{ijs})
\end{align*}

These functions only depend on exogenous parameters specific to the \{i, j, s\} match, \(W_{ijs}, w_{ijs}, A_{ijs}\), and an endogenous parameter, husband’s power, \(p_{ijs}\), which is also specific to the match. \(p_{ijs}\) encapsulates all general equilibrium concerns. Thus if we can find determinants or proxies for \(p_{ijs}\) which are unrelated to \(W_{ijs}, w_{ijs}\) and \(A_{ijs}\), we can estimate the labor supply functions for the subset of marital matches that we have empirical specifications for the labor supply functions and the data to estimate them. These parameter estimates will be consistent even though we ignore other marital matches where we do not have data and or empirical specifications for the labor supply functions.

A similar decentralization result holds for estimating the marital matching equations, total gains and marriage market tightness equations, (30) and (31), or equivalently net gains equations, (28) and (29). For any \{i, j, s\} match, in addition to \(W_{ijs}, w_{ijs}, A_{ijs}\), and \(p_{ijs}\), we also need \(W_{0j}, A_{0j}, w_{0i}\) and \(A_{0i}\). If we have empirical proxies for these variables and empirical specifications for the marital matching equations for a subset of marital matches, we can estimate the marital matching equations for that subset.

As will be shown below in section 5.2, our framework generates empirically testable overidentifying restrictions for the collective marriage matching model using a subset of marital matches.

5.1 Empirical evidence using market tightness

Building on CSSa, I estimate the effect of changes in marriage market tightness on spousal labor supplies using the 2000 US census. Define an individual’s type as a combination of race, age and education. For each gender, there are four contiguous age categories of 5 years each. The ages are staggered by two years across gender to reflect the fact that most men marry younger women. The youngest female and male age categories, are 25-29 and 27-31 respectively. For each gender, there are two schooling categories: high school graduates (at least 12 and up to and including 15 years of education) and college graduates (16 years of education and higher). For each race and gender, there are 8 potential
Table 1: Effects of market tightness on log odds of labor force participation of spouses

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<td>Ti j</td>
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<td>(0.016)**</td>
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Table 2: Effects of market tightness on log usual hours worked per weeks of spouses

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<td>Ti j</td>
<td>0.037</td>
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<td></td>
<td>(0.002)**</td>
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^ Fraction of SD increase of dependent variable due to 1 SD increase in Ti j.
types of individuals. Since I am only consider same race marriages, whites, blacks and hispanics, there are potentially 64 × 3 = 192 types of marital matches for each society.

I treat each state as a separate isolated society. With 50 states, there are potentially 192 × 50 = 9600 cells across all sub-marriage markets. However, the majority of these potential cells (marital match × state) have few or no marriages. To avoid thin cell problems, I delete a cell if the number of marriages in that cell is less than 5. For most regressions, I have about 2400 observations. Most of the missing cells are due to non-white marriages, with large spousal age differences, in states with small populations. I also exclude mixed race couples to mitigate thin cells. Excluding thin cells from the empirical analysis should not affect the consistency of my estimates. As discussed above, my labor supplies regressions have to hold for any subset of marital matches.

In CSSa and the sample here, an observation is a cell. Mean $T_{ij}^{w}$ is -0.129 with a standard deviation of 0.962. About half the cells involve white marriages and the rest are black and hispanic marriages. Mean ages by cell are between 35-39 for women and 37-41 for men. For both gender, there are slightly more high school graduates than college graduates. To control for aggregate labor market conditions in an individual's local marriage market, we define the following three variables to characterize the earnings and asset income distributions. First, conditional on positive annual labor earnings for a type of unmarried individual, we construct the mean and standard deviation of log annual labor earnings for the distribution of unmarried individuals (wage and salary income). The second measure is the fraction of individuals with zero labor earnings for each match type in each marriage market. Finally, I construct the analogous variables for asset earnings, defined from the Census as total personal income minus wage and salary income. Note that I do not include marital wages and asset incomes. In other words, I am using unmarried wages and asset income proxies as control functions for the wages and asset income for both marrieds and unmarrieds individuals.

The labor force participation rates for husbands and wives are 94% and 73%, respectively. I consider two measures of labor supply. My first measure is the log odds of LFP (labor force participation). Conditional on participating in the labor force, my second measure of labor supply is the log of usual hours worked per week. Mean usual hours worked for men and women were 45 and 34 hours respectively.

CSSa shows that after controlling for own sex ratios, variations in substitutes

\footnote{Also market tightness for mixed race couples which include white spouses are very different from own race couples because there are so many more whites than other races in the data. So we would need to have separate coefficients on tightness for each mixed race couples.}

\footnote{This is the difference between individual observations, where 80% of the marriages are among white couples, and observations by cell.}

\footnote{To be precise, we measure the fraction of individuals with non-positive non-labor income rather than zero non-labor income.}

\footnote{I also estimated the supply of log of weeks worked per year. The estimated elasticities were similar to that for usual hours of work per week.}
sex ratios also affect spousal labor supplies. With many different sex ratios in the labor supply regressions and the associated problem of multicollinearity, tightness provide a convenient summary statistic for marriage market conditions. An advantage to using \( T_{ij} \) in spousal labor supplies regressions, rather than sex ratios, is that the estimated coefficient has a clear behavioral interpretation. It alerts us to the conclusion that a labor supply regression with labor market controls and only the own sex ratio as a proxy for marriage market conditions has no clear causal interpretation.

Since tightness is endogenous, I instrument tightness by the population vectors in each state (by age and race and average educational attainment of their parents) twenty years earlier.

Table 1 shows the estimated effect of market tightness on the log odds of labor force participation (LFP) of wives and husbands. All regressions include state effects. These state effects control for state differences in labor market conditions as well as differences in state level invariant gains. Other controls include labor market and asset income controls at the individual type and state level, and individual type effects. I interpret the labor force participation status of a spouse as an living arrangement. In other words, marriages in which the wife works is one living arrangement and marriages in which the wife does not work is another living arrangements. Under this interpretation, the estimates of the coefficient on tightness in Table 1 provide estimates of \( \frac{\partial \ln \mu_{ij}}{\partial T_{ij}} \) in equation (41) where \( s \) denotes marriages with a working wife and \( s' \) those who do not.

Columns (1) to (3) are estimates for wives. \( T_{ij} \) is affected by variations in invariant gains and labor market conditions. I control for these variations with state and individual effects. There are variations in invariant gains that I do not control for. Instrumenting \( T_{ij} \) with sex ratios at birth should mitigate the endogeneity problem. The point estimate of 0.330 in column (1), which only includes state effects as the other covariates, shows that instrumenting \( T_{ij} \) is not sufficient to obtain a “right” estimated sign. Column (2) adds labor market conditions. In this case, the estimated coefficient on \( T_{ij} \) is -0.176 and it is statistically different from zero. Column (3) adds labor market conditions and individual effects. The point estimate on \( T_{ij} \) is -0.167 and the standard error is 0.065. So the point estimate is similar to that without individual effects but the standard error doubles. This fall in estimated precision is the same for the other specifications.

Using the point estimate of -0.176 in column (2) as a benchmark, a one standard deviation increase in \( T_{ij} \) will decrease the log odds LFP of wives by 0.26 standard deviation. Thus variations in tightness is quantitatively important for explaining variations in the log odds LFP of wives across matches and or societies. Ceteris paribus, women and men are more willing to enter into marriages where the wife does not work when \( T_{ij} \) increases. Equation (41) and

\[18\] For each \( \{i,j,s\} \) marriage, I add sex ratios by age, by race, and by education as substitute sex ratios.
the signs of our estimates also imply that husband’s power of husbands with working wives is lower than those without working wives.

Columns (4) to (6) are estimates for husbands. The point estimate of $T_{ijs}$ is -0.034 in column (4) which only includes state effects. Column (5) adds labor market conditions. In this case, the estimated coefficient on $T_{ijs}$ is 0.145 and it is statistically different from zero at the 1% level. Column (6) adds labor market conditions and individual effects. The point estimate on $T_{ijs}$ is 0.015 and the standard error is 0.072. The point estimate is smaller than in column (5) and the standard error is significantly larger. This fall in estimated precision is the same for the other specifications.

Using the point estimate of 0.145 in column (5) as a benchmark, a one standard deviation increase in $T_{ijs}$ will decrease the log odds LFP of husbands by 0.16 standard deviation. Thus variations in tightness has a smaller standardized impact on the log odds LFP of husbands than wives. This smaller impact by standard deviation is driven by the larger standard deviation in log LFP of husbands across cells. Equation (41) and the signs of our estimates also imply that husband’s power of husbands who work is higher than those who do not.

The estimates in Table 1 show that changing marriage market conditions substantially affect spousal LFP. These results are comparable to that found in the literature on the response of female LFP to changes in labor market conditions. Since estimates of LFP equations are standard in the labor literature, it is useful that the collective marriage matching model rationalizes these LFP equations in the context of the collective model. 19

Table 2 columns (1) to (3) show the estimated effects of market tightness on the log usual hours of work per week of working wives. Column (1) adds state effects. The point estimate on $T_{ijs}$ has the wrong sign and is statistically different from zero at the 5% level. Column (2) adds state effects and labor market conditions. The point estimate on $T_{ijs}$ is -0.028 and the estimated standard error is 0.005. Column (6) adds state effects, labor market conditions, individual effects. The point estimate on $T_{ijs}$ is -0.044 and the standard error is 0.010. Thus adding individual effects lower the estimated precision on $T_{ijs}$.

The estimates in Table 2 is qualitatively similar to their counterparts in Table 1. The estimated magnitudes on ln hrs/wk are smaller than for participation. Using the estimate in column (2) as a benchmark, a one standard deviation increase in $T_{ijs}$ results in 0.065 standard deviation decrease in ln hrs/wk for wives. So variation in tightness explains less of the variation in mean log usual hours of work per week of wives compared with log odds of LFP.

Columns (4) to (6) are estimates for husbands. As before, state effects alone in Column (4) results in a estimate that is statistically not different from zero. Column (5) adds state effects and labor market conditions. The point estimate on $T_{ijs}$ is 0.018 and the estimated standard error is 0.003. Column (6) adds state effects, labor market conditions, individual effects. The point estimate on $T_{ijs}$ is not statistically different from zero at the 5% level. Thus adding individual effects lower the estimated precision on $T_{ijs}$. The estimated

19 Blundell, et. al. 2007 provides an alternative interpretation.
elasticities on hrs/wk are smaller than for participation. Using the estimate in column (5) as a benchmark, a one standard deviation increase in $T_{ijs}$ results in 0.068 standard deviation increase in ln hrs/wk for husbands. So variation in tightness explains less of the variation in mean log usual hours of work per week of husbands than log odds LFP.

The empirical results discussed add to the literature on the estimated effects of changes in marriage market conditions on spousal labor supplies. The estimates are qualitatively consistent with both theory and the evidence in the literature. As already noticed in the literature, it is important to control for labor market conditions. Otherwise, the point estimates of the effects of $T_{ijs}$ spousal labor supplies consistently has the wrong sign. The fact that labor force participation is more affected than hours of work per week by variations in tightness is consistent with the finding in the literature that participation is more sensitive than hours of work to determinants of labor supplies. Finally, I find theoretically consistent estimated effects for male labor supplies which is uncommon for this literature.

A caveat is important. The empirical evidence also shows that market tightness is not always a sufficient statistic for own and substitute sex ratios. That is, in some spousal labor supply regressions which include market tightness as an endogenous regressor, I cannot reject the hypothesis that own and substitute sex ratios also affect spousal labor supplies. There are two non-mutually exclusive reasons for this finding. First, the instruments may not be valid. Second, the logit specification of spousal choice may be too restrictive.

At this stage, my interpretation of the empirical results is that market tightness is a first order approximation for marriage market conditions and not a sufficient statistic.

5.2 Do spousal labor supplies clear the marriage market?

While the previous empirical results and the results in the literature on the estimated effects of changes in sex ratios on spousal labor supplies show that changes in marriage market conditions affect spousal labor supplies, we do not know whether the estimated effects are large enough to clear the marriage market. One cannot address this question unless we estimate a structural model of marriage market clearing and spousal labor supplies.

Also as discussed above, the empirical evidence to date shows that $T_{ijs}$ is not always a sufficient statistic for marriage market condition in spousal labor supplies regressions. The exact cause for this finding is not known. Thus it is useful to consider an alternative empirical estimation and testing strategy of the collective marriage matching model. Instead of using market tightness, we exploit proposition 8 and use equilibrium quasi demand and supply equations.

---

20 The instrumental variable results presented here quantitatively and qualitatively similar to the OLS results which raises a question as to how powerful the instruments are.

21 The empirical industrial organization literature usually reject the basic logit specification when estimating discrete demand models.
CSSb does that for a particular set of marriages. Using the model in the previous section, consider the set of marriages, \( \{i, j, s\} \), in which both spouses work and there is no public goods consumption. In this case, CSSb shows that the intra-household allocation model is the same as that in CFL. CFL showed that the wife’s and husband’s labor supplies are:

\[
\begin{align*}
H_{ijs}(p_{ijs}, W_{ijs}, w_{ijs}, A_{ijs}) &= \tilde{H}_{ijs}(W_{ijs}, \tau_{ijs}(p_{ijs})) \\
\tilde{h}_{ijs}(p_{ijs}, W_{ijs}, w_{ijs}, A_{ijs}) &= \tilde{h}_{ijs}(w_{ijs}, A_{ijs} - \tau_{ijs}(p_{ijs}))
\end{align*}
\]

where \( \tilde{H}_{ijs} \) and \( \tilde{h}_{ijs} \) are standard Marshallian labor supply functions with \( \tau_{ijs}(p_{ijs}) \) and \( A_{ijs} - \tau_{ijs}(p_{ijs}) \). being asset incomes for the wife and husband respectively. Let husband’s power, \( p_{ijs} \), be a function of \( R_{ijs} \); \( W_{ijs} \); \( w_{ijs} \); \( A_{ijs} \) where \( R_{ijs} \) is a vector of other factors which affect husband’s power. When we hold invariant gains to marriage, wages and asset incomes constant, \( R_{ijs} \) consists of population vectors, \( M \) and \( F \), and determinants of marital surpluses from other marital matches including remaining unmarried.\(^{22}\)

Let \( \tau_{ijs} \) be an element of \( R_{ijs} \).

Dropping the \( ijs \) subscript for convenience, define \( B_1 = \frac{h_w}{h_A} \), \( D_1 = \frac{h_r}{h_A} \), \( E_1 = \frac{h_w}{h_A} \), and \( F_1 = \frac{h_r}{h_A} \). CFL shows:

**Proposition 9** The partial derivatives of \( \tau(p) \) are given by:

\[
\begin{align*}
\tau_{pA} &= \frac{D_1}{D_1 - F_1} \\
\tau_{pr} &= \frac{D_1 F_1}{D_1 - F_1} \\
\tau_{pw} &= \frac{B_1 F_1}{D_1 - F_1} \\
\tau_{pw} &= \frac{D_1 E_1}{D_1 - F_1}
\end{align*}
\]

CSSb shows that in the marriage market, the equilibrium quasi demand by type \( j \) women for \( \{i, j, s\} \) marriages satisfy:

\[
\ln \frac{\mu_{ij}{s}}{\mu_{0j}} = (\gamma_{ij} - \gamma_{0j}) + Q(W_{ijs}, \tau_{ijs}(p_{ijs})) - Q_{0j}(W_{0j}, A_{0j})
\]

Similarly, for every type of man \( i \), the equilibrium quasi demand of type \( i \) men for \( \{i, j, s\} \) marriages is:

\[
\ln \frac{\mu_{ij}{s}}{\mu_{i0}} = (\gamma_{ij} - \gamma_{i0}) + q_{ijs}(w_{ijs}, A_{ijs} - \tau_{ijs}(p_{ijs})) - q_{i0}(w_{i0}, A_{i0})
\]

\(^{22}\) \( R \) are distributional factors in CFL terminology. They are factors, other than determinants of preferences and budget constraints in \( \{i, j, s\} \) marriages, which affect husband’s power. Due to the set of marriages that they consider, CFL and CSSb work with \( \tau(R, W, w, A) \) notation directly.
Denote the net gains, \( N_{ijs} = \ln \frac{\mu_{ijs}}{\nu_{ijs}} \) and \( n_{ijs} = \ln \frac{\mu_{ijs}}{\nu_{i0}} \). \( N_{ijs} \) and \( n_{ijs} \) are observable by the researcher. Then the net gains reduced forms \( N_{ijs} \) and \( n_{ijs} \) satisfy:

\[
N_{ijs}(W_{ijs}, w_{ijs}, A_{ijs}, w_{i0}, A_{i0}, W_{i0}, A_{i0}, R_{ijs}) = (\Gamma_{ijs} - \Gamma_{0j}) + Q(W_{ijs}, \tau_{ijs}(p_{ijs})) - Q_{0j}(W_{i0}, A_{i0})
\]

\[
n_{ijs}(W_{ijs}, w_{ijs}, A_{ijs}, w_{i0}, A_{i0}, W_{i0}, A_{i0}, R_{ijs}) = (\gamma_{ijs} - \gamma_{i0}) + q_{ijs}(w_{ijs}, A_{ijs} - \tau_{ijs}(p_{ijs})) - q_{0j}(w_{i0}, A_{i0})
\]

Dropping the \( ijs \) subscript for convenience, define \( B_2 = \frac{N}{N_A}, D_2 = \frac{n}{n_A} \).

CSSb shows that:

**Proposition 10** The partial derivatives of \( \tau(p) \) are given by:

\[
\tau_{pA} = \frac{F_2}{F_2 - B_2}
\]

\[
\tau_{pr} = \frac{B_2 F_2}{F_2 - B_2}
\]

\[
\tau_{pw} = \frac{B_2 D_2}{F_2 - B_2}
\]

Propositions 9 and 10 provides us with the following set of over-identifying restrictions:

**Proposition 11**

\[
\tau_{pA} = \frac{D_1}{D_1 - F_1} = \frac{F_2}{F_2 - B_2}
\]

\[
\tau_{pr} = \frac{D_1 F_1}{D_1 - F_1} = \frac{B_2 F_2}{F_2 - B_2}
\]

\[
\tau_{pw} = \frac{B_1 F_1}{D_1 - F_1} = \frac{B_2 D_2}{F_2 - B_2}
\]

Proposition 11 allows us to estimate reduced form spousal labor supplies and net gains, and test whether spousal labor supplies clear the marriage market for the subset of marriages that those propositions apply to. Something like Proposition 11 should be expected. The CS class of marriage matching models assume that individuals make their marital choices based on comparing their indirect utilities from different choices. These indirect utilities depend on private consumption, public consumption and own labor supplies. Thus we should
expect net gains to depend on the determinants of husband’s power as well as own wages and asset incomes. What proposition 11 says is that since we can identify the determinants of husband’s power using spousal labor supplies as per proposition 9, and the determinants of husband’s power using net gains as per proposition 10, these two different ways of estimating these determinants should give the same results.

If one spouse does not work, there is only one spousal labor supply equation and the identification of the determinants of husband’s power fails. CSSb shows that the determinants of husband’s power continued to be identified by estimating the two net gains equations when one spouse does not work.

Preliminary empirical results from CSSb using the data from the 2000 US census show that there is support for the hypothesis that spousal labor supplies of dual working spouses without children clear their marriage market.

6 A lifecycle CS model

Choo Siow 2007 proposes and provide preliminary estimates for a lifecycle CS model with an exogenous divorce hazard and the same population of new entrants in every period. This section sketches a two period lifecycle CS model with no divorce and time varying supplies of new entrants. The objective of the model is to show that the CS framework extends naturally to lifecycle considerations. In general, I cannot get analytic solutions to the determinants of current marital matching behavior. Numerical solutions are of course available. In a special case (where there is no gender asymmetry), I show analytically that the current marriage behavior in this society depends on the past, current and future birth rates.

Every adult lives for two periods, one and two. The type of an individual is their age. The age of a male is indexed by $i$ and the age of a female is indexed by $j$. In any period (year), the type of an adult is defined by his or her age. I may also call age one adults young and age two adults old.

There is no divorce. An adult whose spouse dies will automatically return to the marriage market in the next period. For analytic convenience, widowed individuals and never married individuals of the same age and gender are the same in the marriage market.

Let $\{m^t_1, f^t_1\}$ be the number of young males and females who enter the society at year $t$. Let $\mu^t_{ij}$ be the number of age $i$ males who marry age $j$ females at year $t$. Without divorce, I have the following accounting identities:

$$m^t_{2} = m^t_{1} - \mu^t_{11}$$

$$f^t_{2} = f^t_{1} - \mu^t_{11}$$

There is no discounting.

Let $\pi^t_{ij}$ be the discounted within marriage payoffs where an age $i$ male marries an age $j$ female in year $t$. This is the discounted within marriage payoff which are be divided between the two spouses.
Let \( w^t_i \) be the expected discounted payoff to a type \( i \) male of being available in the marriage market in year \( t \).

When a type \( i \) male marries a type \( j \) female in year \( t \), \( r^t_{ij} \) is his expected discounted marriage termination payoffs. It is defined as:

\[
r^t_{ij} = \delta_{ij} w^{t+1}_{2}
\]

where \( \delta_{12} = 1 \), and \( \delta_{11} = \delta_{2j} = 0 \).

When an \( \{i, j\} \) couple marry, not all within marital payoff accrues to the husband. Let \( \pi^t_{ij} \) be the equilibrium share of the within marriage output of an \( \{i, j\} \) marriages in year \( t \) which accrues to the wife. The systematic expected discounted payoff to a type \( i \) male who enters into a marriage with a type \( j \) female in year \( t \) consists of his systematic expected discounted within marriage payoff, \( \pi^t_{ij} - r^t_{ij} \), and his expected discounted marriage termination payoff, \( r^t_{ij} \):

\[
\pi^t_{ij} - r^t_{ij} + r^t_{ij}
\]

A particular type \( i \) male, \( g \), who chooses to marry a type \( j \) female in year \( t \) will receive an expected discounted payoff of:

\[
w^{t}_{ijg} = \pi^{t}_{ij} - r^{t}_{ij} + r^{t}_{ij} + \varepsilon^{t}_{ijg}
\]

\( \varepsilon^{t}_{ijg} \) is the realization of an i.i.d. random variable with type I extreme value distribution.

Thus his expected discounted payoff, \( w^{t}_{ijg} \), consists of two components, a systematic discounted payoff (which is common to all such marriages) and an idiosyncratic payoffs which applies only to him.

If he chooses not to marry, his expected discounted payoff is:

\[
w^{t}_{i0g} = \pi^{t}_{i0} + r^{t}_{i0} + \varepsilon^{t}_{i0g}
\]

The systematic expected discounted payoff from not marrying at age \( i \) consists of two components, a current payoff \( \pi^{t}_{i0} \) and an expected discounted payoff from being in the marriage market again in the next period, \( r^{t}_{i0} = \delta_{i0} w^{t+1}_{2} \) where \( \delta_{i0} = 1 \) and \( \delta_{20} = 0 \). \( \varepsilon^{t}_{i0g} \) is the realization of an i.i.d. random variable with type I extreme value distribution.

It is important to note that male \( g \), if he chooses not to marry or marry an older spouse, do not know for certain what his marital behavior in the next period will be. \( r^{t}_{i0} \) and \( r^{t}_{12} \) are independent of \( g \).

The objective of male \( g \) in year \( t \) is to choose whether to remain unmarried or who to marry in order to maximize his discounted utility:

\[
w^{t}_{ig} = \max\{w^{t}_{i1g}, w^{t}_{i2g}, w^{t}_{i0g}\}
\]

\[
= \max\{\pi^{t}_{i1} - r^{t}_{i1} + r^{t}_{i1} + \varepsilon^{t}_{i1g}, \pi^{t}_{i2} - r^{t}_{i2} + r^{t}_{i2} + \varepsilon^{t}_{i2g}, \pi^{t}_{i0} + r^{t}_{i0} + \varepsilon^{t}_{i0g}\}
\]

I assume that the numbers of men and women of each type is large. Let \( \mu^{t}_{ij} \) be the number of type \( i \) males who want to marry type \( j \) females in year \( t \). Let \( \mu^{t}_{i0} \) be the number of type \( i \) males who want to remain unmarried in year \( t \).
Then the quasi-demand equation by type $i$ males to marry type $j$ females is:

$$\ln \mu_{ij}^t - \ln \mu_{i0}^t = \pi_{ij}^t - r_{ij}^t + r_{ij}^t - \pi_{i0}^t - r_{i0}^t, \ j = 1, 2 \quad (50)$$

Using the properties of McFadden’s random utility model, the following standard results also obtain (E.g. CS):

$$w_t^i - w_{i0}^t = \ln \left[ \frac{m_t^i}{\mu_{i0}^t} \right] \quad (51)$$

$$w_{i0}^t = c + \pi_{i0}^t \quad (52)$$

I will now derive the equivalent equations to Equation (50) for women.

Let $W^j_t$ be the expected discounted payoff to a type $j$ female of being available in the marriage market in year $t$. Recall that if she marries a type $i$ male in year $t$, she will receive a within marriage output of $r_{ij}^t$.

Let $R_{ij}^t$ denote her expected discounted end of marriage payoff. $R_{ij}^t$ is defined by:

$$R_{ij}^t = \delta_{ij}W_{2}^{t+1} \quad (53)$$

where $\delta_{21} = 1$ and $\delta_{11} = \delta_{e2} = 0$.

So the systematic expected discounted payoff to a type $j$ female who enters into a marriage in year $t$ with a type $i$ male is:

$$r_{ij}^t + R_{ij}^t$$

Female utilities from marriage are modelled similarly as the males. A particular type $j$ female, $k$, who chooses to marry a type $i$ male in year $t$ will receive an expected discounted payoff of:

$$W_{ijk}^t = r_{ij}^t + R_{ij}^t + \varepsilon_{ijk}^t$$

$\varepsilon_{ijk}^t$ is the realization of an i.i.d. random variable with type I extreme value distribution.

If she chooses not to marry, her expected discounted payoff is:

$$W_{0jk}^t = \pi_{0j}^t + R_{0j}^t + \varepsilon_{0jk}^t$$

Her systematic expected discounted payoff from not marrying at age $j$ consists of two components, a current payoff $\pi_{0j}^t$ and an expected discounted payoff from being in the marriage market again in the next period, $R_{0j}^t = \delta_{0j}W_{2}^{t}$ where $\delta_{01} = 1$ and $\delta_{02} = 0$. $\varepsilon_{0jk}^t$ is the realization of an i.i.d. random variable with type I extreme value distribution.

The objective of female $k$ in year $t$ is to choose whether to remain unmarried or who to marry in order to maximize her expected discounted utility:

$$W_{jk}^t = \max\{W_{0jk}^t, W_{ijk}^t, W_{2jk}^t\}$$
Let $\pi_{ij}^t$ be the number of type $j$ females who want to marry type $i$ males in year $t$. Let $\pi_{0j}^t$ be the number of type $j$ females in year $t$ who want to remain unmarried. Then

$$\ln \mu_{ij}^t - \ln \mu_{0j}^t = \tau_{ij}^t + R_{ij}^t - \pi_{0j}^t - R_{0j}^t, \ i = 1, 2 \quad (54)$$

$$W_j^t - W_{j0}^t = \ln \left[ \frac{f_j^t}{\pi_{0j}^t} \right] \quad (55)$$

$$W_{j0}^t = c + \pi_{0j}^t \quad (56)$$

### 6.1 Marriage market clearing

In each year $t$, there are 4 submarriage markets, $ij = \{11, 12, 21, 22\}$. The marriage market clears when all submarriage markets in the current and future years clear.

Assuming that the marriage market clears, given equilibrium shares, $\tau_{ij}^t, \forall \{i, j\}, t$, the demand for spouses by males is equal to the supply of spouses by females for every submarket. So the equilibrium number of $\{i, j\}$ marriages, $\mu_{ij}^t$, satisfies:

$$\mu_{ij}^t = \mu_{ij}^t = \pi_{ij}^t \quad (57)$$

From (57) and (50), the definitions of $\tau_{ij}^t$ and (51)

$$\ln \mu_{ij}^t - \ln \mu_{0j}^t = \pi_{ij}^t - \tau_{ij}^t + r_{ij}^t - \pi_{i0}^t - r_{i0}^t \quad (58)$$

$$= \pi_{ij}^t - \tau_{ij}^t - \pi_{i0}^t + (\delta_{ij} - \delta_{i0})(c + \pi_{0i}^t + \ln \left[ \frac{m_{ij}^{t+1}}{m_{0i}^{t+1}} \right]) \quad (59)$$

Thus I can identify the systematic total gains to a type $i$ male entering an $\{i, j\}$ marriage in year $t$ relative to him not marrying, $\ln \mu_{ij}^t - \ln \mu_{i0}^t$. When $ij = 11$, then $(1 - \delta)(c + \pi_{i0}^t + \ln \left[ \frac{m_{i0}^{t+1}}{m_{0i}^{t+1}} \right])$ is the expected discounted future cost of marriage today. When the marriage rate of old males are higher, the future cost is higher. When $ij \neq 11$, there is no expected discounted future cost of marriage today even for young males because they can re-enter the marriage market when old without cost.

Similarly,

$$\ln \mu_{ij}^t - \ln \mu_{0j}^t = \tau_{ij}^t + R_{ij}^t - \pi_{0j}^t - R_{0j}^t \quad (60)$$

$$= \tau_{ij}^t - \pi_{0j}^t + (\delta_{ij} - \delta_{0j})(c + \pi_{02}^t + \ln \left[ \frac{f_{i2}^{t+1}}{\mu_{02}^{t+1}} \right]) \quad (61)$$

I can identify the systematic total gains to a type $j$ female entering an $\{i, j\}$ marriage in year $t$ relative to her not marrying, $\ln \mu_{ij}^t - \ln \mu_{0j}^t$. 29
Add (58) and (60) to obtain:

\[
\ln \left[ \frac{\mu_{ij}}{\sqrt{\mu_{ij}^0 \mu_{ij}^0}} \right] = \frac{\pi_{ij}^t - (\pi_{i0}^t + \pi_{0j}^t)}{2} + \frac{r_{ij}^t - r_{i0}^t + R_{ij}^t - R_{0j}^t}{2} \tag{62}
\]

The left hand side of (62) is equal to the log of the number of \( \{i, j\} \) marriages divided by the geometric average of the number of unmarrieds of each type.

The right hand side of (62), called the systematic total gain (or total gain) to marriage, is interpretable as the systematic discounted gain per partner from an \( \{i, j\} \) marriage minus the mean discounted gain per partner from remaining unmarried in year \( t \). The total gain surplus will depend on the year \( t \). Put another way, \( \{i, j\} \) couples who marry in different years will get different total gains. \( \ln \mu_{ij}^t - \ln \sqrt{\mu_{ij}^0 \mu_{ij}^0} \) is observable to the econometrician. It is what is estimated in CS.

\( \{i, j\} \) couples who choose to marry will in general get larger payoffs than the systematic total gain. The reason is that the systematic total gain do not include the gains from the idiosyncratic shocks that both individuals obtain from optimizing behavior. That is, they are choosing to marry each other even though other options are available.

The systematic total gain can be decomposed to a systematic static gain (or static gain) to marriage, \( \Gamma_{ij}^t \), and a systematic dynamic cost (or dynamic cost) of marriage, \( \Delta_{ij}^t \).

\[
\ln \left[ \frac{\mu_{ij}^t}{\sqrt{\mu_{ij}^0 \mu_{ij}^0}} \right] = \Gamma_{ij}^t - \Delta_{ij}^t \tag{63}
\]

Using (62), the various definitions of marital termination payoffs and marriage market re-entry payoffs, (51), (55),

\[
\Gamma_{ij}^t = \frac{\pi_{ij}^t - (\pi_{i0}^t + \pi_{0j}^t)}{2} + \frac{(\delta_{ij} - \delta_{i0})(c + \pi_{20}^t) + (\delta_{ij} - \delta_{0j})(c + \pi_{02}^t)}{2} \tag{64}
\]

\[
\Delta_{ij}^t = \frac{(\delta_{00} - \delta_{ij})(\ln \left[ \frac{m_{02}^{t+1}}{\mu_{20}^{t+1}} \right] + \frac{1}{2} \ln \left[ \frac{f_{20}^{t+1}}{f_{02}^{t+1}} \right])}{2} \tag{65}
\]

\( \Gamma_{ij}^t \) does not depend on demand and supply conditions, \( m_{i}^t \) or \( f_{i}^t \) for any \( t' \), \( \Gamma_{ij}^t \), the static gain has the interpretation as the per spouse average gain to marriage for an \( \{i, j\} \) couple in year \( t \) without subsequent remarriage relative to them not marrying now or in the future. I call this the static gain because it is as if the couple is making a static marry once or not at all decision. The first term on the right hand side, \( \pi_{ij}^t - (\pi_{i0}^t + \pi_{0j}^t) \), is the current within marriage payoff to an \( \{i, j\} \) marriage in year \( t \) relative to them not marrying. \( 2c + \pi_{20}^t + \pi_{02}^t \) is the total payoff of an old \( i \) type man and a old \( j \) type woman if they must remain single. So for a young couple, \( ij = 11 \), the next term in (??), \( -(2c + \pi_{20}^t + \pi_{02}^t) \)
is the discounted cost of being unmarried and single in the second period. \( \Gamma_{ij}^t = \{ \pi^t_{ij} - (\pi^t_{i0} + \pi^t_{oj}) \}/2 \) for \( ij \neq 11 \).

\( \Delta_{ij}^t \), the dynamic cost of marriage is the discounted cost of delayed re-entry into the marriage market in the future if the \( \{i, j\} \) couple decide to marry at time \( t \). For \( \{1, 1\} \) marriages, \( \Delta_{ij}^t = \left\{ \ln \sqrt{m_{i+1,j+1}^{t+1}} - \ln \sqrt{\mu_{i+1,j+1}^{t+1}} \right\} \). For a young couple, part of the cost of marriage is that they cannot re-enter the marriage market in the next period. If they do not marry, they can always re-enter the marriage market in the next period. \( \ln \sqrt{m_{i+1,j+1}^{t+1}} - \ln \sqrt{\mu_{i+1,j+1}^{t+1}} \)
is the the geometric average of the gains of being able to participate in the marriage market in period \( t+1 \) relative to not being able to participate. \( \Delta_{ij}^t = 0 \) for \( ij \neq 11 \) because younger spouses can always re-enter when old and older spouses do not care to re-enter.

Using (63) and (64), the static gains to marriage is measured by:

\[
\Gamma_{ij}^t = \ln \left[ \frac{\mu_{ij}^t}{\mu_{i0}^t \mu_{0j}^t} \right] + \left( \frac{\delta_{o0} - \delta_{ij}}{2} \right) \ln \left[ \frac{m_{i+1}^{t+1}}{\mu_{20}^{t+1}} \right] + \left( \frac{\delta_{oj} - \delta_{ij}}{2} \right) \ln \left[ \frac{f_{j+1}^{t+1}}{\mu_{02}^{t+1}} \right] \tag{66}
\]

If marital outputs are year invariant, \( \Gamma_{ij}^t = \Gamma_{ij} \), which is independent of \( t \). Then (66) imposes very strong restriction on the data. It says that:

\[
\ln \left[ \frac{\mu_{11}^t}{\sqrt{\mu_{i0}^{t+1} \mu_{0i}^{t+1}}} \right] + \frac{1}{2} \ln \left[ \frac{m_{2}^{t+1}}{\mu_{20}^{t+1}} \right] + \frac{1}{2} \ln \left[ \frac{f_{2}^{t+1}}{\mu_{02}^{t+1}} \right] = \Gamma_{11} \tag{67}
\]

\[
\ln \left[ \frac{\mu_{12}^t}{\sqrt{\mu_{i0}^{t+1} \mu_{0i}^{t+1}}} \right] = \Gamma_{12} \tag{68}
\]

\[
\ln \left[ \frac{\mu_{21}^t}{\sqrt{\mu_{20}^{t+1} \mu_{0i}^{t+1}}} \right] = \Gamma_{21} \tag{69}
\]

\[
\ln \left[ \frac{\mu_{22}^t}{\sqrt{\mu_{20}^{t+1} \mu_{0i}^{t+1}}} \right] = \Gamma_{22} \tag{70}
\]

\[
m_{1}^{t+1} = m_{11}^t - \mu_{11}^t \tag{71}
\]

\[
f_{2}^{t+1} = f_{11}^t - \mu_{11}^t \tag{72}
\]

for all \( t \). Given \( \Gamma_{11}, \Gamma_{12}, \Gamma_{21} \) and \( \Gamma_{22} \), a sequence of new entrants, \( \{m_1^t, f_1^t\} \), for \( t = 0, \ldots \) and initial values, \( \{m_0^1, f_0^1, m_0^2, f_0^2\} \), the sequence of marital behavior \( \{\mu_{11}^t, \mu_{12}^t, \mu_{21}^t, \mu_{22}^t\} \) can be solved for \( t = 0, \ldots \). Given data from two adjacent time periods, \( t \) and \( t+1 \), \( \Gamma_{11}, \Gamma_{12}, \Gamma_{21} \) and \( \Gamma_{22} \) can be estimated using equations (67) to (70). Thus (67) to (72) constitute a dynamic MMF. Unlike static MMFs, the dynamic MMF is defined by a system of forward looking difference equations. The forward looking aspect of the marriage matching function is due to the assumption that market participants are forward looking and therefore future supplies will matter to their current decisions.
A constant $\Gamma_{ij}^t$ is not an innocuous assumption. Based on static models, CS have already shown that the gains to marriage have fallen substantially between the seventies and eighties in the US. Angrist and Evans 1999 and CS have shown that the legalization of abortion in the US in the early seventies substantially reduced the gains to marriage for young adults. Thus while $\Gamma_{ij}^t = \Gamma_{ij}$ provides a very strong restriction of the model, it is counterfactual for recent US marital history. BSV, summarized here, also shows that a time independent $\Gamma_{ij}^t$ is not plausible for China. If $\Gamma_{ij}^t$ is time varying, then one needs to specify the dynamic process for $\Gamma_{ij}^t$ in the dynamic MMF as well.

Choo Siow 2007 provides preliminary estimates of a dynamic MMF with long lived agents and exogenous divorce, assuming $\Gamma_{ij}^t = \Gamma_{ij}$ and a time invariant distribution of new entrants.

As is well known, most individuals marry when young, spouses tend to be the same age and husbands are generally older than wives by a few years. The dynamic MMF above shows that strong apriori identifying assumptions are needed to estimate the structural parameters of behavioral dynamic marriage matching models. Independent of the fit of these models, welfare simulations of these models should be treated cautiously.

### 6.2 Towards two sex models of population growth

As discussed in section 1, a motivation for constructing MMFs is to use them to construct two sex models of population growth. I will sketch how this can be done using the dynamic MMF above.

Equations (67) to (72) take the sequence of population supply of new entrants (births), $\{m^t_1, f^t_1\}$, as exogenous. Adding a birth rate equation to the above system will generate a two sex model of population growth. For example, consider:

$$m^t_1 = f^t_1 = \kappa_{11}\mu_{ii}^{t-1} + \kappa_{21}\mu_{21}^{t-1} + \xi^t$$  \hspace{1cm} (73)

Equation (73) relates births in the current period to marriages of young females in the period before. It embodies three assumptions. First for convenience, it assumes that the sex ratio of new entrants is one. Second, it assumes that the birth rate is related to the marriage of young females. \footnote{Building on the literature on gender differences in fecundity, Siow 1998 derives restrictions on $\kappa_{11}$, $\kappa_{21}$ and $\Gamma_{ij}$ assuming $\xi^t = 0$.} Third, $\{\xi^t\}$ is a sequence of exogenous birth rate shocks which will generate stochastic population growth. Adding equation (73) to the dynamic MMF will generate a two sex model of population growth. It is beyond this paper to analyze how such models of population growth behaves.

\footnote{Also see Goldin and Katz 2002.}
6.3 Analytic solution to a special case

Returning to the case of exogenous arrivals of new entrants, in general, the number of marriages of the dynamic MMF, equations (67) to (72), cannot be analytically solved for. I will consider a special case. Let preferences be time invariant, i.e. \( \Gamma_{ij}^t = \Gamma_{ij} \). Since the sex ratio at birth is essentially one, let \( f_{t1}^1 = m_{t1}^1 \). Finally, let there be no gender difference in marriage preferences, i.e. \( \Gamma_{21} = \Gamma_{12} \). With these restrictions, (67) to (72) reduce to:

\[
\begin{align*}
\frac{\mu_{11}^t}{m_{t1}^1 - \mu_{12}^t - \mu_{11}^t} \left[ \frac{m_{t1}^1 - \mu_{11}^t}{m_{t1}^1 - \mu_{11}^t - \mu_{12}^t + \mu_{21}^{t+1}} \right] &= \gamma_{11} = \exp \Gamma_{11} \\
\frac{\mu_{22}^t}{(m_{t1}^1 - \mu_{12}^t - \mu_{11}^t)(m_{t1}^1 - \mu_{11}^t - \mu_{12}^t - \mu_{22}^t)} &= \gamma_{12} = \exp 2\Gamma_{12} \\
\frac{\mu_{12}^t}{m_{t1}^1 - \mu_{12}^t - \mu_{11}^t} &= \gamma_{22} = \exp \Gamma_{22} \\
\mu_{12}^t &= \mu_{21}^t
\end{align*}
\]

I can solve for the steady state number of marriages when \( m_{t1}^1 = m \):

\[
\begin{align*}
\mu_{11} &= m_{t1}^1 \frac{\gamma_{11}}{\sqrt{1 + \gamma_{22} + \gamma_{12}^2} + \gamma_{11}} \\
\mu_{22} &= m_{t1}^1 \frac{\gamma_{22}}{(1 + \sqrt{1 + \gamma_{22} + \gamma_{12}^2})^2} + \gamma_{11} \\
\mu_{21} &= m_{t1}^1 \frac{\gamma_{12}(1 + \sqrt{1 + \gamma_{22} + \gamma_{12}^2})}{(\sqrt{1 + \gamma_{22} + \gamma_{12}^2})^2 + \gamma_{11}}
\end{align*}
\]

As expected, the steady state number of marriages are homogenous of degree one in \( m_{t1}^1 \). The steady state number of marriages are also uniquely determined by \( m_{t1}^1 \) and the preference parameters \( \gamma_{11} \), \( \gamma_{12} \) and \( \gamma_{22} \).

Working with the linearized system around the steady state, (74) to (77) can be reduced to a two equation first order difference equation system in \( \mu_{11}^t \) and \( \mu_{22}^t \). \( \mu_{12}^t \) can be obtained as a function of \( \mu_{11}^t \) and \( \mu_{22}^t \) via (76). Let \( G_{ij}(L) \) denote a lag structure of length \( \{i, j\} \). The appendix shows that:

**Proposition 12**

\[
\begin{align*}
\mu_{11}^t &= \phi_{11} + a_{11}\mu_{11}^{t-1} + a_{12}\mu_{22}^{t-1} + \sum_{k=0}^{\infty} \lambda_{12}^{k+1} G_{12}(L)m_{1}^{t+k} \\
\mu_{22}^t &= \phi_{22} + a_{21}\mu_{11}^{t-1} + a_{22}\mu_{22}^{t-1} + \sum_{k=0}^{\infty} \lambda_{22}^{k+1} G_{22}(L)m_{1}^{t+k}
\end{align*}
\]

where \( \lambda_{12} \) and \( \lambda_{22} \), are between 0 and 1, so that the infinite sums converge.

---

25This section applies standard techniques in rational expectations models to the problem at hand.
The equilibrium number of \( f_{i,j} \) marriages depend on past number of marriages, past, present and future population supplies. The past number of marriages and past number of young population supply matters because it gives the number of available old individuals in the current period. Future young population supplies because it predicts how many young adults will be around in the next period which will affect the decisions of the current adults. Standard static marriage matching functions on depend on current supplies of each type of individuals (young and old). The dynamic MMF also includes future supplies of the young.

7 Conclusion

This paper presents the CS MMF. It shows that this framework can be extended to incorporate the collective intra-household allocation model. The framework also can be used to develop dynamic MMFs.

The paper also provides different empirical applications. Much remains to be done. Analytically, the substitution properties in the CS model remains to be worked out. The question of global uniqueness of the CS model and its generalizations needs to be addressed. The framework should be extended to include incomplete contracting and endogenous marital dissolution.

The estimation and testing of the collective marriage matching model is just beginning. Estimating a dynamic MMF with time varying preference parameters and population supplies are important goals. Estimating this class of models using internet dating data as in Hitsch, et. al. is also promising.

Finally, the empirical framework here assumes away the problem of unobserved heterogeneity. Empirical bilateral matching models with unobserved heterogeneity on both sides of the market needs to be developed.

References


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E.g. Lundberg and Pollak 2003; Peters and Siow 2002; Chiappori, Iyigun and Weiss 2008 have results under restrictive marriage matching assumptions.


[29] Francis, Andrew M. “Sex ratios and the red dragon.” Emory University. November 29, 2007


Taking logs of (74) to (77):

\[
\ln \mu_{11} - \ln(m_1^t - \mu_{12}) + \ln(\mu_{10} + \mu_{12}) - \ln(m_1^{t+1} - \mu_{11} - \mu_{22} - \mu_{12}) = \Gamma_{11}
\]

\[
2 \ln \mu_{12} - \ln(m_1^t - \mu_{12} - \mu_{11} - \ln(m_1^{t-1} - \mu_{11} - \mu_{22}) = 2\Gamma_{12}
\]

\[
\ln \mu_{22} - \ln \mu_{20} = \Gamma_2
\]

\[
\ln \mu_{22} - 2 \ln \mu_{12} + \ln(m_1^t - \mu_{12} - \mu_{11}) = \Gamma_{22} - 2\Gamma_{12}
\]

\[
m_{10}^{t-1} + m_{12}^{t-1} = m_{12}^t
\]

\[
m_{1}^{t-1} - m_{12}^{t-1} - m_{22}^{t-1} = m_{12}^t
\]

Linearization of the above gives:

\[
0 = \frac{\mu_{11} - m_{11}^{t-1} - \mu_{12}^{t+1}}{\mu_{11}^{t+1}} = \frac{m_{11}^{t-1} - \mu_{12}^{t+1}}{m_{11}^{t+1}} + \frac{m_{12}^{t-1} - m_{11}^{t+1}}{m_{12}^{t+1}}
\]

(78)

\[
0 = \frac{m_1^{t-1} - m_{12}^{t+1}}{\mu_{12}} = \frac{m_{12}^{t-1} - m_{11}^{t+1}}{\mu_{12}}
\]

(79)

\[
0 = \frac{m_1^{t-1} - m_{12}^{t+1}}{\mu_{11}^{t+1}} = \frac{m_{12}^{t-1} - m_{11}^{t+1}}{\mu_{11}^{t+1}}
\]

(80)

\[\text{Shashi Khatri aided in these derivations.}\]
I can ignore the equation (80) when solving the two equation difference equation system in \( \mu_{11} \) and \( \mu_{22} \), (78) and (79). This two equation system may be written as:

\[
A \mu^{t+1} = B \mu^t + C z^t
\]

where \( \mu^t = \begin{bmatrix} \mu_{11}^t \\ \mu_{22}^t \end{bmatrix} \) and \( z^t = G(L)m^t \).

On simply picking up the coefficients from the above two equations I get the following matrices, \( B \):

\[
\begin{bmatrix}
1 \\
\frac{\mu_{12}}{\mu_{11}} - \frac{1}{m_2} + \frac{1}{\mu_{20}} \mu_{12} \\
-2 \mu_{10} - \frac{1}{m_2} + \frac{1}{\mu_{20}} \mu_{12}
\end{bmatrix}
\]

\( A \):

\[
\begin{bmatrix}
\frac{\mu_{12}}{\mu_{11}} - \frac{1}{m_2} + \frac{1}{\mu_{20}} \mu_{12} \\
\frac{\mu_{12}}{2\mu_{10} + \mu_{12}} - \frac{1}{m_2} + \frac{1}{\mu_{20}} \mu_{12} \\
-2 \mu_{10} - \frac{1}{m_2} + \frac{1}{\mu_{20}} \mu_{12}
\end{bmatrix}
\]

I can rewrite the two equation system as:

\[
\mu^{t+1} = A^{-1} B \mu^t + A^{-1} C z^t
\]

Now I can put \( A^{-1} B = N^{-1} D N \) where \( D \) is the diagonal matrix of eigenvalues of \( A^{-1} B \) and \( N \) are the eigenvectors of \( A^{-1} B \). So I can write

\[
\mu^{t+1} = N^{-1} D N \mu^t + A^{-1} C z^t
\]

\( N \mu^t = \nu^t \) and \( \zeta^t = A^{-1} C z^t \). Then I have

\[
\nu^{t+1} = D \nu^t + \zeta^t
\]

Let \( \lambda_1 \) and \( \lambda_2 \) be the eigenvalues of \( A^{-1} B \). So

\[
\nu_1^{t+1} = \lambda_1 \nu_1^t + \zeta_1^t \\
\nu_2^{t+1} = \lambda_2 \nu_2^t + \zeta_2^t
\]

I will show that \( \lambda_1 > 1 \) and \( 0 < \lambda_2 < 1 \). Then I express:

\[
\nu_1^t = \sum_{k=0}^{\infty} \lambda_1^{-k-1} \zeta_1^{t+k} \\
\nu_2^t = \sum_{l=0}^{\infty} \lambda_2^l \zeta_2^{t-l}
\]

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Now since
\[ n_{11} \mu_{11} + n_{12} \mu_{22} = \nu_1 \]
\[ n_{21} \mu_{11} + n_{22} \mu_{22} = \nu_2 \]

So
\[ \mu_{11}^t = \frac{n_{12}(n_{21} \mu_{11}^{t-1} + n_{22} \mu_{22}^{t-1}) - n_{22} \sum_{k=0}^{\infty} \lambda_1^{-k-1} \xi_{1+k}}{n_{21} n_{12} - n_{11} n_{22}} \]
\[ \mu_{22}^t = \frac{-n_{11}(n_{21} \mu_{11}^{t-1} + n_{22} \mu_{22}^{t-1}) + n_{21} \sum_{k=0}^{\infty} \lambda_1^{-k-1} \xi_{1+k}}{n_{21} n_{12} - n_{11} n_{22}} \]

So I am done if I can show that \( \lambda_1 > 1 \) and \( 0 < \lambda_2 < 1 \).

Now \( B^{-1}A \) is equal to:

\[
\begin{bmatrix}
-\frac{\mu_{12}}{\mu_{11}} & \frac{2 \mu_{12}^2 + \mu_{11} \mu_{22} - 2 \mu_{22}^2 \mu_{11}}{\mu_{11} \mu_{22}} \\
-(2m^2 + 2m_1^2 + 2m_{12}^2 - m_{12} m_{11} - 4m_{11} m_{22}) \mu_{22} & 4m_1^2 + 4m_{11}^2 - 10m_{11} m_{12} + 8m_2^2 \mu_{11} + 3m_{12}^2 \mu_{11} - 4m_2^2 m_{12} + m_1^2 \mu_{22} - 2m_{11} \mu_{12} \mu_{22}
\end{bmatrix}
\]

Substituting in the steady state values, the eigenvalues of \( B^{-1}A \) are in the form of \( \frac{1}{\gamma_{11}}(a + 2\sqrt{b}) \) and \( \frac{1}{\gamma_{11}}(a - 2\sqrt{b}) \) where \( A \) and \( B \) are the following:

\[
a = 2\gamma_{12} \gamma_{11} + 2\sqrt{\gamma_{12} \gamma_{11}} \sqrt{(1 + \gamma_{22})} + 4\gamma_{22} \gamma_{12} + 4\gamma_{12} + 2(\sqrt{\gamma_{12}})^3 \sqrt{(1 + \gamma_{22})} + 2\sqrt{\gamma_{12}} \sqrt{(1 + \gamma_{22})} \sqrt{\gamma_{11}}
\]
\[
b = \gamma_{12} + 2\gamma_{11} \gamma_{12} + 3\gamma_{12} \gamma_{22} + 4\gamma_{11} \gamma_{12} \gamma_{22} + 6\gamma_{12} + \gamma_{22} + 6 \gamma_{11} \gamma_{12} + \gamma_{12} \gamma_{22} + 3 \gamma_{12} \gamma_{22} + 12 \gamma_{12} \gamma_{22} + \gamma_{12} \gamma_{22}^3 + \gamma_{12} \gamma_{22}^3 + \gamma_{11} \gamma_{12} \gamma_{22} + 6 \gamma_{11} \gamma_{12} \gamma_{22} + \gamma_{11} \gamma_{12} \gamma_{22} + 6 \gamma_{12} \gamma_{22} + 6 \gamma_{12} \gamma_{22} + 6 \gamma_{12} \gamma_{22} + 4 \gamma_{12} \gamma_{22} + 4 \gamma_{12} \gamma_{22} \gamma_{12} \gamma_{22} + 1 + 6 \gamma_{11} \gamma_{12} \gamma_{12} \gamma_{22} + 1 + 2 \gamma_{11} \gamma_{12} \gamma_{12} \gamma_{22} + 1 + 8 \gamma_{12} \gamma_{22} \gamma_{22} + 1 + 4 \gamma_{12} \gamma_{22} \gamma_{22} \gamma_{12} \gamma_{22} + 1
\]

I need to show that the first eigenvalue is greater than 1 and the second is less than 1 but greater than zero. The first is self evident, just taking the multiplier \( \frac{1}{\gamma_{11}} \) and multiplying it with \( a \) yields the desired result, we needn’t even use \( 2\sqrt{b} \) for this. Now moving to the second part, showing that second lies between 0 and 1.

First let’s show that \( (a - 2\sqrt{b}) > 0 \) or \( A^2 - 4B > 0 \). Evaluating the left side I get \( a^2 - 4b = 4\gamma_{11}^2 \gamma_{12}^2 > 0 \).

The above shows that eigenvalue is greater than zero. Now I need to show that its less than one or \( (a - 2\gamma_{11} \gamma_{12})^2 - 4b < 0 \). Let me evaluate the left side:

40
\[(a - 2\gamma_{12}\gamma_{11})^2 - 4b = -16\gamma_{11}\gamma_{12}^2 - 16\gamma_{11}\gamma_{12}\gamma_{22} - 8\gamma_{11}\gamma_{12}^2\sqrt{\gamma_{22} + 1} - 8\gamma_{11}\gamma_{12}^2\sqrt{\gamma_{22} + 1} - 8\gamma_{11}\gamma_{12}\gamma_{22}\sqrt{\gamma_{22} + 1} - 8\gamma_{11}\gamma_{12}\gamma_{22}\sqrt{\gamma_{22} + 1}\]

All terms are negative, hence the proof.