Sensitivity Analysis of Distortion Risk Measures

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Abstract

This paper provides a unified statistical framework for the analysis of distortion risk measures (DRM) and of their sensitivities with respect to parameters representing risk aversion and/or pessimism. We derive the general formula for calculating the functional asymptotic distribution of the nonparametric estimator of the functional distortion risk measures. Closed form expressions are provided for special examples such as VaR, Tail-VaR and Proportional Hazard distortion risk measure. Moreover, we analyze the link between Value-at-Risk and Tail-VaR and characterize the underlying distributions under which the two risk measures are linearly related through their risk levels. We apply the results to currency portfolios and observe that this linearity relationship between Value-at-Risk and Tail-VaR is a surprisingly common phenomenon for the portfolios considered.

Keywords: Value-at-Risk, Tail-VaR, Loss-Given-Default, Distortion Risk Measure, Implied Pessimism Parameter, Empirical Process.
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1 Introduction

New rules for fixing the reserves needed to balance a risky investment have recently been introduced both in Finance and Insurance. The level of required capital depends on the type of investment, but also on the selected risk measure. It is recognized that a standard mean-variance approach is generally inappropriate for risk control and other risk measures have been considered. For instance, since 1996, the Basle Committee has proposed to use the Value-at-Risk (VaR), which is a quantile of the profit and loss (P&L) distribution. Since then, a great deal of effort has been devoted to the study of the applications of VaR in related literatures. These include determining capital reserves, portfolio management, hedging and so on [see Gourieroux and Jasiak (2005b) for an overview]. For instance, the capital reserve at a given risk level can be determined by taking the sum of VaR and initial wealth. VaR has a few nice properties as a measure of risk. For example, i) VaR summarizes the risks as a number representing an extreme event with a certain confidence level and thus is easy to understand; ii) by directly measuring the tail of the distribution of the profit and loss of a portfolio, VaR is compatible with non-Gaussian distributional properties such as the fat tail commonly encountered in asset returns. However, focusing only on VaR for measuring risk can be misleading since it only takes into account one point of the distribution. This can lead to counterintuitive behavior. For instance, Boyle et al. (2005) show that under a VaR constraint, a trader has the incentive to hold a riskier portfolio which can be subject to huge losses with small probability. Basak and Shapiro (2001) study portfolio selection under a VaR constraint and find that an agent tends to invest more in the risky asset than she would in the absence of this constraint. Furthermore, the VaR is not convex in general, so that an investor may be better off in some cases if she invests in the individual asset separately. Indeed, the VaR is convex only under additional restrictions on the conditional distribution of asset returns, for instance, when the returns are i.i.d. Gaussian or follow a Gaussian random walk with stochastic volatility [see Gourieroux et al. (2000)].

Better measures of risk are desired for robust risk management. These measures have to take into account not only the probability of a bad event, but also its magnitude. Artzner et al. (1999) follow a systematic approach and define "coherent risk measures". A risk measure is coherent if it satisfies axioms such as monotonicity, invariance with respect to drift, homogeneity and subadditivity. Clearly, the VaR is not coherent and violates the subadditivity for most distributions.[1] The appropriateness of these axioms is still a matter for debate; nevertheless, they build a standard for introducing new risk measures. The authors propose in particular to replace the standard VaR by the Tail-VaR, which takes into account not only the probability

[1]Recently, independent works by Ibragimov (2005) and Garcia et al. (2006) show that VaR may satisfy the subadditivity requirement if the tails of the marginal distributions are reasonably thin and equally asymmetric.

of loss, but also the magnitude of the loss, when a loss occurs. The application of Tail-VaR has gained increasing interest both in the academic literature and in industry. For instance, it is used to derive the efficient portfolio frontier [see Rockafellar and Uryasev (1999), (2000), Bassett et al. (2004) for example] as in the mean-variance framework [Markowitz (1952)], to calculate required capital [Manistre and Hancock (2005)], or to perform a sensitivity analysis of portfolio risk [Tasche (2002), Laurent (2003), Fermanian and Scaillet (2005)]. Tail-VaR has been recommended as the standard measure for calculating the capital requirement. For example, in 2002, the Life capital subcommittee of American Academy of Actuaries has suggested the use of Tail-VaR to set the risk-based capital requirement.

Both VaR and Tail-VaR are closely related to distortion risk measures considered in the insurance literature [Wang (1996), Wang and Young (1998)]. The distortion risk measure (DRM) is a special class of the so-called Choquet expected utility, that is, an expected utility calculated under a modified probability measure [Bassett et al. (2004)]. The distortion risk measure distorts the probability measure while specifying the utility as an identity function. This special class has various alternative names, such as spectral risk measure [Acerbi (2002)], or pessimistic risk measure [Bassett et al. (2004)].

Comprehensive risk analysis requires the joint consideration of not only different risk measures, but also risk measures at several levels. Indeed, many distortion risk measures can be characterized by parameters representing risk aversion and/or pessimism. For example, the loss probability associated with VaR or Tail-VaR represents the risk level selected by the regulator or the investor. Thus, knowledge about the sensitivity of risk measures with respect to a slight modification of the risk level is useful for selecting an appropriate risk management strategy. This can be done by studying the partial derivative of the risk measures with respect to the risk level. The Proportional Hazard measure introduced by Wang (1995) is another example of the class of one parameter distortion risk measures. It is based on a pessimistic view of the loss probability. Therefore, a similar analysis can be implemented to study its sensitivity to the level of pessimism.

This paper provides a unified statistical framework for a nonparametric analysis of the functional distortion risk measures and of their sensitivities with respect to parameters representing risk aversion and/or pessimism. The properties of the estimated functional risk measures are based on the analogy principle and the asymptotic properties of the empirical processes (recalled in Appendix A). Then, we study in detail the relationship between the VaR and Tail-VaR. Indeed, a simple relationship can simplify the computation of the risk measures, extend the range of their interpretation, and facilitate empirical sensitivity analysis. We first show that the Tail-VaR can be approximated by multiplying VaR at the same risk level by an amplifying factor. Alternatively, the VaR and the Tail-VaR can be related through their risk levels. Moreover, we identify the condition such that this relationship is linear. One of the main contributions of this paper is the analysis of the asymptotic properties of an estimator of the function that relates their risk levels; a test statistic is also provided for the null hypothesis of linearity of this function. The analysis is performed in an i.i.d. framework not only for expository purpose, but also since it corresponds to the approach suggested by the regulator (e.g. BIS). First, the regulator proposes to define the risk measure by historical simulation. More precisely, the measure is replaced by its sample counterpart computed on a rolling window basis. This practice assimilates the marginal and conditional distributions, and thus, assumes implicitly i.i.d. returns.[2] Second, the i.i.d. assumption is also required to check for the accuracy of the risk measure and its sensitivity to downturn conditions for instance. Indeed, this is usually done by Monte-Carlo, that is, by i.i.d. drawings from the historical distribution.

The rest of this paper is organized as follows. In Section 2, we describe the distortion risk measures and discuss the relationship between the VaR and Tail-VaR. We further analyze the sensitivity of the distortion risk measures with respect to the distortion parameter. Indeed, the sensitivity has an expression similar to the expression of the distortion risk measure. In Section 3, we derive the functional asymptotic properties of the estimators of the functional distortion risk measures and their sensitivities. Besides the general case, we consider three examples of distortion risk measures (VaR, Tail-VaR, and Proportional Hazard) and provide a closed form expression for their asymptotic variances and covariances. In Section 4, we focus on the function defining the change of risk level to pass from the VaR to the Tail-VaR and propose a test to check if the function is linear. We illustrate our analytical results by considering currency portfolios in Section 5. Concluding remarks are given in Section 6 and proofs are gathered in Appendices.

2 The distortion risk measures and their sensitivities

2.1 Choquet expected utility and distortion risk measures

Before introducing and interpreting the distortion risk measures, it is necessary to fix a convention of profit and loss appropriate for the application to market finance, credit risk and insurance. Let us denote by Y a portfolio value, corresponding to a zero initial investment. There is a profit if Y is positive, a loss, otherwise. Let us now consider the standard way for computing the amount of reserve to hedge this risky investment. For a given loss probability u, the $VaR(u)$, is defined by:

$$P\big[Y < -VaR(u)\big] = u \Leftrightarrow \ P\big[-Y \le VaR(u)\big] = 1 - u.$$

[2] See Appendix A.3 for a discussion of the non i.i.d. case.

The VaR is the negative of the u-quantile of the profit and loss variable Y, as well as the $(1-u)$-quantile of the loss and profit variable $X = -Y$. For applications to insurance or regulation in credit risk, the focus is on the loss and profit variable $X = -Y$. This variable is positive in a lot of applications, such as the study of the loss component in an insurance contract, or the Loss-Given-Default (LGD) in credit risk. In the sequel, the variable of interest is the loss (and profit) variable, $X = -Y$.

2.1.1 Definitions

Expected utility theory was the first coherent approach introduced to compare risk variables. The risks are compared by means of a scalar expected utility:

$$E\,U(Y) = E\,U^*(X) = \int U^*(x)\,d\,F(x),$$

where U is an increasing concave utility function and $U^*(x) = U(-x)$ is its decreasing concave counterpart associated with the loss (and profit) variable. For a continuous one-dimensional risk variable, the expected utility can be written as:

$$E\,U^*(X) = \int_0^1 U^*[Q(v)]\,d\,v = \int_0^1 U^*[Q(1-u)]\,d\,u, \tag{2.1}$$

where the second equality is obtained by the change of variable $u = 1 - F(x)$, and $Q = F^{-1}$ denotes the quantile function. Different authors [Yaari (1987), Schmeidler (1989)] argue that "the independence axiom underlying the von-Neumann-Morgenstern axiomatization may be too powerful to be acceptable" and they propose another independence axiom valid for comonotonic variables. The set of scalar risk measures is enlarged to the so-called Choquet expected utilities:

$$\Pi(U^*, H; Q) = \int_0^1 U^*[Q(1-u)]d\,H(u). \tag{2.2}$$

The risk measure involves a utility function U^* as in the standard expected utility framework and a distorted cumulative distribution function H (also called capacity in Choquet's terminology). Function U^* represents the standard risk aversion (when U^* is concave); the distortion measure defines a change of probability, and represents the more or less pessimistic view on admissible risk levels. The extent of pessimism is determined by the level of concavity of the distortion function H [see e.g. Bassett et al. (2004)].

The limiting case, $U^*(x) = x$, where only the distortion measure matters, has gained increasing attention recently due to its close relationship with many well recognized risk measures [Wang (1995), (1996), (2000), (2001), Acerbi and Simonetti (2002), Bassett et al. (2004)].

Definition 1 (Wang (1996)). *A distortion risk measure (DRM) is defined as*

$$\Pi(H;Q) = \int_0^1 Q(1-u) d\, H(u), \tag{2.3}$$

where H is a cdf on $[0,1]$.

When Q is the quantile function of a loss (and profit) variable, a DRM is simply a weighted sum of VaR at level u. This interpretation explains why DRMs have been proposed to measure the risk and compute risk premiums in the insurance literature in a series of papers by Wang and others [Wang (1995), (1996), (2000), Wang and Young (1998)]. Moreover, when the distortion cdf H is concave, the DRM is a coherent risk measure in the sense of Artzner et al. (1999) [see e.g.Wirch and Hardy (1999)], and a good candidate to define a level of required capital to balance a risky investment.

Finally, a DRM admits different equivalent expressions. Indeed, we get:

$$\begin{aligned}
\Pi(H;Q) &= \int_0^{1-F(0)} Q(1-u) d\, H(u) + \int_{1-F(0)}^1 Q(1-u) d\, [H(u)-1] && \text{(by splitting the interval)} \\
&= -\int_0^{1-F(0)} H(u) dQ(1-u) - \int_{1-F(0)}^1 [H(u)-1] dQ(1-u) && \text{(by integrating by parts)} \\
&= \int_{F(0)}^1 H(1-u) dQ(u) + \int_0^{F(0)} [H(1-u)-1] dQ(u) && \text{(by the change of variable } u \to 1-u).
\end{aligned}$$

These expressions are greatly simplified, when the loss (and profit) variable X is nonnegative. Indeed, we get $F(0)=0$, and:

$$\Pi(H;Q) = \int_0^1 Q(1-u) d\, H(u) = \int_0^1 H(1-u) d\, Q(u). \tag{2.4}$$

So when X is nonnegative, there is a symmetry between functions H and Q.

2.1.2 Families of distortion risk measures

Many risk measures applied in finance and insurance literature, such as the VaR, or the Tail-VaR, are DRMs with carefully selected distortion functions. In practice, several risk measures have to be jointly considered in order to make risk management and risk control robust. This is done by introducing parameterized families of DRMs, or equivalently of distortion functions. Let us consider a family of distortion functions, $H(\cdot\,;p)$, where parameter p belongs to some interval. We get a family of DRMs:

$$\Pi(p\,;Q) = \int_0^1 Q(1-u)\, dH(u\,;p), \quad p \in [a,b],$$

indexed by p. Thus, we are replacing the analysis of the distribution of the risk variable by the analysis of the functional parameter:

$$\Pi(\cdot\,;\,Q) : p \rightarrow \Pi(p\,;\,Q),$$

which is more appropriate for risk control. This functional risk measure can be in a one-to-one relationship with the underlying quantile function Q, or can strictly summarize the corresponding information, if we focus on a special risk feature.

i) **VaR**

When $H(u\,;\,p) = \mathbb{1}_{(u \geq p)}$ for $p \in [0,1]$, the distortion cdf corresponds to a point mass at p. We have:

$$\Pi(p;\,Q) \;=\; Q(1-p), \tag{2.5}$$

which is the VaR at risk level p. Thus, the VaR is a special DRM associated with an indicator distortion function, which is not concave.

ii) **Tail-VaR**

When $H(u;p) = (u/p) \wedge 1$ for $p \in [0,1]$, the distortion function is the cdf of the uniform distribution on $[0,p]$. We get:

$$\Pi(p\,;\,Q) \;=\; \int_0^p \frac{Q(1-u)}{p} d\,u = \frac{1}{p}\int_{Q(1-p)}^{\infty} x d\,F(x) = E\big[X\big|X \geq VaR(p)\big]. \tag{2.6}$$

Thus, $\Pi(p\,;\,Q)$ is the Tail-VaR at level p (denoted by TVaR(p)) as defined in Artzner et al. (1999). Since the function $u \rightarrow \big(u/p\big) \wedge 1$ is concave, the Tail-VaR is a coherent risk measure. The Tail-VaR is an equally weighted average of all VaR at levels smaller than p. Finally, note that the Tail-VaR is in a simple one-to-one relationship with the Lorenz Curve [Gastwirth (1971)], by $\mathfrak{L}(p) = p\Pi(p;Q)/E[X]$.

iii) **Proportional Hazard distortion risk measure**

If $H(u;p) = u^p$ for $p \in [0,\infty]$, the distortion function is the power-law transformation and can be interpreted as a cdf on $[0,1]$. The associated DRM is:

$$\begin{aligned}
\Pi(p\,;\,Q) &= \int_0^1 Q(1-u)\,pu^{p-1}d\,u \\
&= \int_{F(0)}^1 (1-u)^p d\,Q(u) + \int_0^{F(0)} \left[(1-u)^p - 1\right]\,d\,Q(u) \\
&= \int_{-\infty}^0 \left[\big(1-F(x)\big)^p - 1\right]\,d\,x + \int_0^\infty \big(1-F(x)\big)^p d\,x. \qquad (2.7)
\end{aligned}$$

The interpretation of the distortion above is the following: The initial survivor function $S(x) = 1 - F(x)$ is replaced by the transformed survivor function $S_p^*(x) = S(x)^p$. Therefore, we have:

$$\Pi(p;Q) = \int_{-\infty}^0 \left[S_p^*(x) - 1\right]\,d\,x + \int_0^\infty S_p^*(x)\,d\,x = E_p^*[X],$$

where E_p^* denotes the expectation with respect to the distribution with survivor function S_p^*. The relationship between the initial and transformed survivor functions can also be written as: $\log S_p^*(x) = p\,\log S(x)$, and implies $\frac{-d\,\log S_p^*(x)}{d\,x} = p\left(\frac{-d\,\log S(x)}{d\,x}\right)$. Thus, the hazard functions associated with both distributions are proportional, which explains the name of the risk measure. The proportional hazard distortion risk measures are coherent risk measures, if parameter $p < 1$, that is, if the extreme losses are overweighted.

iv) **Exponential distortion risk measure**
If $H(u;p) = (1-e^{-pu})/(1-e^{-p})$, the distortion function is the cdf of the exponential distribution on $[0,1]$. The associated DRM is:

$$\begin{aligned}
\Pi(p;Q) &= \int_0^1 Q(1-u)\frac{p\,e^{-pu}}{1-e^{-p}}d\,u \\
&= \int_0^{F(0)} \left[\frac{1-e^{-p(1-u)}}{1-e^{-p}} - 1\right] d\,Q(u) + \int_{F(0)}^1 \frac{1-e^{-p(1-u)}}{1-e^{-p}} d\,Q(u) \\
&= \int_{-\infty}^0 \left[\frac{1-e^{-p\left(1-F(x)\right)}}{1-e^{-p}} - 1\right] d\,x + \int_0^\infty \frac{1-e^{-p\left(1-F(x)\right)}}{1-e^{-p}} d\,x.
\end{aligned}$$

The exponential distortion risk measure satisfies the coherency conditions when $p > 0$.

2.2 Relationship between VaR and Tail-VaR

A main drawback of VaR is that it ignores the magnitude of loss. This problem may be partially solved by replacing the VaR by a more appropriate risk measure, such as the Tail-VaR. In this section, we analyze the link between VaR and Tail-VaR for different underlying distributions.

The second row in Table 1 provides the ratio between VaR and Tail-VaR for uniform, exponential, Pareto and Gaussian distributions, respectively. These ratios are independent of any scale parameter, are nondecreasing functions in the risk level p, and are larger than 1. Thus, the Tail-VaR is an amplified VaR with an amplifying factor which is a positive nondecreasing function of p, $TVaR(p) = [1 + L(p)]VaR(p)$. The value and pattern of this factor depends on the distribution (see Figures 1 for $p \in (0, 0.2]$).[3] The exponential distribution features the widest range for the factor ($L(p)$ is between 0 and 140%), while the uniform distribution has the narrowest variation ($L(p)$ is between 0 and 12%). The Pareto distribution yields the simplest modification, in which the Tail-VaR is obtained by simply multiplying the VaR by a constant factor ($a/(a-1)$), depending on the shape parameter. This constant factor is a decreasing function of $a \in (1, \infty)$ (See Figure 2). In fact, we have the following result:

Proposition 1. *For a positive variable X, the ratio between Tail-VaR and VaR is constant in p, if and only, if the underlying distribution is Pareto.*

Proof. Let us rewrite the ratio as:

$$\begin{aligned}
\frac{E[X|X>\eta]}{\eta} &= \frac{\frac{1}{1-F(\eta)}\int_\eta^\infty x d\, F(x)}{\eta} \\
&= \frac{\frac{-1}{S(\eta)}\int_\eta^\infty x d\, S(x)}{\eta} \\
&= \frac{\eta + \frac{1}{S(\eta)}\int_\eta^\infty S(x) d\, x}{\eta} \quad \text{(by integrating by part)} \\
&= 1 + \frac{1}{\eta\, S(\eta)}\int_\eta^\infty S(x) d\, x. \qquad (2.8)
\end{aligned}$$

The ratio between Tail-VaR and VaR is constant, if and only, if

$$\frac{1}{S(\eta)}\int_\eta^\infty S(x) d\, x = c\eta, \qquad (2.9)$$

where c is a positive constant. By integrating both sides of equation (2.9), we see that:

$$\frac{d}{d\eta}\log\left(\int_\eta^\infty S(x) d\, x\right) = -\frac{1}{c}\frac{d}{d\eta}\log(\eta).$$

[3] All ratios approach 1 for $p \to 0$. Since we are generally interested in risk levels less than 10%, our range of p is wide enough to cover all meaningful situations.

Thus, there exists a positive constant A, such that:

$$\int_{\eta}^{\infty} S(x)d\,x = A\,\eta^{-1/c}. \tag{2.10}$$

Taking derivative of both sides of (2.10) with respect to η, we get:

$$S(\eta) = \frac{A}{c}\eta^{-(c+1)/c}, \tag{2.11}$$

which corresponds to a $Pareto(a, b)$ distribution with $a = (c+1)/c$ and $b = \left(\frac{A}{c}\right)^{c/(c+1)}$. $\square$

An alternative way to describe the relationship between the VaR and Tail-VaR is based on the link between their risk levels. Indeed, the Tail-VaR at risk level p can be viewed as a VaR at a more constraining risk level p^*. This defines an increasing function $p^* = g(p)$ smaller than p, which depends on the underlying distribution, and satisfies $TVaR(p) = VaR(p^*)$ (see the third row of Table 1). Except for the standard normal distribution, $g(p)$ is proportional to p. Its behavior for the standard normal distribution is plotted in Figure 3. In fact, for small value of p, say less than 0.5, the function $g(p)$ is hardly distinguishable from linearity even under Gaussian assumption. This is a desired property from a practical point view. Indeed, after calculating the VaR at several risk levels, the related Tail-VaRs are obtained automatically, which simplifies the computation procedures. In addition, an internal or external regulator can interpret an extreme quantile value either as an amount that a given portfolio's losses will not be likely to exceed under normal market conditions or as the expected loss of the same portfolio under adverse market conditions.[4]

Let us characterize the distributions such that the function g is linear with coefficient α. We get:

$$TVaR(p) = VaR(\alpha p) \Leftrightarrow \int_0^p Q(1-u)d\,u = p\,Q(1-\alpha p). \tag{2.12}$$

In particular, by taking $p = 1$, we get an interpretation of the slope parameter α as: $Q(1-\alpha) = VaR(\alpha) = E[X]$, and note that $TVaR(p) = VaR\left[pVaR^{-1}\big(E[X]\big)\right]$. Typically, the level p has to be divided by 2, if the mean is equal to the median, by a number strictly larger than 2 (resp. smaller than 2) if the mean is smaller (resp. larger) than the median, that is, if the distribution is "right skewed" (resp. "left skewed"). By differentiating both sides of (2.12), we get:

$$Q(1-\alpha p) - Q(1-p) = \alpha p\; q(1-\alpha p),$$

where $q(u) = \partial Q(u)/\partial u$ is the quantile density. From Table 1, we see that the uniform,

[4] The reported p^* are smaller than half the risk level p. This is expected since all distributions considered here (except the uniform distribution) have tails skewed to the right.

exponential and Pareto distributions satisfy the condition with $\alpha = 1/2$ (since the mean is equal to the median), $1/e$ and $((a-1)/a)^a$, respectively. The dependence of the slope coefficient α with respect to the shape parameter a is given in Figure 4. This coefficient varies between 0 and $1/e$. Since the tail of a Pareto distribution is thinner as a rises, the fatter the tail, the smaller is α.

2.3 Sensitivity of a distortion risk measure with respect to a distortion parameter

The sensitivity of the DRM is:

$$\begin{aligned}
\frac{\partial \Pi}{\partial p}(p;Q) &= \frac{\partial}{\partial p}\left[\int_0^1 Q(1-u)d\,H(u;p)\right] \\
&= \int_0^1 Q(1-u)d\left[\frac{\partial}{\partial p}H(u;p)\right] \\
&= -\int_0^1 \frac{\partial}{\partial p}H(u\,;\,p)dQ(1-u) \\
&= \int_0^1 \frac{\partial H}{\partial p}(1-u\,;\,p)d\,Q(u), \qquad (2.13)
\end{aligned}$$

since $\frac{\partial}{\partial p}H(1;p) = \frac{\partial}{\partial p}H(0;p) = 0$.

This expression is similar to the expression of a DRM except that the distortion function $H^*(u;p) = \frac{\partial}{\partial p}H(u;p)$ is not a cdf, since $H^*(1;p) = H^*(0;p) = 0$. Moreover, the alternative expressions (2.4) of the sensitivity are still valid even if the loss (and profit) variable is not necessarily positive:

$$\frac{\partial \Pi}{\partial p}(p;Q) = \Pi(H^*;Q) = \int_0^1 Q(1-u)d\,H^*(u;p) = \int_0^1 H^*(1-u;p)dQ(u). \qquad (2.14)$$

The examples below illustrate the computation and interpretation of the sensitivity.

i) **Tail-VaR**

We have:

$$\begin{aligned}
\frac{\partial}{\partial p} TVaR(p) &= -\int_0^1 \frac{1-u}{p^2} \mathbb{1}_{(1-u \leq p)} dQ(u) \\
&= -\int_{1-p}^1 \frac{1-u}{p^2} dQ(u) \\
&= -\frac{1}{p^2}\left(1-u\right)Q(u)\Big]_{1-p}^1 + \frac{1}{p}\int_{1-p}^1 Q(u) d\,\frac{1-u}{p} \\
&= \frac{1}{p} VaR(p) - \frac{1}{p}\int_{Q(1-p)}^{\infty} x\, d\,\frac{F(x)}{p} \\
&= \frac{1}{p}\big[VaR(p) - TVaR(p)\big]. \qquad (2.15)
\end{aligned}$$

The sensitivity of Tail-VaR with respect to the distortion parameter is the opposite of the difference between the conditional expected loss and the lower bound of the loss per unit of risk level. This derivative is negative and the Tail-VaR increases when the risk level diminishes. As seen in the next section, this value measures the accuracy of the nonparametric estimator of Tail-VaR. Indeed, a large (absolute) sensitivity of the Tail-VaR can induce substantial estimation errors at small p.

The sensitivities of the VaR and Tail-VaR with respect to risk level p are provided in Table 2 for the uniform, exponential, Pareto and standard normal distributions, respectively.

ii) **Proportional Hazard distortion risk measure**

We have $H^*(u;p) = \frac{\partial}{\partial p}(u^p) = u^p(\log\, u)$ and deduce that:

$$\begin{aligned}
\frac{\partial}{\partial p} PH(p) &= \int_0^1 Q(1-u)\, d\left[u^p(\log\, u)\right] \qquad (2.16) \\
&= \int_0^1 Q(1-u)\, u^{p-1}\left(p\,\log\, u + 1\right)\, d\,u \\
&= \int_0^1 Q(1-u)\, w(u,p)\, d\,u, \quad \text{say.}
\end{aligned}$$

When PH is interpreted as a risk premium, the sensitivity is the marginal response of this premium to a slight adjustment of the pessimism level. This marginal response is a weighted expectation of VaR with the weighting function $w(u,p) = u^{p-1}\left(p\,\log\, u + 1\right)$ depending on the pessimism parameter p. Figure 5 provides two examples of weighting functions when $p =$ 0.2, and 2, respectively.

As expected, the marginal response is negative. Indeed, by integrating (2.16) by part, we get:

$$\begin{aligned}
\frac{\partial}{\partial p} PH(p) &= -\int_0^1 u^p (\log u) d\,Q(1-u) \qquad \text{(integration by part)} \\
&= \int_{-\infty}^{\infty} \big[1 - F(x)\big]^p \log\big[1 - F(x)\big]\, d\,x \qquad \text{(change of variable)} \\
&< 0.
\end{aligned}$$

iii) **Exponential distortion risk measure**

With exponential distortion function, we get:

$$H^*(u;p) = \frac{e^{-pu} u}{1 - e^{-p}} - \frac{e^{-p}(1 - e^{-pu})}{(1 - e^{-p})^2}.$$

Thus, the sensitivity of the exponential distortion risk measure is given by:

$$\frac{\partial}{\partial p} EX(p) = \int_0^1 Q(1-u)\, w(u,p)\, d\,u,$$

which is a weighted expectation of VaR with weighting function,

$$w(u,p) = \frac{e^{-pu}}{1 - e^{-p}} - \frac{e^{-p-pu} p}{(1 - e^{-p})^2} - \frac{e^{-pu} p u}{1 - e^{-p}}.$$

Two examples of the shape of this weighting function are plotted in Figure 6.

3 Nonparametric estimation of functional distortion risk measures and their sensitivities

Let us consider a set of i.i.d. one-dimensional observations $x_1, ..., x_T$, with common cdf F_0 and quantile function $Q_0 = F_0^{-1}$. The quantile function Q_0 can be estimated by the sample quantile $\widetilde{Q}_T$ defined by:

$$\widetilde{Q}_T(u) = \inf\{x : \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}_{(x_t \leq x)} \geq u\}, \text{ for } u \in [0,1]. \tag{3.1}$$

We first recall the asymptotic distribution of $\widetilde{Q}_T$. Then, we introduce the distortion risk measures and their expressions in terms of quantile function. By applying the analogy principle (see Appendix A), we deduce functional nonparametric estimators of distortion risk measures and of their sensitivities with respect to parameters.

3.1 Asymptotic distribution of the nonparametric quantile estimator

The analysis is based on the Bahadur representation of the quantile estimator, which provides the expressions of the estimated quantiles in terms of the associated cdf [Koenker (2005), Section 4.3]. We get,

$$\sqrt{T}\big[\widetilde{Q}_T(u) - Q_0(u)\big] = -\frac{1}{f_0\big(Q_0(u)\big)}\sqrt{T}\left[\widetilde{F}_T\big(Q_0(u)\big) - u\right] + o_p(1), \tag{3.2}$$

where $\widetilde{F}_T$ is the sample cdf, Q_0 the true quantile function and f_0 the true density, and the proposition below.

Proposition 2. *For an i.i.d. random sample from a distribution with quantile function Q_0 and pdf f_0, we have:*

$$\sqrt{T}\left[\widetilde{Q}_T(\cdot) - Q_0(\cdot)\right] \Rightarrow -\frac{1}{f_0\big(Q_0(\cdot)\big)}B(\cdot),$$

where $B(u)$ is a univariate Brownian bridge and $\Rightarrow$ denotes weak convergence of stochastic processes (see the Functional Limit Theorem in Appendix A).

3.2 Estimation of distortion risk measure

By the analogy principle, a nonparametric estimator of the DRM is defined by:

$$\widetilde{\Pi}_T(p) = \Pi(p;\widetilde{Q}_T), \quad \text{p varying.}$$

For a given sample $x_1, \ldots, x_T$, the observations can be ranked in an ascending order such that $x_1^* \leq x_2^* \cdots \leq x_T^*$, and the estimated DRM is simply:

$$\widetilde{\Pi}_T(p) = \sum_{i=1}^{T} x_i^* \left[H\left(1 - \frac{i-1}{T}\right) - H\left(1 - \frac{i}{T}\right)\right]. \tag{3.3}$$

Thus, the nonparametric estimator of the DRM is a linear combination of the order statistics x_i^* and, for each value of the pessimism parameter, this is an example of $L-$statistics [see e.g. Jones and Zitikis (2003), (2005)]. For instance, the nonparametric estimator of VaR at risk level p can be written as:

$$\widetilde{VaR}_T(p) = \sum_{i=1}^{T} x_i^* \left[\mathbb{1}_{\left(\frac{i-1}{T} \le 1-p\right)} - \mathbb{1}_{\left(\frac{i}{T} \le 1-p\right)} \right] = \begin{cases} x^*_{T(1-p)}, & \text{if } T(1-p) \text{ is integer,} \\ x^*_{[T(1-p)]+1}, & \text{otherwise,} \end{cases}$$

where $[a]$ denotes the integer part of a. For the Tail-VaR, the estimator is directly related to the estimator introduced in the literature for the Lorenz Curve or the Gini Index [Gastwirth (1972), Barrett and Donald (2000), Zitikis (2003)].

The proposition below is a direct consequence of the expression of the DRM, $\Pi(H;Q) = \int_0^1 Q(1-u)d\,H(u;p)$, and of the results of Section 3.1. The asymptotic behavior is not only a pointwise convergence result [see e.g. Jones and Zitikis (2003)], but concerns the process of DRM indexed by pessimism parameter. This functional result is needed for further analysis of links between the VaR and Tail-VaR for instance.

Proposition 3. *For an i.i.d. random sample from a distribution with quantile function Q_0 and pdf f_0, we have:*

$$\sqrt{T}\left[\widetilde{\Pi}_T(p) - \Pi(p;Q)\right] \;\; \Rightarrow \;\; \int_0^1 \frac{B(1-u)}{f_0\big(Q_0(1-u)\big)} d\,H(u;p),$$

where $B(\cdot)$ is a standard Brownian bridge. The process is asymptotically Gaussian with pointwise variance equal to:

$$\begin{aligned} V\big(\sqrt{T}\,[\widetilde{\Pi}_T(p) - \Pi(p)]\big) &= \int_0^1 \int_0^1 \frac{(1-u_1)\wedge(1-u_2) - (1-u_1)(1-u_2)}{f_0\big(Q_0(1-u_1)\big) f_0\big(Q_0(1-u_2)\big)} d\,H(u_1;p) d\,H(u_2;p) \\ &= 2 \int_0^1 \frac{u_2\, A(u_2,p)}{f_0\big(Q_0(1-u_2)\big)} d\,H(u_2;p), \end{aligned}$$

where

$$A(v,p) = \int_v^1 \frac{1-u}{f_0\big(Q_0(1-u)\big)} d\,H(v;p).$$

Replacing the quantile function and the density by their nonparametric estimators, we get the Corollary below about the estimation of the asymptotic variance of the estimated DRM.

Corollary 1. *For an i.i.d. random sample, the asymptotic variance of estimated DRM can be consistently estimated by:*

$$\widehat{V}\big(\sqrt{T}\,[\widetilde{\Pi}_T(p) - \Pi(p)]\big) = 2 \int_0^1 \frac{u_2\, \widehat{A}(u_2,p)}{\hat{f}\big(\widetilde{Q}_T(1-u_2)\big)} d\,H(u_2;p),$$

where

$$\widehat{A}(v,p) = \int_v^1 \frac{1-u}{\hat{f}\big(\widetilde{Q}_T(1-u)\big)} d\,H(v;p),$$

and $\hat{f}$ is a nonparametric consistent estimator of the density function.

A common choice of the density estimator is a kernel estimator. The estimated asymptotic variance of the estimated DRM can be computed numerically. However, a kernel estimator of the density converges rather slowly, which may render the application of the asymptotic theory questionable in finite sample. Fortunately, except for the VaR, the density function can be eliminated from the variance expression above.

Corollary 2. *When H is continuous and almost everywhere differentiable,*[5] *we have*

$$\begin{aligned} V\big(\sqrt{T}\,[\widetilde{\Pi}_T(p) - \Pi(p)]\big) &= \int_{\mathbb{R}^2} \frac{F(x_1)\wedge F(x_2) - F(x_1)F(x_2)}{f_0(x_1)f_0(x_2)} \frac{\partial H(1-F(x_1);p)}{\partial u}\frac{\partial H(1-F(x_2);p)}{\partial u} d\,F(x_1)\,d\,F(x_2) \\ &= \int_{\mathbb{R}^2} \big(F(x_1)\wedge F(x_2) - F(x_1)F(x_2)\big)\frac{\partial H(1-F(x_1);p)}{\partial u}\frac{\partial H(1-F(x_2);p)}{\partial u} d\,x_1\,d\,x_2 \\ &= 2\int_{\mathbb{R}} \big(1-F(x_2)\big)A(x_2,p)\frac{\partial H(1-F(x_2);p)}{\partial u} d\,x_2, \end{aligned}$$

where

$$A(y,p) = \int_{-\infty}^{y} F(x)\frac{\partial H(1-F(x);p)}{\partial u} d\,x.$$

Thus, this pointwise variance can be estimated by substituting the empirical distribution function $\widetilde{F}_T(x)$ in the expression [see Jones and Zitikis (2003), Theorem 3.2].[6] The asymptotic variance above is estimated by:

$$\widehat{V}\big(\sqrt{T}\,[\widetilde{\Pi}_T(p) - \Pi(p)]\big) = \sum_{i=1}^{T-1}\sum_{j=1}^{T-1}\left(\frac{i}{T}\wedge\frac{j}{T} - \frac{i}{T}\frac{j}{T}\right) w\left(1-\frac{i}{T};p\right) w\left(1-\frac{j}{T};p\right)\left(x^*_{i+1}-x^*_i\right)\left(x^*_{j+1}-x^*_j\right),$$

where

$$w(u;p) = \frac{\partial}{\partial u}H(u;p).$$

Similarly, it is easy to derive the estimated covariance between either the estimators of a DRM with different values of p, or the estimators of two DRMs. For instance, we get:

$$COV\Big(\sqrt{T}\big[\Pi_T p - \Pi(p)\big], \sqrt{T}\big[\Pi_T p' - \Pi(p')\big]\Big) = Q_{F,F}\left(\frac{\partial H}{\partial u}(1-\cdot;p), \frac{\partial H}{\partial u}(1-\cdot;p')\right),$$

[5] with respect to Lebesgue measure on $[0,1]$.

[6] The quantity, $\int_{\mathbb{R}^2}[\min(F(x_1),F(x_2)) - F(x_1)F(x_2)]\Psi_1(F(x_1))\Psi_2(F(x_2))dx_1dx_2$, can be denoted as $Q_{F,F}(\Psi_1,\Psi_2)$. Thus, the pointwise variance is $Q_{F,F}\left(\frac{\partial H}{\partial u}(1-\cdot;p), \frac{\partial H}{\partial u}(1-\cdot;p)\right)$.

and,

$$\widehat{COV}\left(\sqrt{T}\,[\widetilde{\Pi}_T(p)-\Pi(p)],\sqrt{T}\,[\widetilde{\Pi}_T(p')-\Pi(p')]\right)$$
$$=\sum_{i=1}^{T-1}\sum_{j=1}^{T-1}\left(\frac{i}{T}\wedge\frac{j}{T}-\frac{i}{T}\frac{j}{T}\right)w\left(1-\frac{i}{T};p\right)w\left(1-\frac{j}{T};p'\right)\left(x^*_{i+1}-x^*_i\right)\left(x^*_{j+1}-x^*_j\right).$$

Since the derivative $\frac{\partial H}{\partial u}(\cdot;p)$ is positive for any p, we deduce from the expression of $Q_{F,F}$, that two estimated DRM are always positively correlated.

3.3 Estimation of the sensitivity

From expression (2.14), the sensitivity of the distortion risk measure has a similar expression as the DRM, except that the distortion function is replaced by its first-order derivative with respect to parameter p. The limiting properties of their estimators are also similar. They are given in the Corollary below.

Corollary 3. *If $H(u;p)$ is differentiable in p, for an i.i.d. random sample with quantile function Q_0 and pdf f_0, we have:*

$$\sqrt{T}\left[\widetilde{\Pi}_T(p;H^*)-\Pi(p;H^*)\right]\quad\Rightarrow\quad\int_0^1\frac{B(1-u)}{f_0\big(Q_0(1-u)\big)}d\,H^*(u;p).$$

$\widetilde{\Pi}_T(p;H^*)$ is asymptotically Gaussian with pointwise variance given by:

$$V\left(\sqrt{T}\,[\widetilde{\Pi}_T(p;H^*)-\Pi(p;H^*)]\right)=2\int_0^1\frac{u_2\,A^*(u_2,p)}{f_0\big(Q_0(1-u_2)\big)}d\,H^*(u_2;p),$$

where

$$A^*(v,p)=\int_v^1\frac{1-u}{f_0\big(Q_0(1-u)\big)}d\,H^*(u;p).$$

The asymptotic variance of the sensitivity can be estimated in the same way as for the DRM with or without density estimation. Indeed, if the cross-derivative $\frac{\partial^2 H}{\partial u\partial p}$ exists, we have: $V\left(\sqrt{T}\left[\widetilde{\Pi}_T(p;H^*)-\Pi(p;H^*)\right]\right)=Q_{F,F}\left(\frac{\partial^2 H}{\partial u\partial p}(1-\cdot;p),\frac{\partial^2 H}{\partial u\partial p}(1-\cdot;p)\right)$. The function H^* associated with the sensitivity of Tail-VaR is noncontinuous and the estimation of the density function cannot be avoided. On the contrary, it is not necessary to estimate the density function for examples such as Proportional Hazard and Exponential distortion risk measures.

The nonparametric estimator of the sensitivity of VaR is not well defined even though the quantile function is differentiable in p analytically. However, the sensitivity analysis of VaR may be approximated by the sensitivity analysis of a Tail-VaR. For instance, if the two risk measures are related by $TVaR(p)=VaR(\alpha p)$, the marginal change of $VaR(\alpha p)$ with respect to αp is approximated by the marginal change of the $TVaR(p)$ with respect to p divided by α, which has a well defined nonparametric estimator.

3.4 Examples

The closed form of the asymptotic variance can be derived for specific distortion functions. We consider below the examples of VaR, Tail-VaR, and PH and illustrate the accuracy of estimation by studying the asymptotic variances of their nonparametric estimators. The detailed proofs are provided in Appendices C and D.

i) **VaR**

For $H(u;p) = \mathbb{1}_{(u \geq p)}$, $p \in [0,1]$, the asymptotic variances of the estimator of VaR is:

$$V\left(\sqrt{T}\,[\widetilde{VaR}_T(p) - VaR(p)]\right) = \frac{p(1-p)}{\left[f_0\left(Q_0(1-p)\right)\right]^2} = \left[q_0(1-p)\right]^2 p(1-p), \qquad (3.4)$$

where $q_0 = 1/f_0(Q_0)$ is the quantile density function. The tail behavior is classified into different categories in practice. For example, the Gaussian distribution has a thin Gaussian tail; exponential, Laplace and logistic distributions have thick exponential tails; Pareto, Lévy and Cauchy distributions have thicker Pareto tails. It is important to understand how tail behavior influences the estimation accuracy of risk measures. The asymptotic variance of the nonparametrically estimated VaR and its relative accuracy are given in Table 3 for uniform, exponential, Pareto, Gaussian, Lévy and Cauchy distributions, respectively. Since distributions with large absolute values for the limits of the support are likely to yield noisy estimates of the extreme quantiles, relative accuracy may be more informative. In Figure 7, we plot the asymptotic variance and relative accuracy as functions of p. Distributions with unbounded support tend to induce large estimation errors at the tails. This is evidenced by the variance patterns for both tails of Gaussian and Cauchy distributions and for the right tail of exponential, Pareto and Lévy distributions. Moreover, distributions with extremely heavy tails tend to cause huge estimation error, which is evidenced by the magnitude of variance associated with both Lévy and Cauchy distributions.

ii) **Tail-VaR**

If $H(u;p) = (u/p) \wedge 1$, for $p \in [0,1]$, the asymptotic variance of the nonparametrically estimated

Tail-VaR is given by:

$$V\left(\sqrt{T}\,[\widetilde{TVaR}_T(p) - TVaR(p)]\right) = \frac{V\big(X|X \geq VaR(p)\big) + (1-p)\,\big[TVaR(p) - VaR(p)\big]^2}{p}. \tag{3.5}$$

Column 3 and 4 of Table 4 provide the asymptotic variance and relative accuracy of the nonparametric estimator for uniform, exponential, Pareto and Gaussian distributions, respectively. The variance for Lévy and Cauchy distributions cannot be derived, since the associated moments are not defined. In fact, for these distributions, the Tail-VaR may not even exist.[7] The second component in the decomposition of $V\big[\sqrt{T}\,\widetilde{TVaR}_T(p)\big]$ is proportional to the square of the sensitivity of Tail-VaR. Thus, larger sensitivity tends to imply larger variance for the nonparametric estimator. This effect is seen on Figure 8, which plots the accuracies as function of the pessimism parameter. With unbounded right tails, the exponential, Pareto and Gaussian distributions imply large estimation error, both in terms of variance and relative accuracy, for small value of p.

iii) PH

For the power-law distortion function, $H(u;p) = u^p$, the asymptotic variances of the estimated PH are:

$$\begin{aligned} &V\left(\sqrt{T}\,[\widetilde{PH}_T(p) - PH(p)]\right) \\ &\quad = p^2 E^*_{p-1}(X)E^*_p(X) - p^2\left[E^*_p(X)\right]^2 + \\ &\quad\begin{cases} \frac{p^2}{2p-1}\, E^*_{2p-1}(X^2), & \text{if } p \geq 0.5 \\ p^2 \int_{-\infty}^{\infty} F(X)\big(1-F(X)\big)^{2p-1} d\,X^2 - 2\,p^2\left[\frac{p-1}{2p-1} E^*_{2p-1}(X^2) - \frac{1}{2}E^*_{2p}(X^2)\right], & \text{if } 0 < p < 0.5. \end{cases} \end{aligned}$$

Figure 9 displays the accuracy of the nonparametric estimator of Proportional Hazard distortion risk measure for various distributions. Both variance (panel (a)) and relative accuracy (panel (b)) are considered for $0.5 < p < 1$. The more pessimistic, the less accurate is the estimator. The only exception occurs for the standard normal distribution, where the denominator of the relative accuracy goes to zero as $p \to 1$. Intuitively, a smaller p induces on average a larger modification weight both for the distorted mean and for the distorted variance.

[7] It can be verified that the asymptotic variances reduces to the unconditional variances when $p = 1$.

3.5 Implied pessimism parameter

The regulator receives the reserve levels reported by banks on a regular basis to check the capital adequacies. These data can be used to get information on the behavior of a bank concerning the risk, and in particular to estimate the level of the selected pessimism parameter. More precisely, if we observe a reserve level Π^o, the implied pessimism parameter is defined by:

$$\Pi(p; Q_0) = \Pi^o,$$

and is consistently estimated by:

$$\widehat{p}_T = \Pi^{-1}(\Pi^o; \widetilde{Q}_T).$$

The asymptotic property of the estimated implied parameter $\widehat{p}_T$ is determined by the limiting behavior of the estimated distortion risk measure. More precisely, we have (see Appendix E for a derivation):

$$\sqrt{T}(\widehat{p}_T - p) = -\left(\frac{\partial \Pi}{\partial p}(p; Q)\right)^{-1} \sqrt{T}\big[\Pi(p; \widetilde{Q}_T) - \Pi(p; Q)\big] + o_p(1). \tag{3.6}$$

These estimated implied pessimism parameters can be computed for any reported reserve level, that is, for different dates and banks. Their comparison allows to follow how pessimism varies in time, or to get a segmentation of the banks in terms of pessimism.

4 Tail-VaR versus VaR

The aim of this section is to introduce a nonparametric estimator of the function g, which links the VaR and Tail-VaR, and to derive its asymptotic properties. In a second step, we explain how to test for the linearity of function g and estimate the associated slope coefficient.

4.1 Nonparametric estimator of g

Function g is defined by: $TVaR(p) = VaR[g(p)]$, or equivalently:

$$TVaR(p) = Q\big(1 - g(p)\big) \Leftrightarrow g(p) = 1 - F\big[TVaR(p)\big].$$

By the analogy principle, a nonparametric estimator of function g is:

$$\widetilde{g}_T(p) = 1 - \widetilde{F}_T\left[\widetilde{TVaR}_T(p)\right]. \tag{4.1}$$

Under the appropriate regularity conditions, the estimator $\widetilde{g}_T$ is consistent and such that:

$$\sqrt{T}\,[\widetilde{g}_T(p)-g(p)] = -\sqrt{T}\,\Big[\widetilde{F}_T\big[TVaR(p)\big]-F_0\big[TVaR(p)\big]\Big]$$
$$-f_0\big[TVaR(p)\big]\sqrt{T}\,\Big[\widetilde{TVaR}_T(p)-TVaR(p)\Big]+o_p(1).$$

We deduce the proposition below.

Proposition 4. *For an i.i.d. random sample, we have*

$$\sqrt{T}\,[\widetilde{g}_T(p)-g(p)] \Rightarrow -B\left[F_0\big[TVaR(p)\big]\right]+\frac{f_0\big[TVaR(p)\big]}{p}\int_{1-p}^{1}\frac{B(u)}{f_0\big[Q_0(u)\big]}d\,u,$$

which is asymptotically Gaussian with zero mean and pointwise variance

$$V\Big[\sqrt{T}\,\big(\widetilde{g}_T(p)-g(p)\big)\Big] = [1-g(p)]g(p)+f_0\big[TVaR(p)\big]^2 V\Big(\sqrt{T}\,\widetilde{TVaR}_T(p)\Big)$$
$$-2\,f_0\big[TVaR(p)\big]\frac{1}{p}\Bigg\{g(p)\Big[\big[1-g(p)\big]\,TVaR\big(g(p)\big)-VaR(p)(1-p)\Big]$$
$$-\big[p-g(p)\big]E\,\Big[X\big|VaR(p)\le X\le VaR\big(g(p)\big)\Big]\Bigg\}$$

with $V\Big(\sqrt{T}\,\widetilde{TVaR}_T(p)\Big)$ *given by* (3.5).

4.2 Test of the linearity hypothesis

Let us now consider the null hypothesis of linearity of function g in a given risk window (p_0,p_1). This hypothesis concerns the underlying distribution of returns and the portfolio allocation. As seen in the examples, it can be specified for some distributions of portfolio returns, but of course it cannot be satisfied by all possible distributions and portfolio allocations; otherwise, the VaR would be a coherent risk measure. The linearity hypothesis $H_0=\big\{\exists\alpha_0 : g(p)=\alpha_0 p \ \text{ for any } p\in(p_0,p_1)\big\}$ can be tested as follows.

Let us introduce a measure of the distance to the linearity hypothesis:

$$\widetilde{L}_T(\mu)=\min_{\alpha}\int_{p_0}^{p_1}\big(\widetilde{g}_T(p)-\alpha p\big)^2\mu(p)\,d\,p, \tag{4.2}$$

where μ is a weighting function and accept the null hypothesis if the measure is sufficiently small. More precisely, let us first consider the optimal value of the slope parameter α:

$$\widetilde{\alpha}_T = \arg\min_{\alpha} \int_{p_0}^{p_1} \left(\widetilde{g}_T(p) - \alpha p\right)^2 \mu(p)\, d\, p$$
$$= \frac{\int_{p_0}^{p_1} p\, \widetilde{g}_T(p)\mu(p) d\, p}{\int_{p_0}^{p_1} p^2 \mu(p) d\, p}. \tag{4.3}$$

We get the following result (see Appendix C):

Proposition 5. *Under the null hypothesis, the estimator $\widetilde{\alpha}_T$ is consistent, asymptotically Gaussian:*

$$\sqrt{T}(\widetilde{\alpha}_T - \alpha_0) \xrightarrow{a} N\left(0, \eta^2\right),$$

where the variance η^2 is given by

$$\eta^2 = 2\, A^2 \left\{ \int_{p_0}^{p_1} A^* \left[(1 - g(\widetilde{p}))\,(1 - g(p))\right] d\,\widetilde{p} - 2 \int_{p_0}^{p_1} A^* \left[f_0 \left[TVaR(p)\right]^2 COV\left(\sqrt{T}\, \widetilde{VaR}_T(g(p)), \sqrt{T}\, \widetilde{TVaR}_T(\widetilde{p})\right)\right] \right.$$
$$\left. \int_{p_0}^{p_1} A^* \left[f_0 \left[TVaR(\widetilde{p})\right] f_0 \left[TVaR(p)\right] COV\left(\sqrt{T}\, \widetilde{TVaR}_T(p), \sqrt{T}\, \widetilde{TVaR}_T(\widetilde{p})\right)\right] d\,\widetilde{p} \right\}, \tag{4.4}$$

where

$$A = \frac{1}{\int_{p_0}^{p_1} p^2 \mu(p) d\, p}, \quad \text{and} \quad A^*[\xi] = \int_{p_0}^{\widetilde{p}} \widetilde{p}\, p\, \mu(\widetilde{p})\mu(p)\xi d\, p.$$

Then, the optimal value of the criterion function is:

$$\widetilde{L}_T(\mu) = \int_{p_0}^{p_1} \left(\widetilde{g}_T(p) - \widetilde{\alpha}_T p\right)^2 \mu(p) d\, p = \int_{p_0}^{p_1} \left(\widetilde{g}_T(p)\right)^2 \mu(p) d\, p - \frac{\left[\int_{p_0}^{p_1} p\, \widetilde{g}_T(p)\mu(p) d\, p\right]^2}{\int_{p_0}^{p_1} p^2 \mu(p) d\, p}.$$

By applying the Functional Limit Theorem (see Appendix A), we deduce the asymptotic behavior of the test criterion.

Proposition 6. *Under the null hypothesis of linearity,*

$$T\, \widetilde{L}_T(\mu) = T \int_{p_0}^{p_1} \left(\widetilde{g}_T(p) - g(p) - (\widetilde{\alpha}_T - \alpha)p\right)^2 \mu(p) d\, p$$
$$\Rightarrow \int_{p_0}^{p_1} \left\{ B_g(p) - \frac{p \int_{p_0}^{p_1} \widetilde{p}\, B_g(\widetilde{p})\mu(\widetilde{p}) d\, \widetilde{p}}{\int_{p_0}^{p_1} \widetilde{p}^2 \mu(\widetilde{p}) d\, \widetilde{p}} \right\}^2 d\, p,$$

where

$$B_g(p) = -B\left[F_0\left[TVaR(p)\right]\right] + \frac{f_0\left[TVaR(p)\right]}{p} \int_{1-p}^{1} \frac{B(u)}{f_0\left[Q_0(u)\right]} d\, u.$$

This is the distribution of a series of weighted χ_1^2 random variables [see e.g. Freitag et al. (2003), Remark 2.7].

5 Application to currency portfolio

In this section, we apply the results of the previous sections to currency portfolios. The currencies introduced in the portfolio are the Hongkong Dollar and Japanese Yen. The US Dollar (resp. the Singapore Dollar) is chosen as the basic numeraire of the investor. Indeed, the financial features of a currency portfolio can depend on the numeraire. The main part of the literature on currency portfolios [see e.g. Akgiray and Booth (1988); Breymann et al. (2003); Chen et al. (2004); Patton (2006)] consider portfolios written in US Dollar, and exhibit a number of stylized facts such as asymmetry, fat tail and stochastic jumps. It is important to see if these stylized facts are due either to the currencies introduced in the portfolio, or to the portfolio allocation, or if they come from the chosen numeraire. The data set consists of daily data from November 1993 to December 2005 which provides about 3200 observations. Denoting $S^i_{u,t}$ (resp. $S^i_{s,t}$) the exchange rate (at date t) of currency i ($i = 1, 2$ representing Hongkong Dollar and Japanese Yen, respectively) in US Dollar (resp. Singapore Dollar), the daily returns are:

$$\widetilde{x}^i_{u,t} = \frac{S^i_{u,t-1} - S^i_{u,t}}{S^i_{u,t-1}}, \quad \widetilde{x}^i_{s,t} = \frac{S^i_{s,t-1} - S^i_{s,t}}{S^i_{s,t-1}}.$$

The returns are computed at daily and monthly (20 trading days) horizons. To comply with the independence assumption on returns, we avoid overlapping in constructing monthly horizon returns and thus, get $160 = 3200/20$ monthly observations. Summary statistics on equally weighted (negative) portfolio returns are provided in Table 5. Whereas the (negative) mean portfolio returns under both numeraires and both horizons are not statistically different from zero, their tails behave differently. At daily horizon, the portfolio exhibits a rather symmetric pattern when it is written in US Dollar and is more skewed to the left when it is written in Singapore Dollar. Although less significant, the reverse is observed at horizon 20 days, corresponding to a trading month. Finally, both daily returns display fatter tails than Gaussian with much fatter tails for the portfolio written in Singapore Dollar. This fat tail phenomenon is substantially reduced for monthly returns.

For both numeraires, the VaR and Tail-VaR of the portfolios are estimated nonparametrically for daily and monthly horizons (see Figures 10 and 11). In each figure, panel (a) (resp. panel (b)) corresponds to the portfolio written in US Dollar (resp. Singapore Dollar). The solid line represents the estimated risk measure and the dashed lines are the lower and upper bounds of its confidence band. The symmetry of the distribution is evidenced for the estimated VaR in all cases. Moreover, the estimation errors are larger at the (upper) extreme tail. This feature is consistent with the variance pattern implied by exponential, Pareto and Gaussian distributions. Although data under both horizons are symmetric about zero, the monthly (negative) portfolio returns are more likely to reach higher values. Indeed, with longer holding period, the price tends to be more volatile and increase the possibility for extreme changes. Due to the smaller number of observations, estimations performed on monthly data have wider confidence intervals. A similar pattern is illustrated in the Tail-VaR estimation. The confidence bands are large when p is small and shrink gradually as the Tail-VaR converges to the mean, that is when p tends to 1. The Tail-VaR at monthly horizon are at least about twice those under daily horizon. Moreover, the estimations under daily (negative) returns are more accurate than their monthly counterparts. For instance, the estimation of the Tail-VaR for the portfolio written in Singapore Dollar underestimate the loss by about 40 basis points when the risk level is extreme and the holding period is 20 days.

The amplifying factor $TVaR(p)/VaR(p)$ is plotted for both numeraires in Figure 12, with thicker line representing daily data. The amplifying factor can be used as an alternative tool to kurtosis for identifying the distributional behavior of the tail. For the daily (negative) returns, the amplifying factor of the portfolio written in US Dollar starts with a roughly concave shape and increases linearly in p afterward, which is compatible with the pattern implied by exponential distribution. The behavior of the portfolio written in Singapore Dollar exhibits patterns closer to the standard normal distribution, that is, concave at the beginning and slightly convex afterwards (see Figures 1(b) and 1(c) for a visual comparison). Since the sample size is too small to accurately estimate the distribution characteristics, it is more difficult to identify the distribution patterns when the holding period is 20 days. However, the amplifying factor calculated for monthly data are smaller than for daily data. This is consistent with the thinner tail featured by the long horizon data.

Figure 13 displays the shape of function g used to pass from the Tail-VaR to the VaR for the equally weighted currency portfolio. Similarly, the thicker solid line represents the result at daily horizon. For a one-day period, the function g is close to a linear function under both numeraires. However, the function g features steps when the horizon increases to 20 days. This is due to the smaller number of observations.

To analyze the sensitivity of the result above to portfolio allocation, we now consider different portfolio allocations. More precisely, the (negative) portfolio return is constructed as:

$$x_{j,t} = -(a^1 \, \widetilde{x}^1_{j,t} + (1 - a^1) \, \widetilde{x}^2_{j,t}), \quad \text{for } j = u, s.$$

The weight a^1 is chosen to get portfolios with only Japanese Yen ($a^1 = 0$), more Japanese Yen ($a^1 = 0.2$), more Hongkong Dollar ($a^1 = 0.8$) and only Hongkong Dollar ($a^1 = 1$). The patterns of function g are provided in Figure 14. Figures of the first column correspond to the results written in US Dollar and those of the second column show the outputs written in Singapore-Dollar. Figures in rows 1,2,3,4 display the patterns associated with portfolios when $a^1 = 0, 0.2, 0.8$ and 1, respectively. The near-linearity feature is preserved for almost all portfolios at daily horizon. The only exception occurs in the portfolio including Hongkong Dollar only. When using US Dollar as basic numeraire, we identify a few jumps in function g. For comparison, their twenty-day counterparts are also plotted. Because of the small sample size, it is not surprising that we observe step functions in all cases.

Finally, the slope parameter α is estimated for the equally weighted (negative) portfolio return with $p_0 = 0.005$ and $p_1 = 0.2$. We consider various horizons, that are $k = 1, 5, 10, 15$ and 20 days. The estimated values are provided in Figure 15. Returns with longer horizons tend to have higher α, even though the pattern may not be monotonic. This can be due to the thinner tail featured by longer horizon data.

6 Concluding remarks

This paper provides a unified framework for analyzing distortion risk measures, including as special risk measures the VaR and Tail-VaR. Indexing the distortion risk measure as a function of the distortion parameter p, we study the sensitivity of the risk measure with respect to a change of p. Since p can be interpreted as parameter representing risk aversion and/or pessimism, the sensitivity measures the marginal effects on risk measures of slight adjustment of risk (or pessimism) level. Moreover, for special examples such as Tail-VaR, the sensitivity also serves as partial measure of the accuracy for its nonparametric estimator. Applying a Functional Limit Theorem, we derive the asymptotic properties of the nonparametric estimators of distortion risk measures and their sensitivities with respect to the pessimism parameter. Under standard regularity conditions, both distortion risk measures and their sensitivities are asymptotically Gaussian. Closed-form expressions for the asymptotic variances are derived for specific examples such as VaR, Tail-VaR and Proportional Hazard distortion risk measure.

Robust risk management requires control of various risk measures. Thus, the knowledge of relationship between different risk measures is important for selecting appropriate risk control strategies. In this paper, we emphasize the link between the VaR and Tail-VaR. On the one hand, for a given risk level p, the Tail-VaR can be derived by multiplying the VaR with an amplifying factor. We show that this amplifying factor is independent of p ,if and only, if the underlying distribution is Pareto. Defining the amplifying factor as function of p, we observe that different distributions usually imply different patterns of the amplifying factor. Thus, the shape of the amplifying factor can be a criterion for identifying the proper underlying distribution. On the other hand, the VaR and Tail-VaR are related through their risk levels by some transformation g. We introduce a nonparametric estimator of this transformation, derive its asymptotic properties and propose a specification test for the hypothesis of linear transformation g.

The results are illustrated by considering currency portfolios written in different numeraires.

The linearity of the transformation of risk levels is observed for a large range of portfolio allocations.

Whereas the analysis considered in this paper is based on the assumption of i.i.d. observations, the extensions to the dynamic setting can be considered. First, we can still consider the historical DRMs suggested by the regulators, but derive their asymptotic behaviors when the portfolio returns are serially dependent. Second, we can introduce dynamic version of the DRM based on dynamic quantile functions. These extensions will likely be based on parametric specifications such as the dynamic additive quantile model (DAQ) proposed by Gourieroux and Jasiak (2005a).

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Appendix A Analogy principle and empirical process

Let us briefly review the relevant results of analogy principle and empirical process. For a systematic analysis, including the appropriate regularity conditions, we refer to Pollard (1984), Shorack and Wellner (1986), Manski (1988), and van der Vaart and Wellner (1996).

A.1 Analogy principle

The analogy principle has been popularized in econometrics by Manski (1988). Let us consider i.i.d. observations $x_1, ..., x_T$ with common cumulative distribution function F, and a parameter of interest θ. The analogy principle looks for an interpretation of parameter θ, that is, a relationship explaining how θ is related to distribution F. This relationship takes the form: $h(F; \theta) = 0$, where h is a known function. Then, this relation is used to get an estimator $\hat{\theta}_T$ of θ by replacing F by the sample cdf $\widetilde{F}_T$. Thus, $\hat{\theta}_T$ is defined as a solution of $h(\widetilde{F}_T; \hat{\theta}_T) = 0$. If h is invertible with respect to θ and "continuous" with respect to (F, θ), this approach provides a consistent estimator of θ; if h is first-order "differentiable" with respect to (F, θ), the estimator $\hat{\theta}_T$ is asymptotically Gaussian. However, the estimator $\hat{\theta}_T$ is not necessarily asymptotically efficient. Indeed, a given parameter can admit a lot of alternative interpretations. In a second step, it is important to look for an "optimal" interpretation of the parameter, that is, an interpretation leading to asymptotic (semi-parametric or parametric) efficiency. This is the so-called empirical likelihood approach introduced in econometrics by Kitamura (1997), Kitamura and Stutzer (1997).

A.2 Empirical process

Let $\boldsymbol{x}_1, ..., \boldsymbol{x}_T$ be a random sample of i.i.d. one-dimensional observations. Their common cumulative distribution function (cdf) is denoted by $F(\boldsymbol{x}) = \mathcal{P}[\boldsymbol{x_t} \leq \boldsymbol{x}]$. The observations $\boldsymbol{x}_1, ..., \boldsymbol{x}_T$ can be used to define the empirical process $\widetilde{F}_T$ by:

$$\widetilde{F}_T(\boldsymbol{x}) = \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}_{(\boldsymbol{x}_t \leq \boldsymbol{x})}, \tag{A-1}$$

where $\mathbb{1}_{(A)}$ is the indicator function of event A.

The use of the analogy principle is based on the limiting behavior of the empirical process $\widetilde{F}_T$. Loosely speaking, under standard regularity conditions, the empirical process is consistent and asymptotically Gaussian. The convergence in distribution of the empirical process to a Gaussian process is with respect to the notion of weak convergence on $\mathcal{D}[0,1]$, the Skorohod space of right-continuous functions on $[0,1]$ with left limits (see e.g. van der Vaart and Wellner (1996)). This type of convergence is denoted by $\Rightarrow$.

Functional Limit Theorem. *Let $x_1, ..., x_T$ be i.i.d. one-dimensional random observations, we have:*

$$\sqrt{T}\big[\widetilde{F}_T(x) - F(x)\big] \Rightarrow B\big(F(x)\big),$$

where $B(u) = W(u) - uW(1)$ is a Brownian bridge, and $W(u)$ a Brownian motion on $[0,1]$.

Defined as a linear combination of values of standard Brownian motion, the process B is also Gaussian with zero-mean. Its covariance operator is:

$$COV\big(B(u_1), B(u_2)\big) = u_1 \wedge u_2 - u_1 u_2, \quad \text{for } u_1, u_2 \in [0,1],$$

where $u_1 \wedge u_2$ denotes the minimum of u_1 and u_2. These properties of the Brownian bridge are useful in deriving the limiting distribution of sample moments. More precisely, the asymptotic normality of the empirical cdf implies the asymptotic normality of any sample moments (under integrability conditions). By analogy principle, any theoretical moment of $\int_{\mathbb{R}} g(x) dF(x)$ of a p-dimensional integrable function g can be estimated by the following stochastic integral, which equals the associated sample moment:

$$\int_{\mathbb{R}} g(x) d\widetilde{F}_T(x) = \frac{1}{T} \sum_{t=1}^{T} g(x_t). \tag{A-2}$$

By applying the Functional Limit Theorem, the sample moments are such that:

$$\sqrt{T}\left[\int_{\mathbb{R}} g(x)d\widetilde{F}_T(x) - \int_{\mathbb{R}} g(x)dF(x)\right] \Rightarrow \int_{\mathbb{R}} g(x)d\,B\big(F(x)\big). \tag{A-3}$$

The stochastic integral $\int_{\mathbb{R}} g(x)d\,B\big(F(x)\big)$ is Gaussian, zero-mean, with variance-covariance matrix $V\left[\int_{\mathbb{R}} g(x)d\,B\big(F(x)\big)\right] = Vg(x)$, which is the standard Central Limit Theorem.

A.3 Relaxation of i.i.d. assumption

A regulator is often interested in risk measures calculated from the marginal empirical distribution. However, it is shown in the literature that financial (negative) returns are often serially dependent at least for the second-order moment. The empirical process under dependent time series still converges in distribution to a Gaussian process, whenever the time series is stationary and satisfies appropriate ergodicity condition [see e.g. Arcones and Yu (1994)]. The stationary version of the functional limit theorem is given below.

Functional Limit Theorem for stationary process. *For a stationary sequence $x_1, \dots, x_T$ with marginal cdf F, we have:*

$$\sqrt{T}\,[\widetilde{F}_T(x) - F(x)] \Rightarrow Z\big(F(x)\big),$$

where $Z\big(F(x)\big)$ is a zero-mean Gaussian process with variance:

$$V\big[Z\big(F(x)\big)\big] = F(x)\big(1-F(x)\big) + 2\sum_{j=2}^{k} E\left[\big(\mathbb{1}_{(X_1\leq x)} - F(x)\big)\big(\mathbb{1}_{(X_j\leq x)} - F(x)\big)\right],$$

and covariance:

$$COV\big[Z\big(F(x)\big), Z\big(F(x')\big)\big] = F(x)\wedge F(x') - F(x)F(x') + 2\sum_{j=2}^{k} E\left[\big(\mathbb{1}_{(X_1\leq x)} - F(x)\big)\big(\mathbb{1}_{(X_j\leq x')} - F(x')\big)\right],$$

where $k << T$ denotes the largest lag where $Cov(\mathbb{1}_{X_t\leq x}, \mathbb{1}_{X_{t-k}\leq x}) \neq 0$.

This theorem can be used to extend the result of the paper to serially dependent data.

Appendix B Preliminary lemmas

Lemma 1. *Let us consider a random variable X with continuous cdf F and quantile function $Q = F^{-1}$, we have:*

(i).
$$\int_{Q(1-p)}^{\infty} \frac{1}{p}\left[1-F(x)\right] d\,x = TVaR(p) - VaR(p);$$

(ii).
$$2\int_{Q(1-p)}^{\infty} \frac{1}{p^2} x\left[1-F(x)\right] d\,x \;=\; \frac{1}{p}\Big[E\big[X^2|X \geq VaR(p)\big] - \big(VaR(p)\big)^2\Big];$$

(iii).
$$2\,\frac{1}{p^2}\int_{Q(1-p)}^{\infty} Q(1-p)\big[1-F(x)\big]\,d\,x = \frac{2\,VaR(p)}{p}\Big[TVaR(p) - VaR(p)\Big].$$

Proof. (i). By integrating by part, we get:

$$\begin{aligned}
\int_{Q(1-p)}^{\infty} \frac{1}{p}\left[1-F(x)\right] d\,x &= \frac{1}{p}\,x\left[1-F(x)\right]\Big]_{Q(1-p)}^{\infty} + \int_{Q(1-p)}^{\infty} x\,d\,\frac{F(x)}{p} \\
&= E\big[X|X \geq Q(1-p)\big] - Q(1-p) \\
&= TVaR(p) - VaR(p),
\end{aligned}$$

by definition of VaR(p) and TVaR(p).

(ii). We have:

$$\begin{aligned}
2\int_{Q(1-p)}^{\infty} \frac{1}{p^2} x\left[1-F(x)\right] d\,x &= \frac{1}{p^2}\int_{Q(1-p)}^{\infty} \big[1-F(x)\big]\,d\,x^2 \\
&= \frac{1}{p}\left\{ x^2\,\frac{\big[1-F(x)\big]}{p}\Bigg]_{Q(1-p)}^{\infty} - \int_{Q(1-p)}^{\infty} x^2 d\,\frac{\big[1-F(x)\big]}{p} \right\} \\
&= \frac{1}{p}\Big[E\big[X^2|X \geq VaR(p)\big] - \big(VaR(p)\big)^2\Big].
\end{aligned}$$

(iii). By integrating by part, we have:

$$\begin{aligned}
&2\,\frac{1}{p^2}\int_{Q(1-p)}^{\infty} Q(1-p)\big[1-F(x)\big]\,d\,x \\
&= 2\,\frac{1}{p^2} Q(1-p) x\left[1-F(x)\right]\Big]_{Q(1-p)}^{\infty} - 2\,\frac{Q(1-p)}{p^2}\int_{Q(1-p)}^{\infty} x\,d\,\left[1-F(x)\right] \\
&= \frac{2\,VaR(p)}{p}\Big[TVaR(p) - VaR(p)\Big].
\end{aligned}$$

$\square$

Appendix C Variances of $\sqrt{T}[\widetilde{TVaR_T}(p) - TVaR(p)]$

We have:

$$
\begin{aligned}
&V\big(\sqrt{T}\,[\widetilde{TVaR_T}(p) - TVaR(p)]\big)\\
&= \int_{Q(1-p)}^{\infty}\int_{Q(1-p)}^{\infty} \frac{F(x_1)\wedge F(x_2) - F(x_1)F(x_2)}{f(x_1)f(x_2)}\,\frac{1}{p^2}\,d\,F(x_1)\,d\,F(x_2)\\
&= \int_{Q(1-p)}^{\infty}\int_{Q(1-p)}^{\infty} \big[F(x_1)\wedge F(x_2) - F(x_1)F(x_2)\big]\frac{1}{p^2}d\,x_1\,d\,x_2\\
&= 2\int_{Q(1-p)}^{\infty}\int_{Q(1-p)}^{x_2} F(x_1)\big(1-F(x_2)\big)\frac{1}{p^2}d\,x_1\,d\,x_2.\\
&= 2\int_{Q(1-p)}^{\infty}\int_{Q(1-p)}^{x_2} \big[1-F(x_2)\big]\frac{1}{p^2}d\,x_1\,d\,x_2 - 2\int_{Q(1-p)}^{\infty}\int_{Q(1-p)}^{x_2}\big[1-F(x_1)\big]\big[1-F(x_2)\big]\frac{1}{p^2}d\,x_1\,d\,x_2\\
&= 2\,\frac{1}{p^2}\int_{Q(1-p)}^{\infty}\big[x - Q(1-p)\big]\,\big[1-F(x)\big]d\,x - \left[\frac{1}{p}\int_{Q(1-p)}^{\infty}\big[1-F(x)\big]d\,x\right]^2. \qquad \text{(C-1)}
\end{aligned}
$$

Thus, using the results of Lemma 1, we deduce:

$$
\begin{aligned}
&V\big(\sqrt{T}\,[\widetilde{TVaR_T}(p) - TVaR(p)]\big)\\
&= \frac{1}{p}\Big[E\big[X^2|X\geq VaR(p)\big] - \big(VaR(p)\big)^2\Big] - \frac{2\,VaR(p)}{p}\Big[TVaR(p) - VaR(p)\Big] - \big[TVaR(p) - VaR(p)\big]^2\\
&= \frac{V\big(X|X\geq VaR(p)\big) + (1-p)\,\big[TVaR(p) - VaR(p)\big]^2}{p}.
\end{aligned}
$$

It is straightforward to calculate the covariance between two Tail-VaR at different levels. For $\widetilde{p} > p$, we get:

$$
\begin{aligned}
&COV\big(\sqrt{T}\,\widetilde{TVaR_T}(p), \sqrt{T}\,\widetilde{TVaR_T}(\widetilde{p})\big)\\
&= \int_{Q(1-p)}^{\infty}\int_{Q(1-\widetilde{p})}^{\infty}\big[F(x_1)\wedge F(x_2) - F(x_1)F(x_2)\big]\frac{1}{p\widetilde{p}}d\,x_1\,d\,x_2\\
&= 2\int_{Q(1-p)}^{\infty}\int_{Q(1-\widetilde{p})}^{x_2} F(x_1)\big(1-F(x_2)\big)\frac{1}{p\widetilde{p}}d\,x_1\,d\,x_2.\\
&= 2\int_{Q(1-p)}^{\infty}\int_{Q(1-\widetilde{p})}^{x_2}\big[1-F(x_2)\big]\frac{1}{p\widetilde{p}}d\,x_1\,d\,x_2 - 2\int_{Q(1-p)}^{\infty}\int_{Q(1-\widetilde{p})}^{x_2}\big[1-F(x_1)\big]\big[1-F(x_2)\big]\frac{1}{p\widetilde{p}}d\,x_1\,d\,x_2\\
&= 2\,\frac{1}{p\widetilde{p}}\int_{Q(1-p)}^{\infty}\big[x - Q(1-\widetilde{p})\big]\,\big[1-F(x)\big]d\,x - \left[\frac{1}{p}\int_{Q(1-p)}^{\infty}\big[1-F(x)\big]d\,x\right]\left[\frac{1}{\widetilde{p}}\int_{Q(1-\widetilde{p})}^{\infty}\big[1-F(x)\big]d\,x\right].
\end{aligned}
$$

By applying Lemma 1, we get:

$$
\begin{aligned}
&COV\big(\sqrt{T}\,\widetilde{TVaR}_T(p), \sqrt{T}\,\widetilde{TVaR}_T(\widetilde{p})\big)\\
&\quad = \frac{1}{\widetilde{p}}\left[E\left[X^2\middle|X > VaR(p)\right] - \big(VaR(p)\big)^2\right] - \frac{2\,VaR(\widetilde{p})}{\widetilde{p}}\big[TVaR(p) - VaR(p)\big]\\
&\quad - \Big[TVaR(\widetilde{p}) - VaR(\widetilde{p})\Big]\Big[TVaR(p) - VaR(p)\Big]\\
&\quad = \frac{1}{\widetilde{p}}\left[E\left[X^2\middle|X > VaR(p)\right] - \big(VaR(p)\big)^2\right] - \frac{(2-\widetilde{p})\,VaR(\widetilde{p})}{\widetilde{p}}\big[TVaR(p) - VaR(p)\big]\\
&\quad - TVaR(\widetilde{p})\Big[TVaR(p) - VaR(p)\Big]. \qquad \text{(C-2)}
\end{aligned}
$$

Moreover, the covariance between $\widetilde{VaR}_T\big(g(p)\big)$ and $\widetilde{TVaR}_T(\widetilde{p})$ is given by:

$$
\begin{aligned}
&COV\Big(\sqrt{T}\,\widetilde{VaR}_T\big(g(p)\big), \sqrt{T}\,\widetilde{TVaR}_T(\widetilde{p})\Big)\\
&= \frac{1}{\widetilde{p}}\int_{Q(1-\widetilde{p})}^{\infty} \frac{\big(1-g(p)\big)\wedge F(x) - \big(1-g(p)\big)F(x)}{f\big(Q(1-g(p)\big)} d\,x\\
&= \frac{1}{\widetilde{p}f\big(Q(1-g(p)\big)} \int_{Q(1-\widetilde{p})}^{Q\left(1-g(p)\right)} g(p)F(x)d\,x + \frac{1}{\widetilde{p}f\big(Q(1-g(p)\big)} \int_{Q\left(1-g(p)\right)}^{\infty} \big(1-g(p)\big)\big(1-F(x)\big)d\,x\\
&= \frac{g(p)}{\widetilde{p}f\big(Q(1-g(p)\big)}\Big\{\big(1-g(p)\big)VaR\big(g(p)\big) - (1-\widetilde{p})VaR(\widetilde{p}) - E\left[X\middle|VaR(\widetilde{p}) < X < VaR\big(g(p)\big)\right]\Big\}\\
&\quad + \frac{g(p)\big(1-g(p)\big)}{\widetilde{p}f\big(Q(1-g(p)\big)}\Big[TVaR\big(g(p)\big) - VaR\big(g(p)\big)\Big]\\
&= \frac{g(p)}{\widetilde{p}f\big(Q(1-g(p)\big)}\Big\{\big(1-g(p)\big)TVaR\big(g(p)\big) - (1-\widetilde{p})VaR(\widetilde{p}) - E\left[X\middle|VaR(\widetilde{p}) < X < VaR\big(g(p)\big)\right]\Big\}\\
&\qquad \text{(C-3)}
\end{aligned}
$$

Appendix D Variance of $\sqrt{T}[\widetilde{PH}_T(p) - PH(p)]$

For Proportional Hazard distortion, we define the expectation and probability operators as:

$$
E_p^*[g(X)] = -\int_{\mathbb{R}} g(x)\,d\,\big(1-F(x)\big)^p.
$$

The asymptotic variance of $\widetilde{PH}_T(p)$ is given by:

$$\begin{aligned}
&V\big(\sqrt{T}\,[\widetilde{PH}_T(p) - PH(p)]\big)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big[F(x_1)\wedge F(x_2) - F(x_1)F(x_2)\big]\,p^2\big(1-F(x_1)\big)^{p-1}\big(1-F(x_2)\big)^{p-1}d\,x_1\,d\,x_2\\
&= 2\,p^2\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}F(x_1)\big(1-F(x_1)\big)^{p-1}\big(1-F(x_2)\big)^{p}d\,x_1\,d\,x_2.
\end{aligned}$$

We can express variance in terms of the distortion operators with varying p, to make the estimation straightforward. Let us integrate by parts with respect to x_1, we get:

$$\begin{aligned}
&V\big(\sqrt{T}\,[\widetilde{PH}_T(p) - PH(p)]\big)\\
&= 2\,p^2\int_{-\infty}^{\infty} x_1F(x_1)\big(1-F(x_1)\big)^{p-1}\Big]_{-\infty}^{x_2}\big(1-F(x_2)\big)^{p}d\,x_2\\
&\quad -2\,p^2\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}x_1 d\,\Big[F(x_1)\big(1-F(x_1)\big)^{p-1}\big(1-F(x_2)\big)^{p}\Big]\,d\,x_2\\
&= 2\,p^2\int_{-\infty}^{\infty}xF(x)\big(1-F(x)\big)^{2p-1}d\,x - 2\,p^2\int_{-\infty}^{\infty}x_1\int_{x_1}^{\infty}\big(1-F(x_2)\big)^{p}d\,x_2\,d\,F(x_1)\big(1-F(x_1)\big)^{p-1}\\
&= p^2\int_{-\infty}^{\infty}F(x)\big(1-F(x)\big)^{2p-1}d\,x^2 - 2\,p^2\int_{-\infty}^{\infty}x_1\int_{x_1}^{\infty}\big(1-F(x_2)\big)^{p}d\,x_2\,d\,F(x_1)\big(1-F(x_1)\big)^{p-1}\\
&= -p^2\int_{-\infty}^{\infty}x^2 d\,F(x)\big(1-F(x)\big)^{2p-1} - 2\,p^2\int_{-\infty}^{\infty}x_1\int_{x_1}^{\infty}\big(1-F(x_2)\big)^{p}d\,x_2\,d\,F(x_1)\big(1-F(x_1)\big)^{p-1}\\
&= p^2\left[\int_{-\infty}^{\infty}x^2 d\,\big(1-F(x)\big)^{2p} - \int_{-\infty}^{\infty}x^2 d\,\big(1-F(x)\big)^{2p-1}\right]\\
&\quad -2\,p^2\int_{-\infty}^{\infty}x_1\int_{x_1}^{\infty}\big(1-F(x_2)\big)^{p}d\,x_2\,d\,F(x_1)\big(1-F(x_1)\big)^{p-1},
\end{aligned}$$

which is equal to:

$$\begin{cases}
p^2\left[E^*_{2p-1}(X^2) - E^*_{2p}(X^2)\right] - 2\,p^2\int_{-\infty}^{\infty}x_1\int_{x_1}^{\infty}\big(1-F(x_2)\big)^{p}d\,x_2\,d\,F(x_1)\big(1-F(x_1)\big)^{p-1} & p \geq 0.5,\\
p^2\int_{-\infty}^{\infty}F(x)\big(1-F(x)\big)^{2p-1}d\,x^2 - 2\,p^2\int_{-\infty}^{\infty}x_1\int_{x_1}^{\infty}\big(1-F(x_2)\big)^{p}d\,x_2\,d\,F(x_1)\big(1-F(x_1)\big)^{p-1} & p < 0.5.
\end{cases}$$

By integrating by part the second term with respect to x_2, we have:

$$\begin{aligned}
&2\,p^2\int_{-\infty}^{\infty}x_1\int_{x_1}^{\infty}\big(1-F(x_2)\big)^{p}d\,x_2\,d\,F(x_1)\big(1-F(x_1)\big)^{p-1}\\
&= -2p^2\int_{-\infty}^{\infty}x^2\big(1-F(x)\big)^{p}d\,F(x)\big(1-F(x)\big)^{p-1} - 2p^2\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1x_2 d\big(1-F(x_2)\big)^{p}d\,F(x_1)\big(1-F(x_1)\big)^{p-1}\\
&= -2p^2\left[\frac{1}{2}\,E^*_{2p}(X^2) - \frac{p-1}{2p-1}E^*_{2p-1}(X^2)\right] - p^2\left[2\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1x_2 d\,\big(1-F(x_2)\big)^{p}d\,\big(1-F(x_1)\big)^{p-1}\right.\\
&\quad\left. -2\int_{-\infty}^{\infty}\int_{x_1}^{\infty}x_1x_2 d\,\big(1-F(x_2)\big)^{p}d\,\big(1-F(x_1)\big)^{p}\right]
\end{aligned}$$

$$= 2p^2\left[\frac{p-1}{2p-1}E^*_{2p-1}(X^2) - \frac{1}{2}\,E^*_{2p}(X^2)\right] - p^2E^*_{p-1}(X)E^*_p(X) + p^2\left[E^*_p(X)\right]^2.$$

Putting together, this complete the proof.

Appendix E Asymptotic expansion of the implied pessimism parameter

Since:

$$\Pi^o = \Pi(p;Q) = \Pi(\widehat{p}_T;\widetilde{Q}_T),$$

we get:

$$\begin{aligned}
0 &= \Pi(p;Q) - \Pi(\widehat{p}_T;\widetilde{Q}_T) \\
&= \Pi(p;Q) - \Pi(p;\widetilde{Q}_T) + \Pi(p;\widetilde{Q}_T) - \Pi(\widehat{p}_T;\widetilde{Q}_T) \\
&= \Pi(p;Q) - \Pi(p;\widetilde{Q}_T) + \Pi(p;Q) - \Pi(\widehat{p}_T;Q) + o_p(1) \\
&= \sqrt{T}[\Pi(p;Q) - \Pi(p;\widetilde{Q}_T)] + \frac{\partial \Pi}{\partial p}(p;Q)\sqrt{T}(p-\widehat{p}_T) + o_p(1),
\end{aligned}$$

which can be rewritten as:

$$\sqrt{T}(\widehat{p}_T - p) = -\left(\frac{\partial \Pi}{\partial p}(p;Q)\right)^{-1}\sqrt{T}\big[\Pi(p;\widetilde{Q}_T) - \Pi(p;Q)\big] + o_p(1).$$

Appendix F Proof of Proposition 4

The limiting process comes from the properties of the empirical process and the asymptotic Gaussian distribution follows immediately. Let us now derive the asymptotic variance.

$$\begin{aligned}
V\Big(\sqrt{T}\left[\widetilde{g}_T(p) - g(p)\right]\Big) &= \big(1-g(p)\big)g(p) + f\big[TVaR(p)\big]^2 V\Big(\sqrt{T}\,[\widetilde{TVaR}_T(p) - TVaR(p)]\Big) \\
&\quad - 2\,f\big[TVaR(p)\big]\frac{1}{p}\int_{1-p}^{1}\frac{\big(1-g(p)\big)\wedge u - \big(1-g(p)\big)\,u}{f\big[Q(u)\big]}\,d\,u.
\end{aligned}$$

Since $g(p) \leq p$, the covariance term can be rewritten as:

$$\begin{aligned}
&2\,f\big[TVaR(p)\big]\frac{1}{p}\int_{1-p}^{1}\frac{\big(1-g(p)\big)\wedge u - \big(1-g(p)\big)\,u}{f\big[Q(u)\big]}\,d\,u \\
&= 2\,f\big[TVaR(p)\big]\frac{1}{p}\left\{\int_{Q(1-p)}^{Q\left(1-g(p)\right)} F(x)\,g(p)d\,x + \int_{Q\left(1-g(p)\right)}^{\infty}\big(1-g(p)\big)\big(1-F(x)\big)d\,x\right\}
\end{aligned}$$

$$
\begin{aligned}
&= 2\,f\big[TVaR(p)\big]\frac{1}{p}\Bigg\{g(p)\Bigg[x\,F(x)\Big]_{Q(1-p)}^{Q\left(1-g(p)\right)} - \int_{Q(1-p)}^{Q\left(1-g(p)\right)} x\,d\,F(x)\Bigg] \\
&\qquad + g(p)\big[1-g(p)\big]\,\big[TVaR\big(g(p)\big) - VaR\big(g(p)\big)\big]\Bigg\} \\
&= 2\,f\big[TVaR(p)\big]\frac{1}{p}\Bigg\{g(p)\Bigg[VaR\big(g(p)\big)\big(1-g(p)\big) - VaR(p)(1-p) \\
&\qquad - \big[p-g(p)\big]E\left[X\middle|VaR(p)\le X\le VaR\big(g(p)\big)\right]\Bigg] + g(p)\big[1-g(p)\big]\,\big[TVaR\big(g(p)\big) - VaR\big(g(p)\big)\big]\Bigg\} \\
&= 2\,f\big[TVaR(p)\big]\frac{1}{p}\Bigg\{-g(p)VaR(p)(1-p) + g(p)\big[1-g(p)\big]\,TVaR\big(g(p)\big) \\
&\qquad - \big[p-g(p)\big]E\left[X\middle|VaR(p)\le X\le VaR\big(g(p)\big)\right]\Bigg\} \\
&= 2\,f\big[TVaR(p)\big]\frac{1}{p}\Bigg\{g(p)\Big[\big[1-g(p)\big]\,TVaR\big(g(p)\big) - VaR(p)(1-p)\Big] \\
&\qquad - \big[p-g(p)\big]E\left[X\middle|VaR(p)\le X\le VaR\big(g(p)\big)\right]\Bigg\}.
\end{aligned}
$$

The result follows.

Table 1: Relationship between VaR and Tail-VaR

Distribution	$U(a,b)$	$\gamma(1,\lambda)$	$Pareto(a,b)$	$N(0,1)$
$\frac{TVaR(p)}{VaR(p)}$	$\frac{b(2-p)+ap}{2[b(1-p)+ap]}$	$1-\frac{1}{\log(p)}$	$\frac{a}{a-1}$	$\frac{\phi\left(\Phi^{-1}(1-p)\right)}{p\Phi^{-1}(1-p)}$
$g(p)$	$p/2$	p/e	$\left(\frac{a-1}{a}\right)^a p$	$1-\Phi\left[\frac{1}{p}\phi\left[\Phi^{-1}(1-p)\right]\right]$

The first row gives the distribution specification. The second row provides the ratio between $TVaR(p)$ and $VaR(p)$. In the third row, we provide the value of $p^* = g(p)$ such that $TVaR(p) = VaR(p^*)$. Φ and ϕ denote the cdf and pdf of standard normal distribution, respectively. The parameter a of the Pareto distribution is strictly larger than 1 to ensure that the $TVaR$ exists.

Table 2: Sensitivity of the VaR and Tail-VaR

Distribution	$U[a,b]$	$\gamma(1,\lambda)$	$Pareto(a,b)$	$N(0,1)$
$\frac{\partial VaR(p)}{\partial p}$	$a-b$	$\frac{-1}{\lambda p}$	$\frac{-b}{a}p^{-(a+1)/a}$	$\frac{-1}{\phi\left[\Phi^{-1}(1-p)\right]}$
$\frac{\partial TVaR(p)}{\partial p}$	$(a-b)/2$	$\frac{-1}{\lambda p}$	$\frac{-b}{a-1}p^{-(a+1)/a}$	$-\frac{1}{p^2}\phi\left(\Phi^{-1}(1-p)\right)+\frac{1}{p}\Phi^{-1}(1-p)$

The first row gives the distribution specification. The second and third rows provide the sensitivity of Value-at-Risk and Tail-VaR, respectively. Φ and ϕ denote the cdf and pdf of standard normal distribution, respectively.

Table 3: Asymptotic variance and relative accuracy of the estimated $VaR(p)$

Distribution	$VaR(p)\big(Q_0(1-p)\big)$	Variance	$\sigma_{VaR(p)}/VaR(p)$
Uniform $U[a,b]$	$b(1-p)+ap$	$(b-a)^2 p(1-p)$	$\frac{(b-a)\sqrt{p(1-p)}}{b(1-p)+ap}$
Exponential $\gamma(1,\lambda)$	$\frac{-\log(p)}{\lambda}$	$\frac{1}{\lambda^2}\frac{1-p}{p}$	$-\sqrt{\frac{1-p}{p}}/\log(p)$
Pareto $Pareto(a,b)$	$b\,p^{-1/a}$	$\left(\frac{b}{a}\right)^2 \frac{1-p}{(p)^{\frac{2+a}{a}}}$	$\frac{1}{a}p^{1/a}\sqrt{\frac{1-p}{(p)^{\frac{2+a}{a}}}}$
Gaussian $N(\mu,\sigma^2)$	$\mu+\sigma\Phi^{-1}(1-p)$	$\frac{\sigma^2}{\left[\phi\left(\Phi^{-1}(1-p)\right)\right]^2}p(1-p)$	$\frac{\sigma\sqrt{p(1-p)}}{\left[\mu+\sigma\Phi^{-1}(1-p)\right]\phi\left[\Phi^{-1}(1-p)\right]}$
Lévy $Lévy(c)$	$\frac{c}{\left[\Phi^{-1}\left(\frac{1+p}{2}\right)\right]^2}$	$\frac{2\pi c^2 p(1-p)}{\left[\Phi^{-1}\left(\frac{1+p}{2}\right)\right]^6}\exp\left(\left[\Phi^{-1}\left(\frac{1+p}{2}\right)\right]^2\right)$	$\frac{\sqrt{p(1-p)}}{\phi\left(\frac{1+p}{2}\right)\Phi^{-1}\left(\frac{1+p}{2}\right)}$
Cauchy $Cauchy(m,b)$	$m+b\tan\left[\pi\left(\frac{1}{2}-p\right)\right]$	$(b\pi)^2\left\{\sec\left[\pi\left(\frac{1}{2}-p\right)\right]\right\}^4 p(1-p)$	$\frac{b\pi\left\{\sec\left[\pi\left(\frac{1}{2}-p\right)\right]\right\}^2\sqrt{p(1-p)}}{m+b\tan\left[\pi\left(\frac{1}{2}-p\right)\right]}$

The first column gives the distribution specifications with their parameters. The second column lists the quantile functions and the third column provides the asymptotic variances for the nonparametric estimators of VaR. Relative accuracy measured by the ratio between the standard deviation and the $VaR(p)$ are listed in column 4. ϕ (resp. Φ) is the pdf (resp. cdf) of standard normal distribution. sec is the secant function $\sec[x]=1/\cos[x]$. The variance is defined for the estimator scaled by $\sqrt{T}$.

Table 4: Asymptotic variance and relative accuracy of the estimated $TVaR(p)$

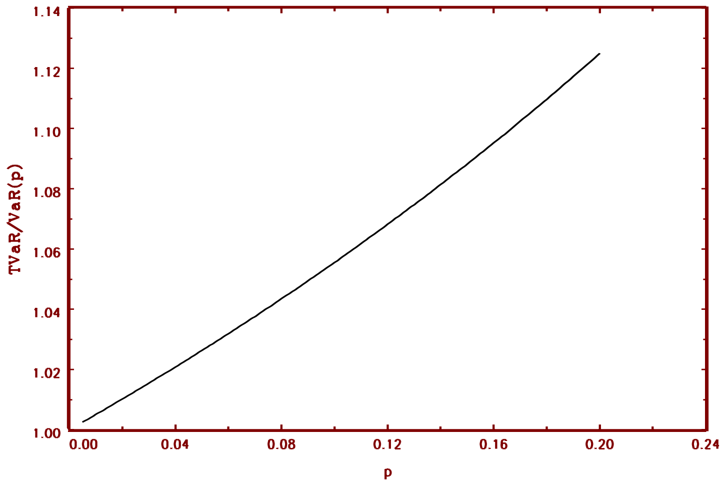
Distribution	$TVaR(p)$	Variance	$\sigma(\widetilde{TVaR_T}(p))/TVaR(p)$
Uniform $U[a,b]$	$\frac{b(2-p)+ap}{2}$	$\frac{(a-b)^2 p(4-3p)}{12}$	$\frac{(b-a)\sqrt{\frac{p(4-3p)}{3}}}{2b+(a-b)p}$
Exponential $\gamma(1,\lambda)$	$\frac{1-\log(p)}{\lambda}$	$\frac{2-p}{\lambda^2 p}$	$\sqrt{\frac{2-p}{p}}/\big(1-\log(p)\big)$
Pareto $Pareto(a,b)$	$\frac{ab}{a-1}p^{-1/a}$	$b^2 p^{-\frac{2+a}{a}}\left\{\frac{a}{a-2}-\frac{a^2}{(a-1)^2}+(1-p)\left[\frac{a}{a-1}-1\right]^2\right\}$	$\frac{(a-1)p^{-1/2}\sqrt{\frac{a}{a-2}-\frac{a^2}{(a-1)^2}+(1-p)\left[\frac{a}{a-1}-1\right]^2}}{a}$
Gaussian $N(0,1)$	$\frac{1}{p}\phi\big(\Phi^{-1}\big)$	$\frac{1}{p^2}\left[p+(2p-1)\phi\big(\Phi^{-1}\big)\Phi^{-1}+(1-p)p\big[\Phi^{-1}\big]^2-\big[\phi\big(\Phi^{-1}\big)\big]^2\right]$	$\frac{\sqrt{\left[p+(2p-1)\phi\left(\Phi^{-1}\right)\Phi^{-1}+(1-p)p\left[\Phi^{-1}\right]^2-\left[\phi\left(\Phi^{-1}\right)\right]^2\right]}}{\phi\left(\Phi^{-1}\right)}$

The first column gives the distribution specifications with their parameters. The second column provides $TVaR(p)$ and the third column lists the asymptotic variance of the nonparametric estimator of $TVaR(p)$. The relative accuracy $(\sigma(\widetilde{TVaR_T}(p))/TVaR(p))$ is provided in column 4. Φ and ϕ denote the cdf and pdf of standard normal distribution, respectively. For shortening the expressions associated with the standard normal distribution, we denote $\Phi^{-1}(1-p)$ by Φ^{-1}. The variance is defined for the estimator scaled by $\sqrt{T}$.

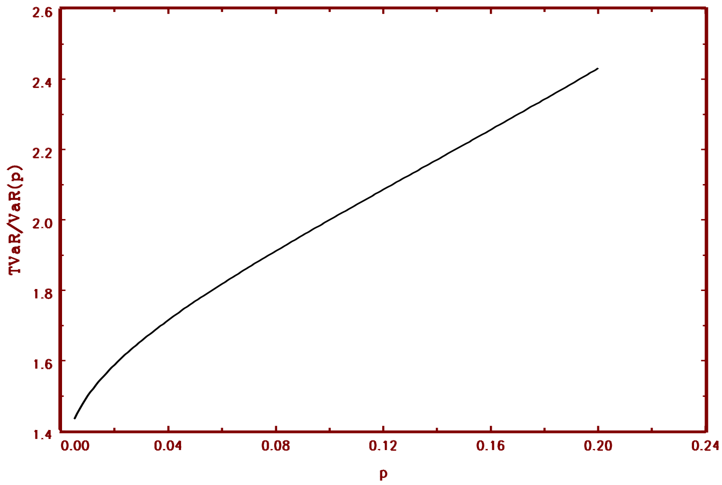
Table 5: Summary statistics of currency portfolio returns

Basic	Mean	Variance	Skewness	Excess Kurtosis
1 day				
US$	0.000002	0.000049	-0.680470*	5.195544*
SIN$	-0.000008	0.000042	-1.195269*	12.987396*
20 days				
US$	0.000204	0.000968	-0.981831*	2.356374*
SIN$	-0.000014	0.000697	-0.483116*	0.142152

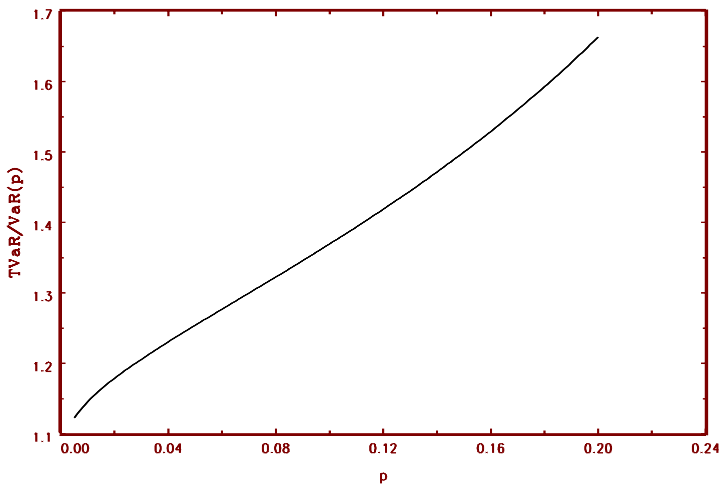
The star (*) introduced for mean and skewness indicates significant results and for excess kurtosis result significantly different from 3, that is the kurtosis of a standard normal distribution.



(a) $X \sim U(0,1)$



(b) $X \sim \gamma(1,2)$



(c) $X \sim N(0,1)$

Figure 1: $\frac{TVaR(p)}{VaR(p)}$ as function of p.

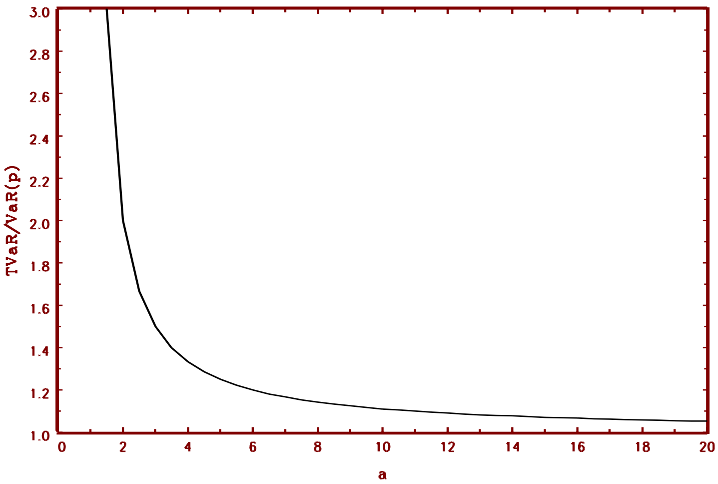


Figure 2: $\frac{TVaR(p)}{VaR(p)}$ as function of a when $X \sim Pareto(a, b)$.

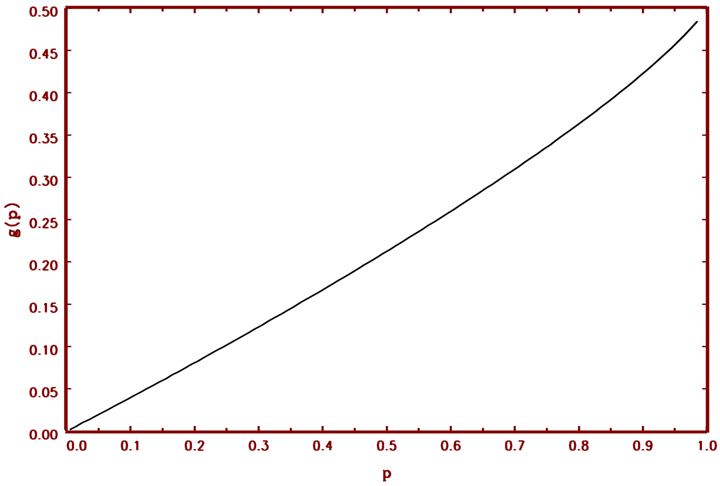


Figure 3: $g(p)$ when $X \sim N(0, 1)$.

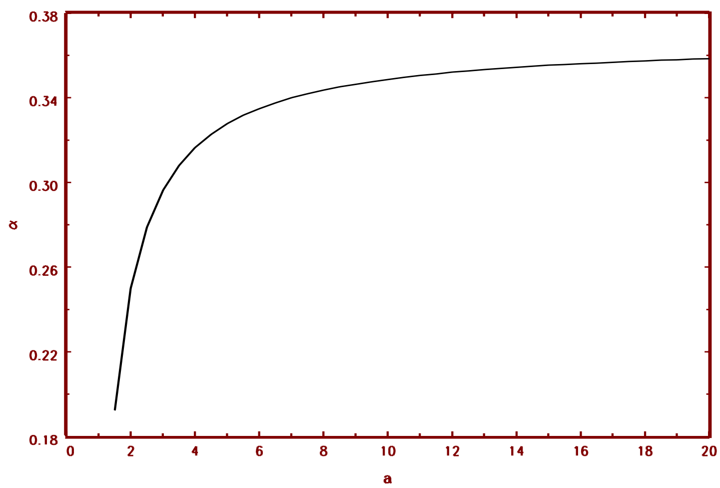
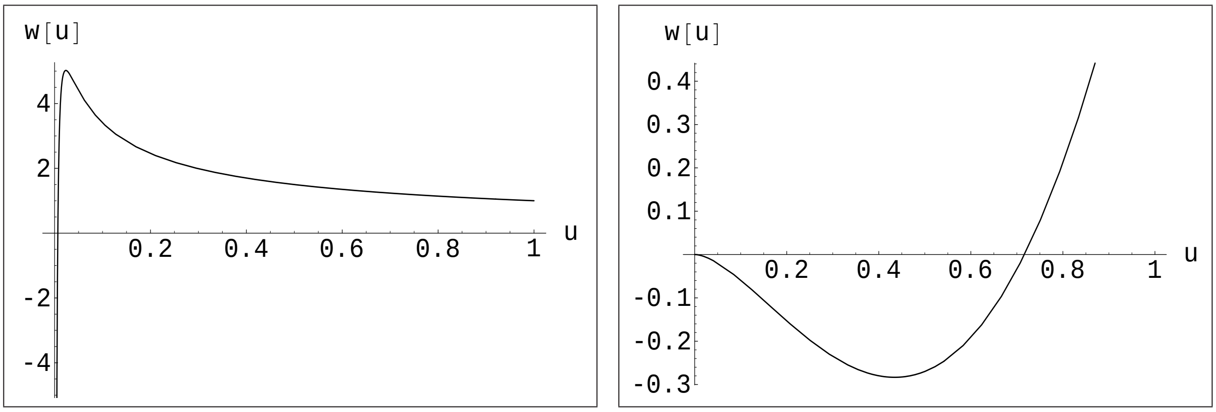
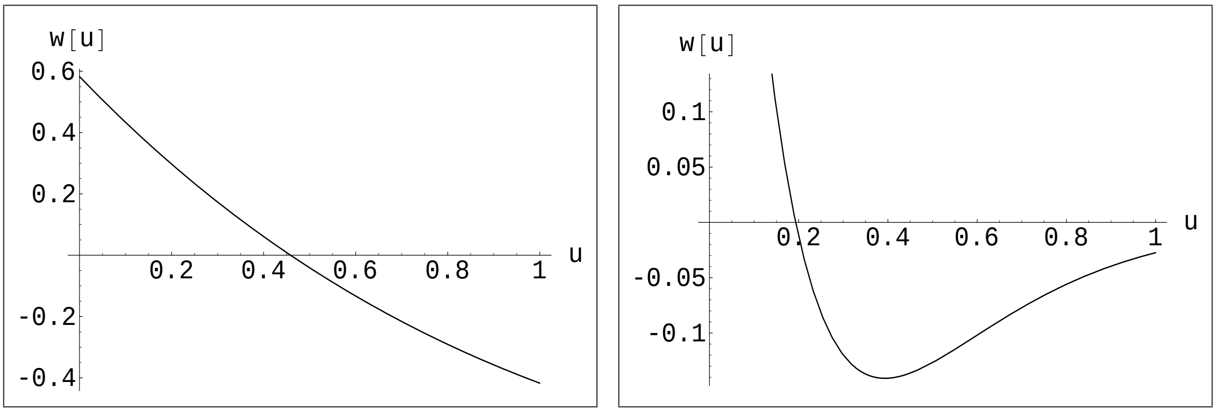


Figure 4: α as function of a when $X \sim Pareto(a, b)$.



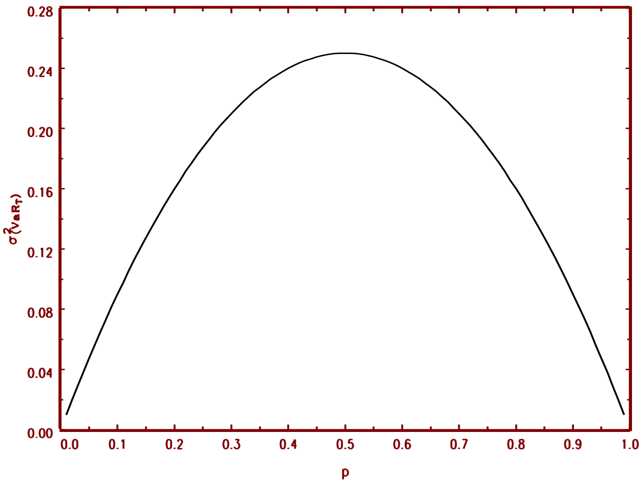
(a) $p = 0.2$ (b) $p = 2$

Figure 5: Plot of the weights $w(u, p)$ as a function of u in the sensitivity of $PH(p)$.

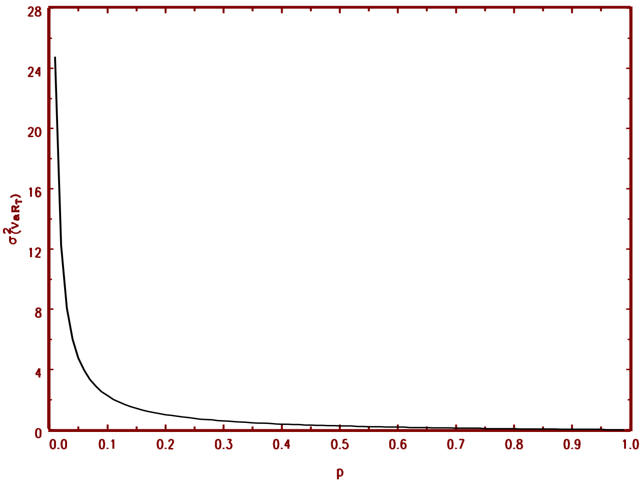


(a) $p = 0.5$ (b) $p = 5$

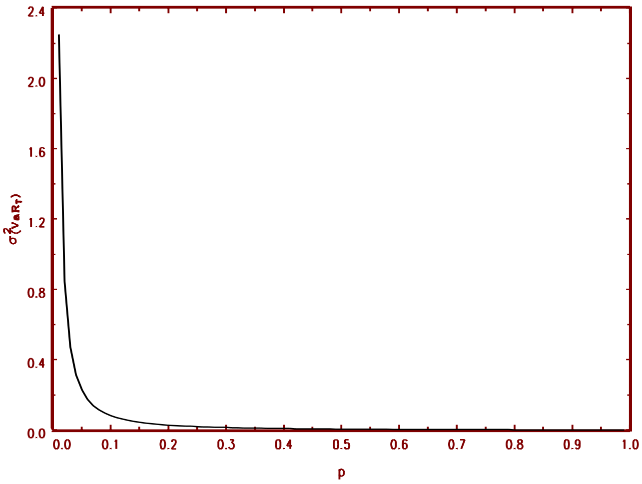
Figure 6: Plot of the weights $w(u, p)$ as a function of u in the sensitivity of $EX(p)$.



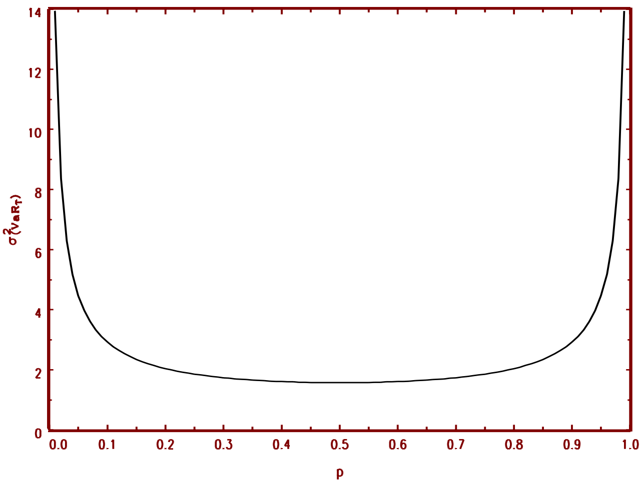
$U(0,1)$



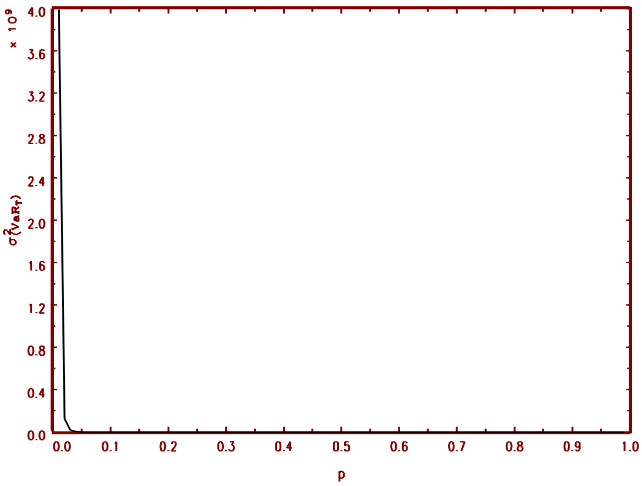
$\gamma(1,\lambda)$



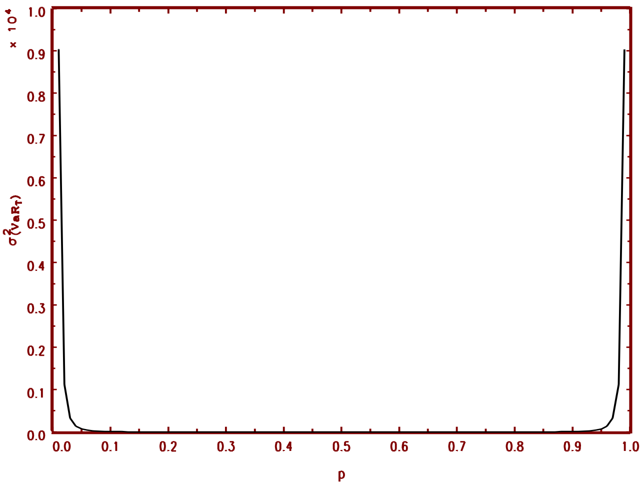
$Pareto(5,0.3)$



$N(0,1)$

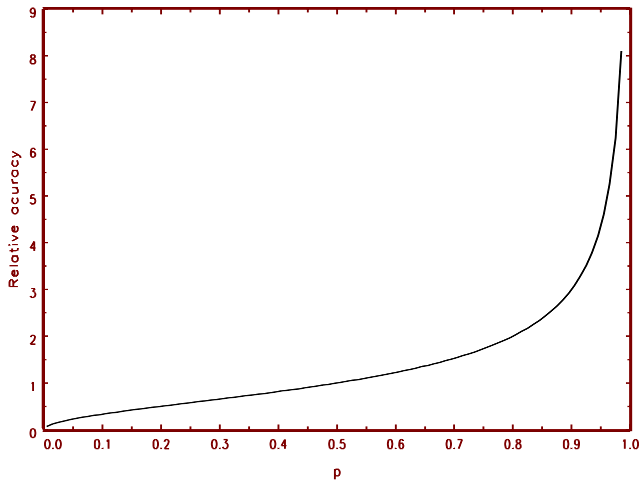


$Lévy(1/2)$

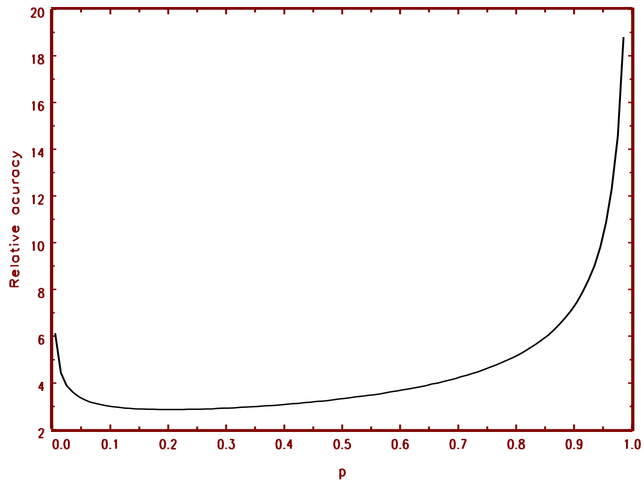


$Cauchy(0,1)$

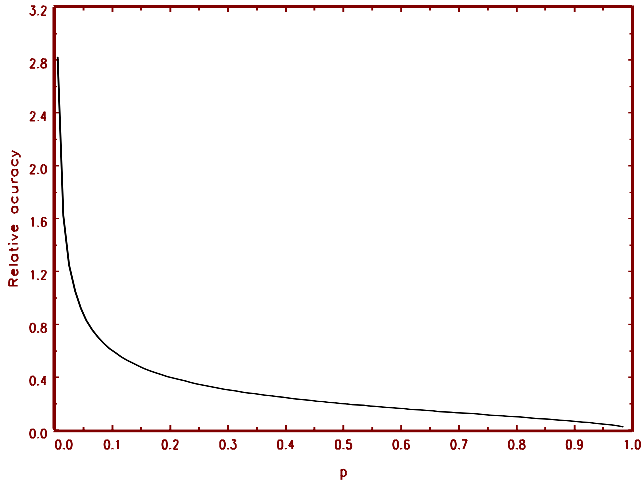
(a) Variance



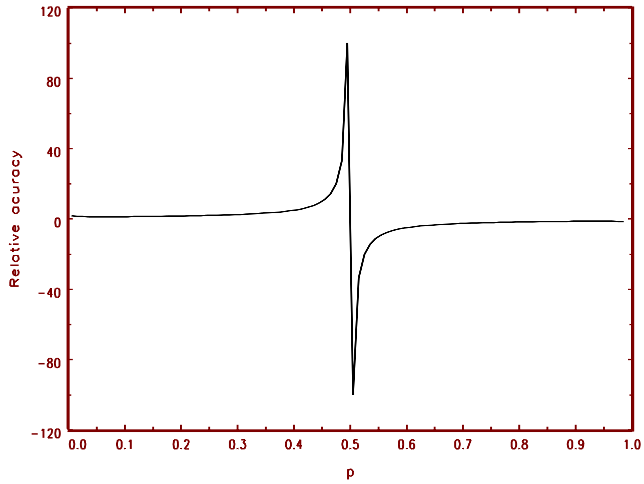
$U(0,1)$



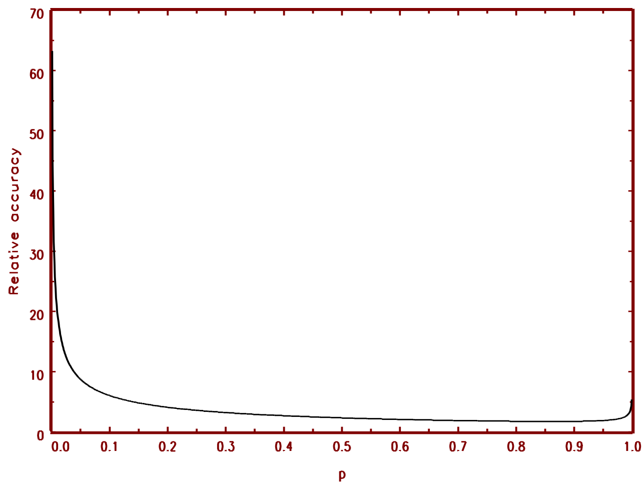
$\gamma(1,\lambda)$



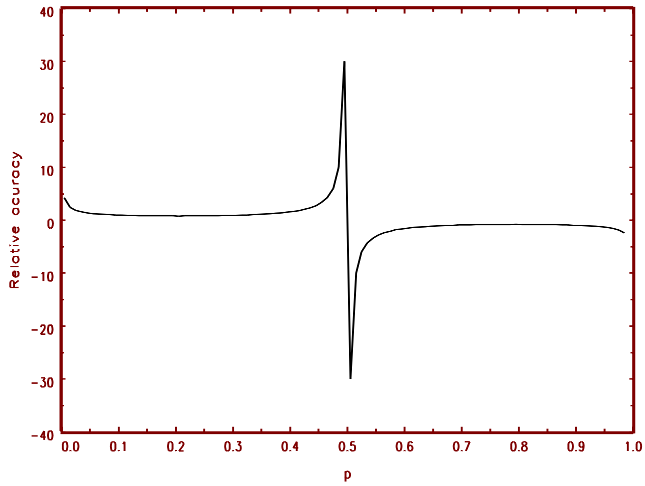
$Pareto(5, 0.3)$



$N(0,1)$



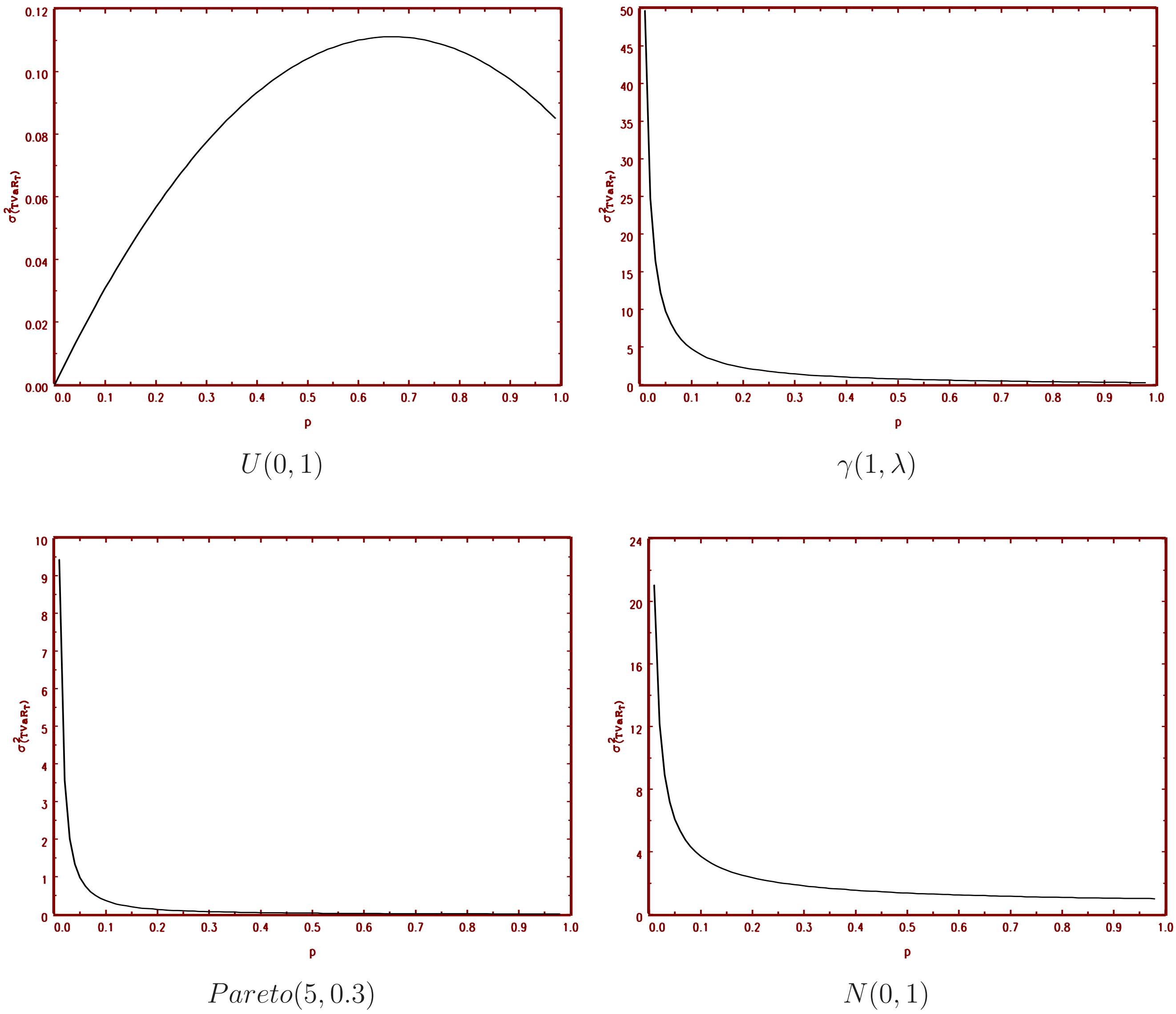
$Lévy(1/2)$



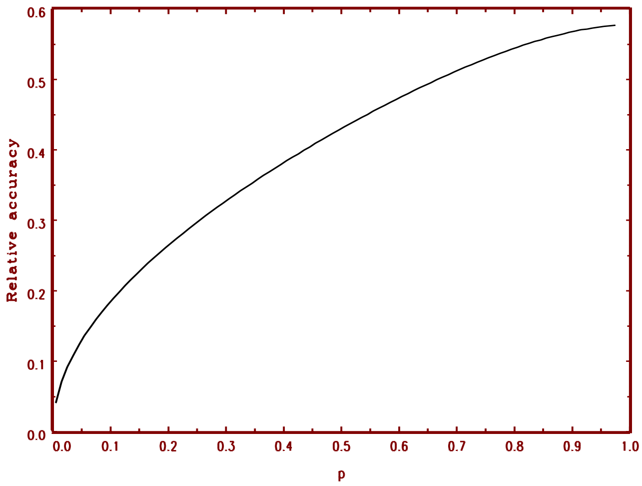
$Cauchy(0,1)$

(b) Relative accuracy

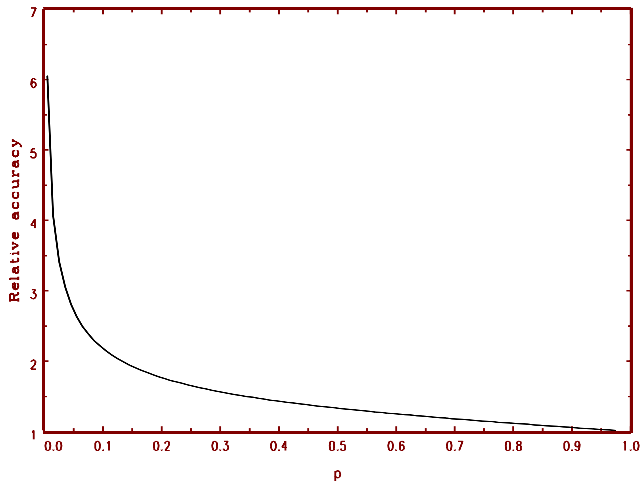
Figure 7: Variance and relative accuracy for $\widetilde{VaR_T}(p)$.



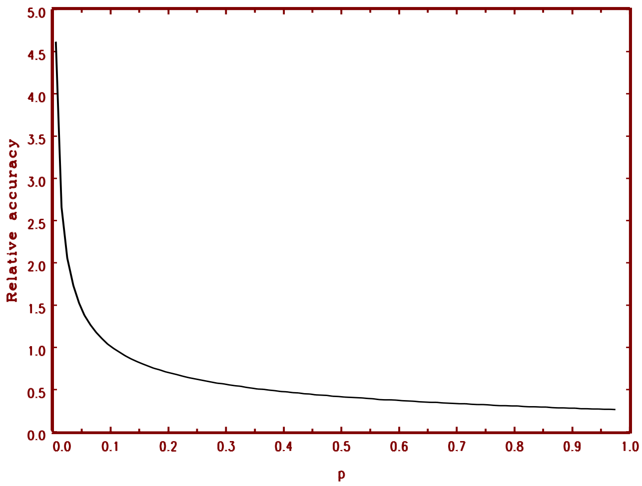
(a) Variance



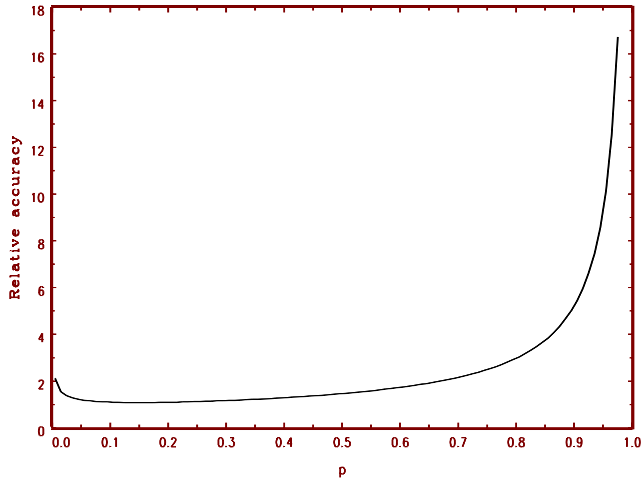
$U(0, 1)$



$\gamma(1, \lambda)$



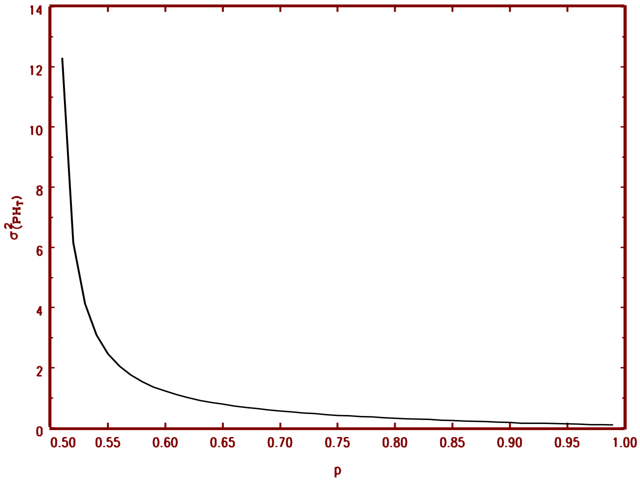
$Pareto(5, 0.3)$



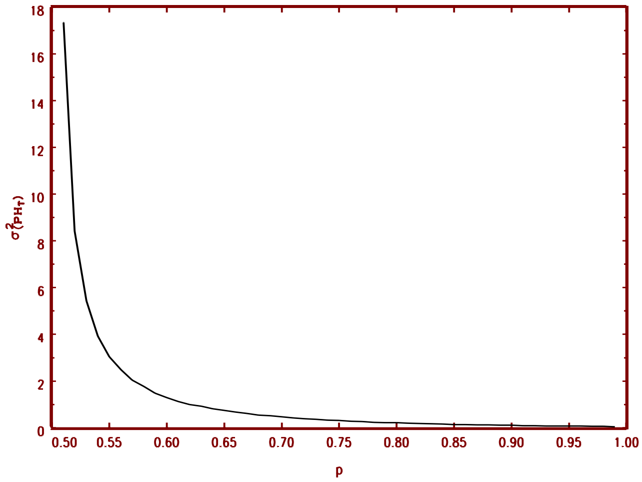
$N(0, 1)$

(b) Relative accuracy

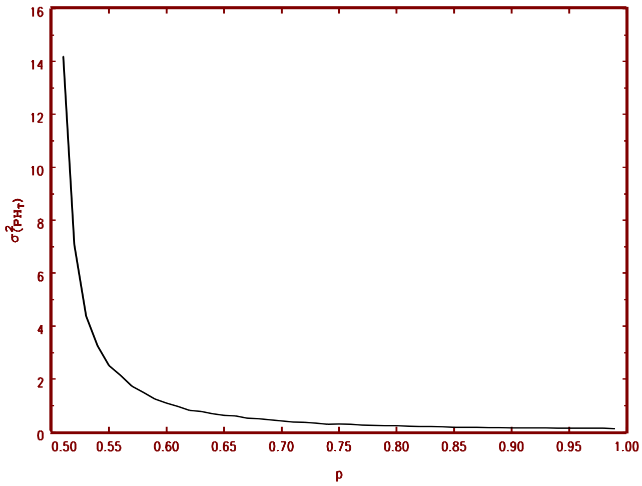
Figure 8: Variance and relative accuracy for $\widetilde{TVaR}_T(p)$.



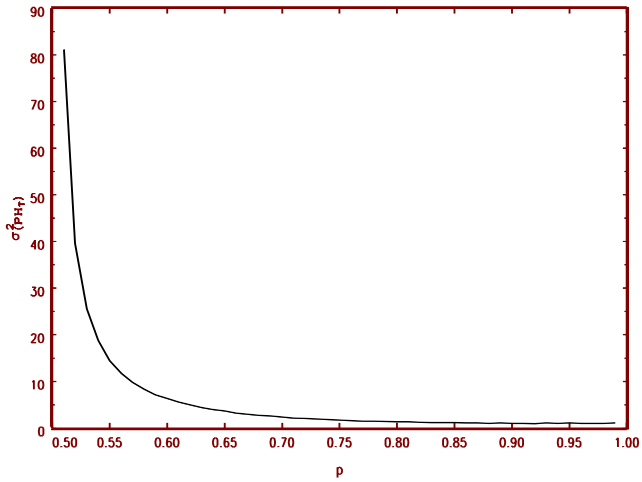
$U(0,1)$



$\gamma(1,\lambda)$

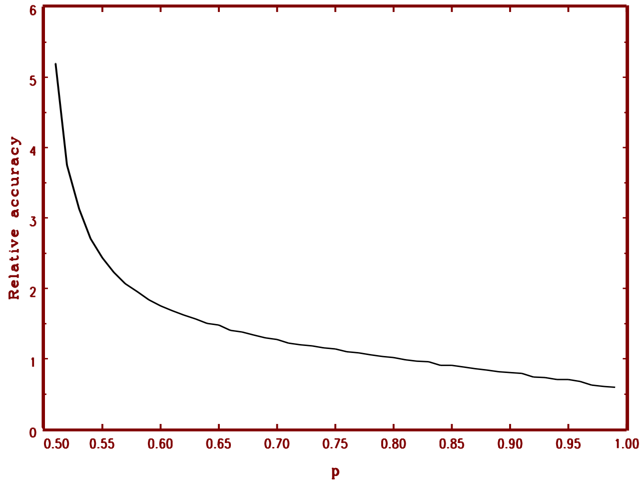


$Pareto(5,0.3)$

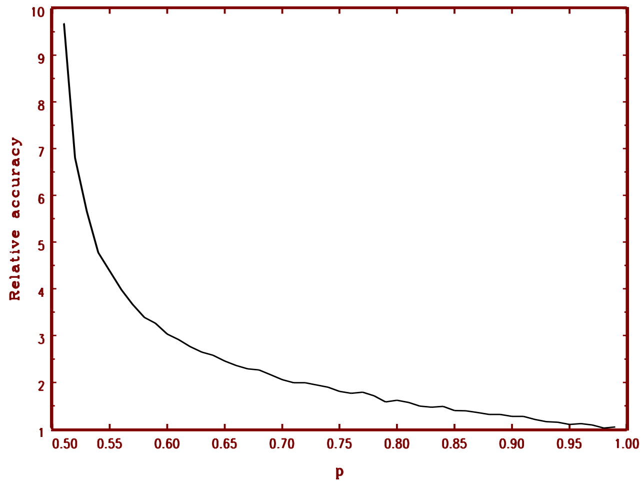


$N(0,1)$

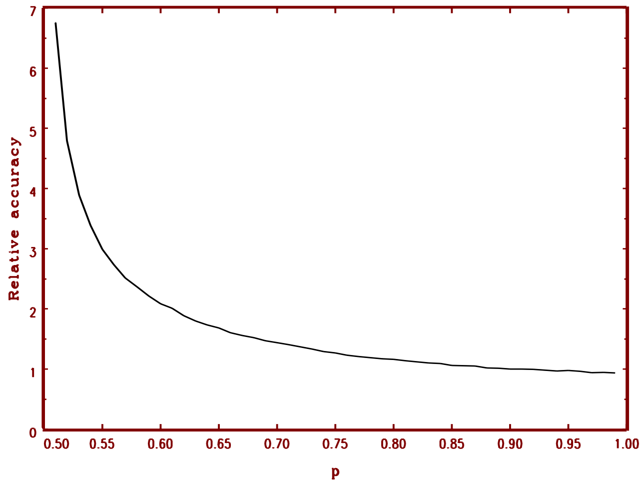
(a) Variance



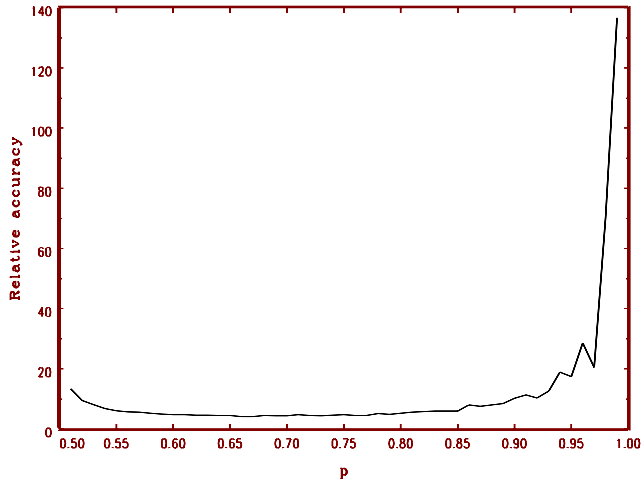
$U(0, 1)$



$\gamma(1, \lambda)$



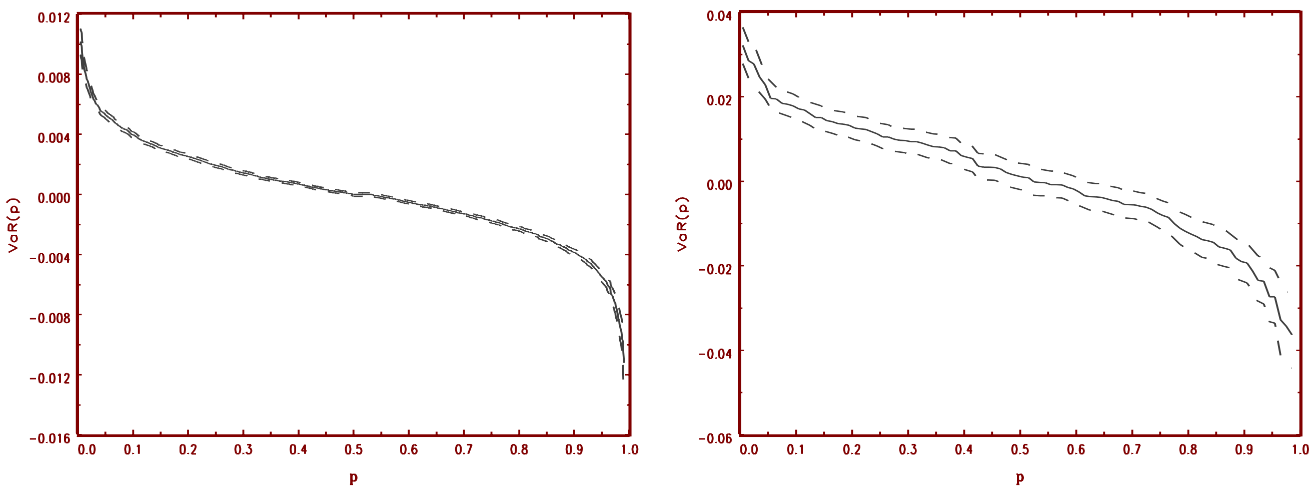
$Pareto(5, 0.3)$



$N(0, 1)$

(b) Relative accuracy

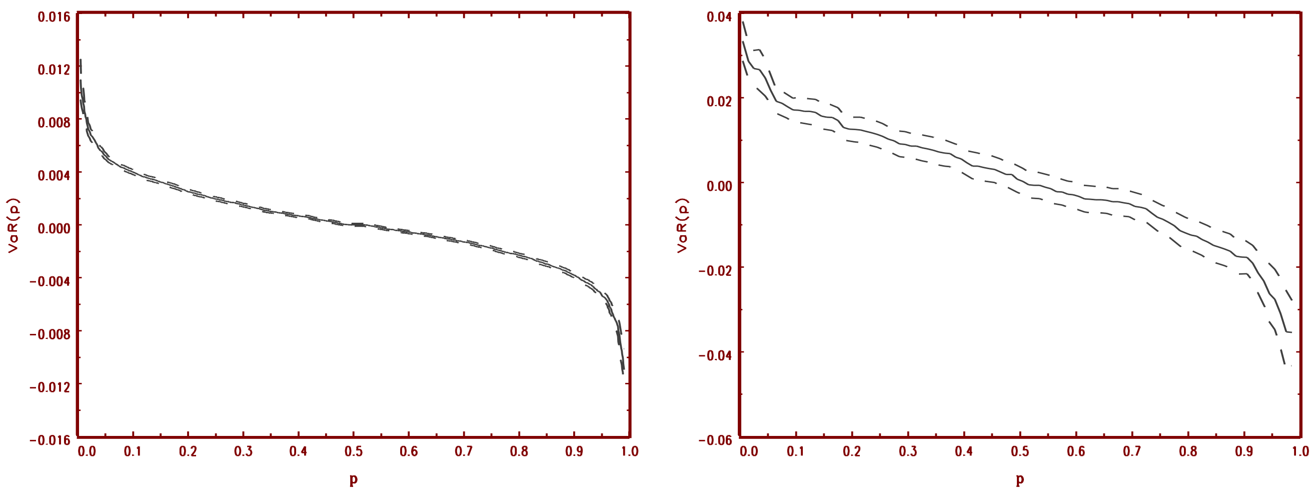
Figure 9: Variance and relative accuracy for $\widetilde{PH}_T(p)$.



Horizon = 1 day

Horizon = 20 days

(a) Base:US$

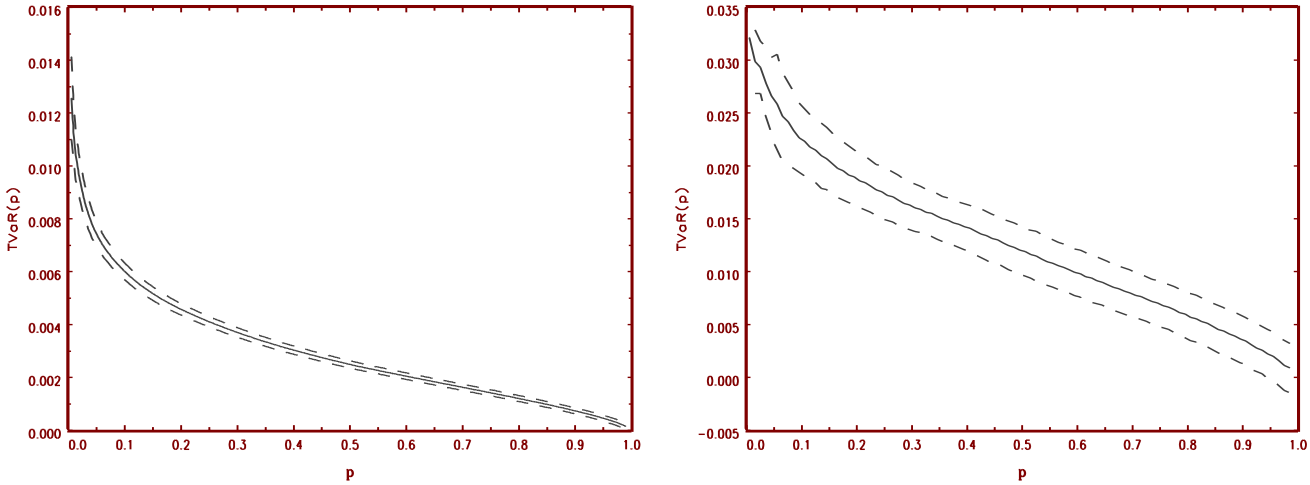


Horizon = 1 day

Horizon = 20 days

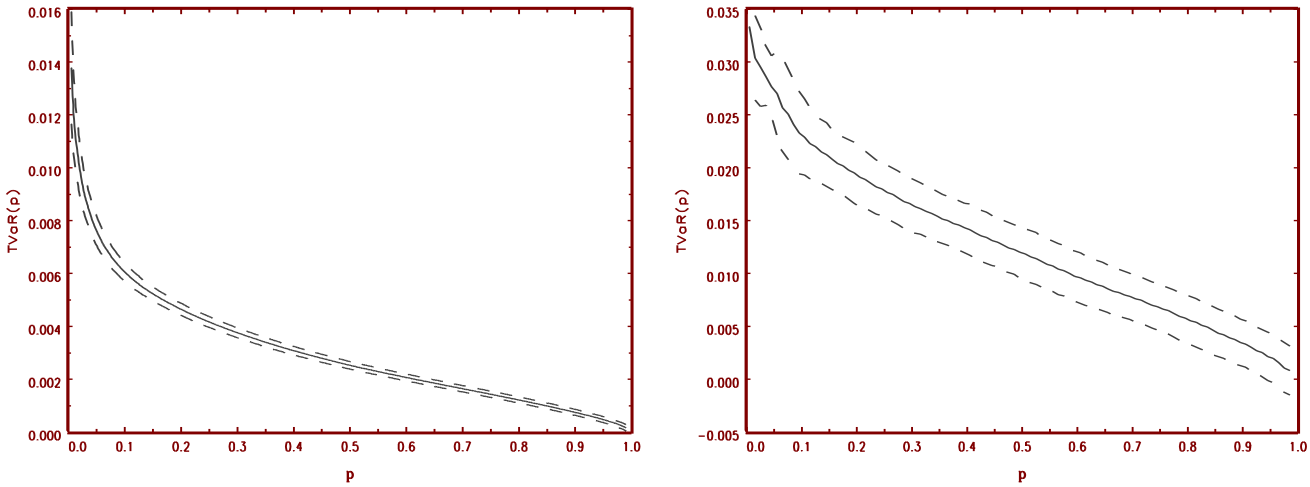
(b) Base:SIN$

Figure 10: Value-at-Risk of the currency portfolio.



Horizon = 1 day　　Horizon = 20 days

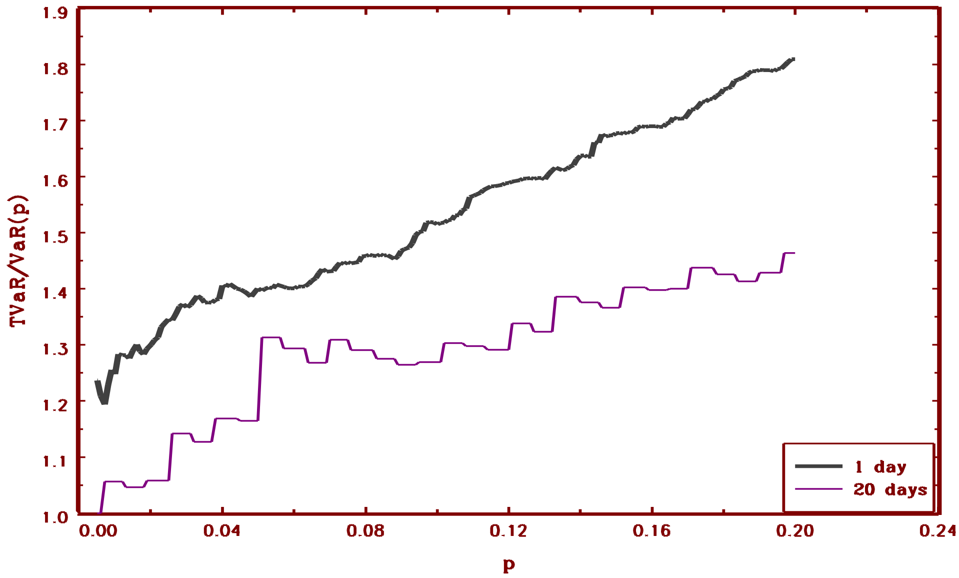
(a) Base:US$



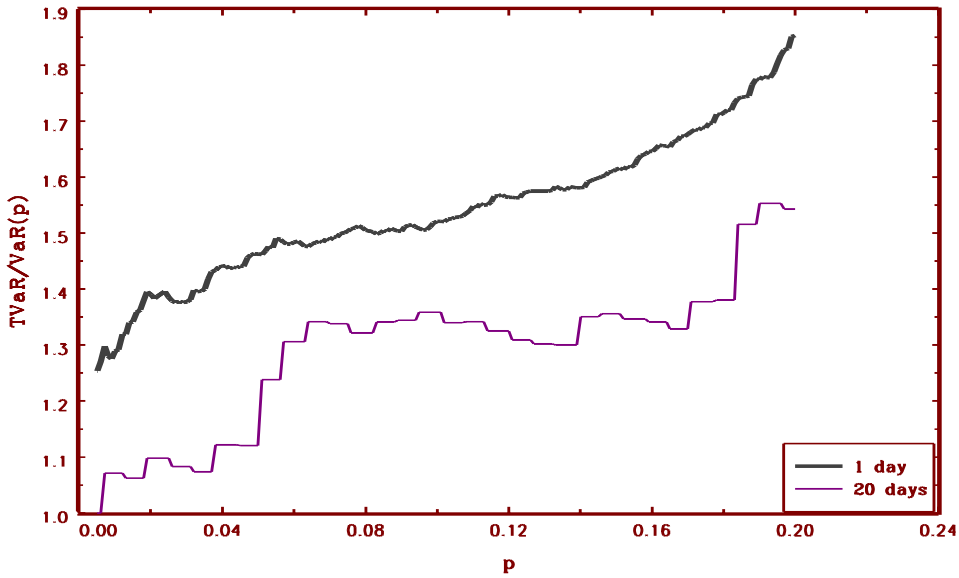
Horizon = 1 day　　Horizon = 20 days

(b) Base:SIN$

Figure 11: Tail-VaR of the currency portfolio.

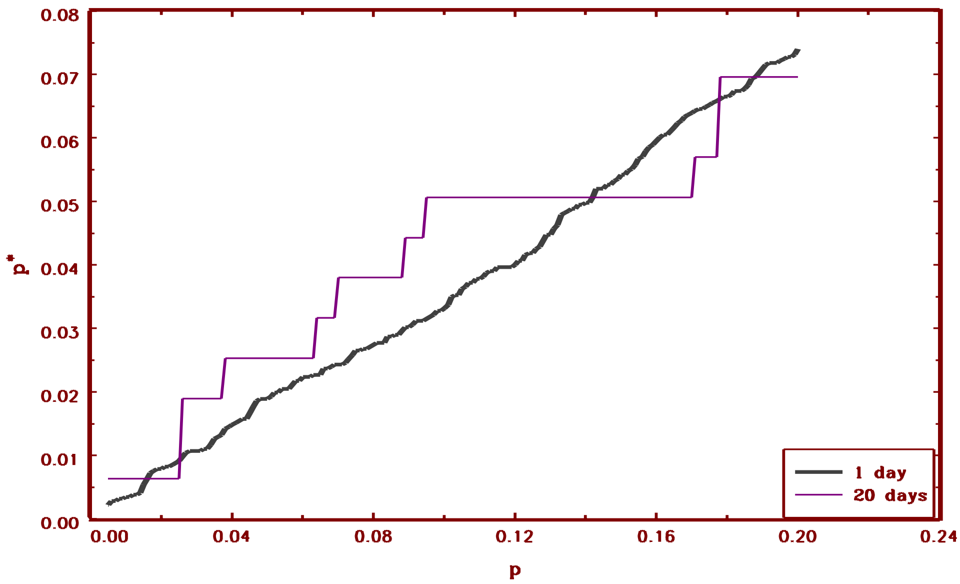


(a) Base: US$

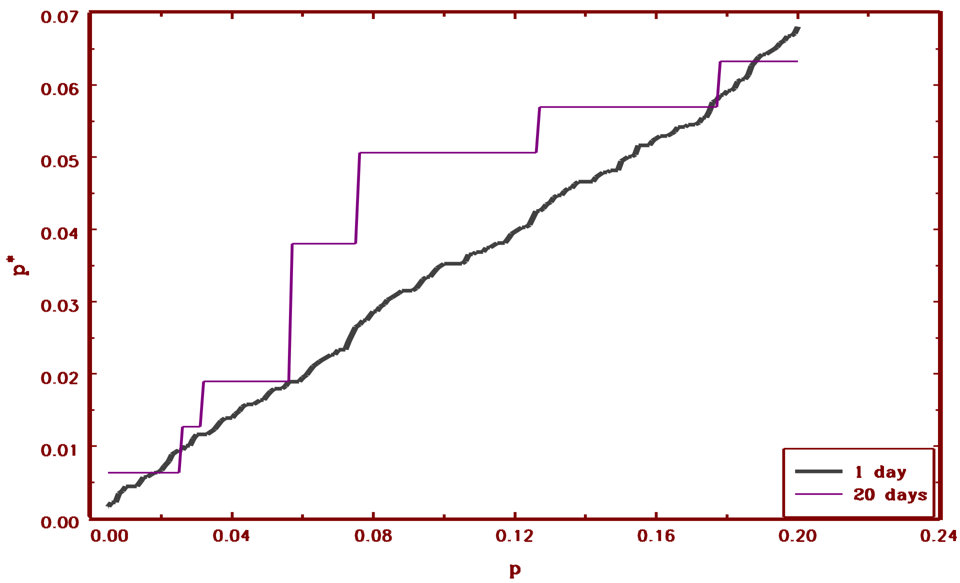


(b) Base: SIN$

Figure 12: $\frac{TVaR(p)}{VaR(p)}$ of the equally weighted currency portfolio.



(a) Base: US$



(b) Base: SIN$

Figure 13: Estimated link $g(p)$ for the equally weighted currency portfolio.

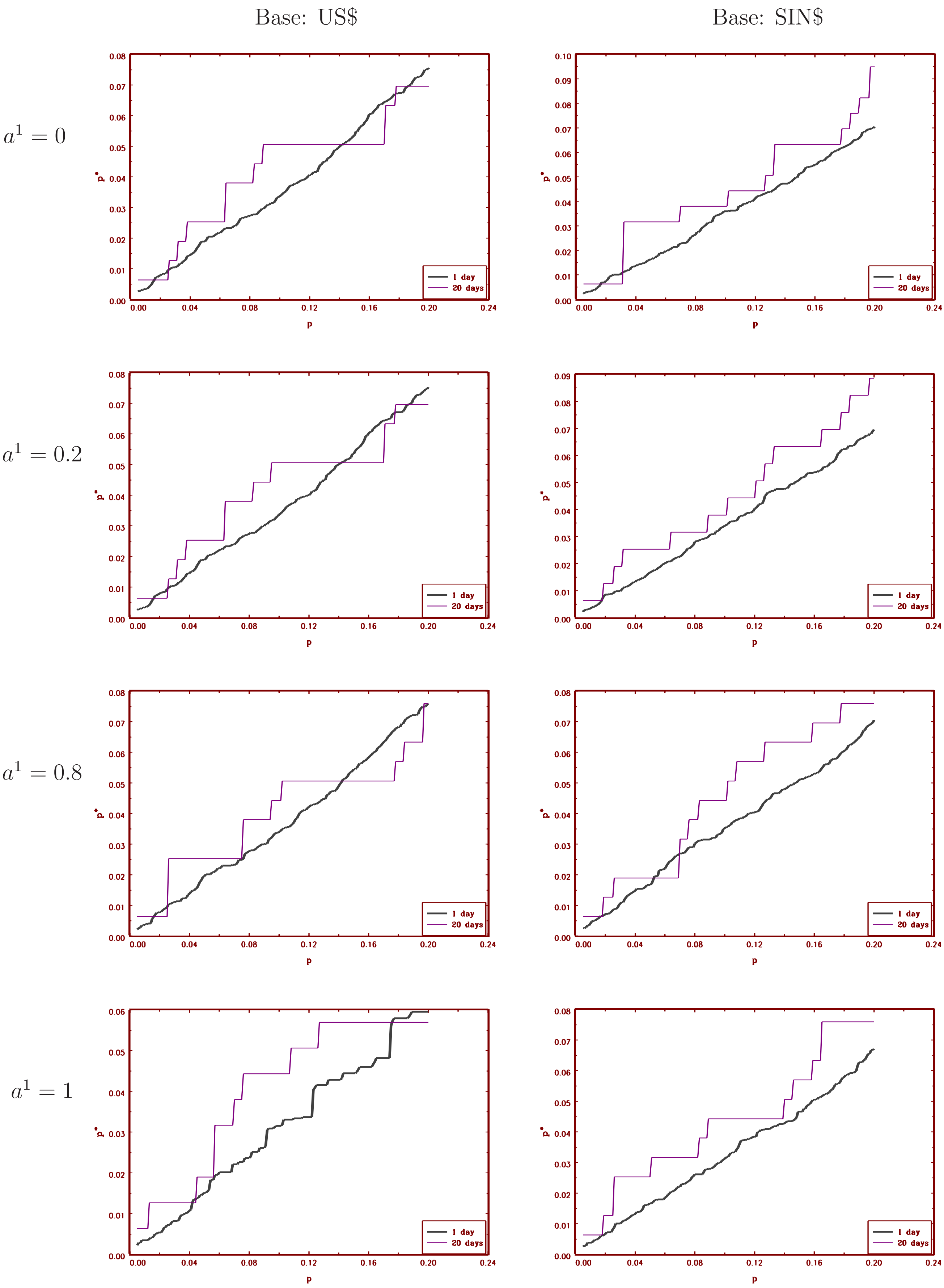
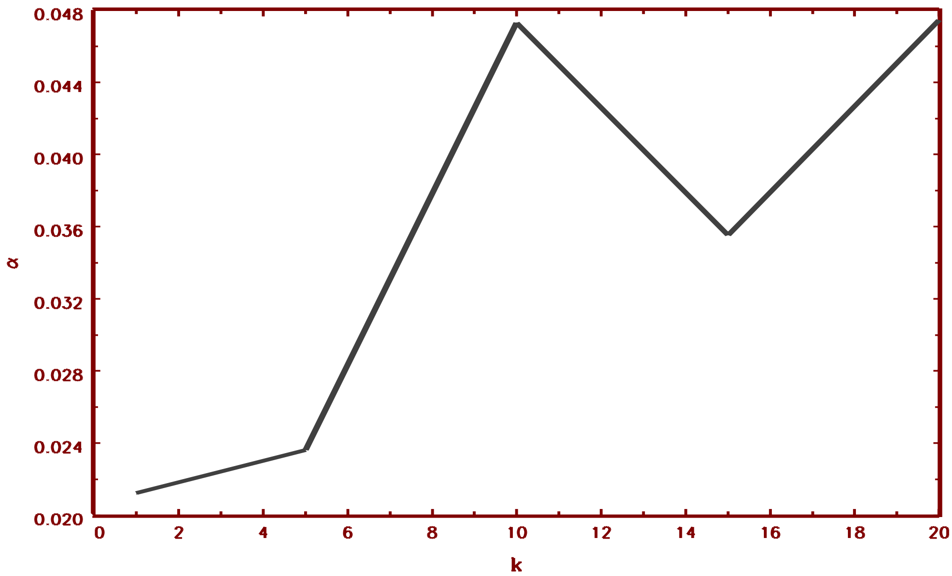
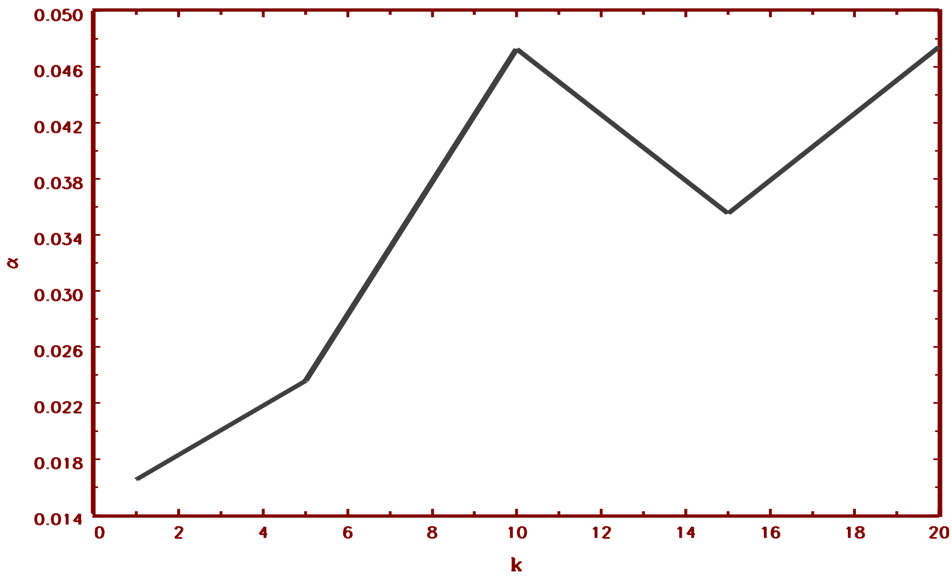


Figure 14: Estimated link $g(p)$ for the currency portfolios and varying allocations.



(a) base: US$



(b) base: SIN$

Figure 15: Term structure of slope parameter α